Antiderivative: Integration

$\int f'(x)\ dx = f(x) + c$	$\int (x^n)\ dx = \frac{x^{n+1}}{n+1} + c$
$\int (\sin x)\ dx = -\cos x + c$	$\int (\cos x)\ dx = \sin x + c$
$\int (\sec^2 x)\ dx = \tan x + c$	$\int (\sec x\ .\ \tan x)\ dx = \sec x + c$
$\int (\csc^2 x)\ dx = -\cot x + c$	$\int (\csc x\ .\ \cot x)\ dx$ $= -\csc x + c$
$\int (\cosh x)\ dx = \sinh x + c$	$\int (\sinh x)\ dx = \cosh x + c$

Inverse Trigonometric Functions

$\int \left(\frac{1}{\sqrt{1-x^2}}\right) dx = \sin^{-1} x + c$	$\int \left(\frac{1}{\sqrt{a^2-x^2}}\right) dx = \sin^{-1}\left(\frac{x}{a}\right) + c$
$\int \left(\frac{1}{1+x^2}\right) dx = \tan^{-1} x + c$	$\int \left(\frac{1}{a^2+x^2}\right) dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
$\int \left(\frac{1}{x\sqrt{x^2-1}}\right) dx = \sec^{-1} x + c$	$\int \left(\frac{1}{x\sqrt{x^2-a^2}}\right) dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$

Remember

$\sin\theta = \dfrac{1}{\csc\theta}$	$\cos\theta = \dfrac{1}{\sec\theta}$
$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\cot\theta}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$
$\sec\theta = \dfrac{1}{\cos\theta}$	$\csc\theta = \dfrac{1}{\sin\theta}$

$\boxed{\sin^2 x = \frac{1}{2}(1 - \cos 2x)}$, $\boxed{\cos^2 x = \frac{1}{2}(1 + \cos 2x)}$

• Example:

1) $\int (x-1)(x+3)dx$	2) $\int (x^2 + \cancel{2x+} 3)^2 dx$
3) $\int \left(\dfrac{x^2-4}{x-2}\right) dx$	4) $\int \left(\dfrac{x+1}{\sqrt{x}}\right) dx$
5) $\int (3\cos x + \csc^2 x)\, dx$	6) $\int \left(\dfrac{2}{\cos^2 x}\right) dx$
7) $\int \left(\dfrac{2\tan x}{\sec x}\right) dx$	8) $\int \left(\dfrac{\sin x}{\cos^2 x}\right) dx$
9) $\int \left(\dfrac{\cos x - 1}{\sin^2 x}\right) dx$	10) $\int \left(\dfrac{1}{1+\sin x}\right) dx$

Solutions:

1) $\int (x-1)(x+3)dx = \int (x^2+3x-x-3)dx$

$= \int (x^2+2x-3)dx = \frac{x^3}{3} + \frac{2x^2}{2} - 3x + c$

$= \frac{1}{3}x^3 + x^2 - 3x + c$

2) $\int (x^2+3)^2 dx = \int (x^4+6x^2+9)\,dx$

$= \frac{x^5}{5} + \frac{6x^3}{3} + 9x + c = \frac{1}{5}x^5 + 2x^3 + 9x + c$

3) $\int \left(\frac{x^2-4}{x-2}\right) dx = \int \left(\frac{(x-2)(x+2)}{(x-2)}\right) dx = \frac{x^2}{2} + 2x + c$

4) $\int \left(\frac{x+1}{\sqrt{x}}\right) dx = \int \left(\frac{x}{x^{\frac{1}{2}}} + \frac{1}{x^{1/2}}\right) dx$

$= \int \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) dx = \frac{2}{3} x^{3/2} + 2x^{\frac{1}{2}} + C$

5) $\int (3\cos x + \csc^2 x)\,dx = \int 3\cos x\,dx + \int \csc^2 x\,dx$

$= 3\sin x + (-\cot x) + c$

$= 3\sin x - \cot x + c$

6) $\int \left(\frac{2}{\cos^2 x}\right) dx = 2\int \sec^2 x\,dx$

$= 2\tan x + c$

7) $\int \frac{2\tan x}{\sec x}\,dx = 2\int \left(\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}}\right) dx = 2\int \sin x\,dx$

$= 2(-\cos x) + c = -2\cos x + c$

8) $\int \frac{\sin x}{\cos^2 x}\,dx = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\,dx$

$= \int \tan x \cdot \sec x\,dx = \sec x + c$

9) $\int \frac{\cos x - 1}{\sin^2 x}\,dx = \int \left(\frac{\cos x}{\sin^2 x} - \frac{1}{\sin^2 x} \right) dx$

$= \int \left(\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} - \csc^2 x \right) dx$

$= \int (\cot x \cdot \csc x - \csc^2 x)\,dx$

$= -\csc x - (-\cot x) + c = -\csc x + \cot x + c$

10) $\int \frac{1}{1+\sin x}\,dx = \int \frac{1}{1+\sin x} \cdot \frac{1-\sin x}{1-\sin x} \cdot dx$

$= \int \frac{1-\sin x}{1-\sin^2 x}\,dx = \int \frac{1-\sin x}{\cos^2 x} \cdot dx$

$= \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx = \int \left(\sec^2 x - \tan x \cdot \frac{1}{\cos x} \right) dx$

$= \int (\sec^2 x - \tan x \cdot \sec x)\,dx$

$= \tan x - \sec x + c$

The Definite Integral:

$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$$

$$\boxed{\int_a^a f(x)\,dx = 0}$$

***a* is <u>lower limit</u>, *b* is <u>upper limit</u>**

- ***<u>Properties of The Integral: ((c is constant))</u>***

1. $\boxed{\int_a^b c\,dx = c(b - a)}$

2. $\boxed{\int_a^b [\,f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx}$

3. $\boxed{\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx}$

4. $\boxed{\int_a^b [\,f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx}$

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

- ## *Comparison Properties Of the Integral:*

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a)$$

The Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(x)\Big]_a^b = f(b) - f(a)$$

- **Examples:**

1) $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{8}{1+x^2}\,dx$

Solution:

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{8}{1+x^2}\,dx = 8\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+x^2} = 8\left[\tan^{-1} x\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$= 8\left[\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= 8\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = 8\left[\frac{2\pi - \pi}{6}\right] = 8\left[\frac{\pi}{6}\right]$$

$$= \frac{4}{3}\pi$$

Indefinite integrals and the Net Change Theorem:

$\int \frac{1}{x} dx = \ln\|x\| + c$	$\int \frac{f'(x)}{f(x)} dx = \ln\|fx\| + c$
$\int e^x\, dx = \frac{e^x}{\ln(e)} + c = e^x + c$	$\int a^x\, dx = \frac{a^x}{\ln(a)} + c$

- ***Examples:***

1) $\int_{-1}^{1} e^{u+1}\, du$	2) $\int \frac{x+1}{x} dx$	3) $\int \frac{dx}{e^x}$
4) $\int \frac{x}{x^2+1} dx$	5) $\int \frac{e^x}{e^x+1} dx$	6) $\int \frac{3x+2}{x^2+9} dx$

Solutions:

1) $\int_{-1}^{1} e^{u+1}\, du = e^{u+1}\Big]_{-1}^{1} = e^{1+1} - e^{-1+1} = e^2 - e^0$

$= e^2 - 1$

2) $\int \frac{x+1}{x} dx = \int \left(\frac{x}{x} + \frac{1}{x}\right) dx = \int \left(1 + \frac{1}{x}\right) dx$

$= x + \ln|x| + c$

3) $\int \frac{dx}{e^x} = \int \frac{1}{e^x} dx = \int e^{-x} dx = \frac{e^{-x}}{\ln e^{-1}}$

$= \frac{e^{-x}}{-\ln e} + c = -e^{-x} + c$

4) $\int \frac{x}{x^2+1}\,dx = \frac{2}{2}\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$

$= \frac{1}{2}\ln|x^2+1| + c$

5) $\int \frac{e^x}{e^x+1}\,dx = \ln|e^x+1| + c$

6) $\int \frac{3x+2}{x^2+9}\,dx = \int\left(\frac{3x}{x^2+9} + \frac{2}{x^2+9}\right)dx$

$= 3\int \frac{x}{x^2+9}\,dx + 2\int \frac{1}{x^2+9}\,dx$

$= \frac{3}{2}\int \frac{2x}{x^2+9}\,dx + 2\int \frac{1}{x^2+9}\,dx$

$= \frac{3}{2}\ln|x^2+9| + 2\left(\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right) + c$

$= \frac{3}{2}\ln|x^2+9| + \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$

- ***Remark:***

$$\int f'(ax+b)\,dx = \frac{f(ax+b)}{a} + c$$

- ***Examples:***

1) $\int (1-\sin 3x)dx$	2) $\int \sec^2(7x+3)dx$
3) $\int (\sin^2 x)\,dx$	4) $\int \frac{dy}{\csc 4y}$

Solutions

1) $\int (1-\sin 3x)\,dx = x + \frac{\cos(3x)}{3} + c$

$= x + \frac{1}{3}\cos(3x) + c$

2) $\int \sec^2(7x+3)\,dx = \frac{1}{7}\tan(7x+3) + c$

3) $\int \sin^2(x)\,dx = \int \left(\frac{1}{2}(1-\cos 2x)\right) dx$

$= \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right) = \frac{x}{2} - \frac{1}{4}\sin(2x) + c$

4) $\int \frac{dy}{\csc 4y} = \int \frac{1}{\csc 4y}\,dy = \int \sin(4y)\,dy$

$= \frac{-\cos(4y)}{4} + c$

- ***Remark:***

$\int (\tan x)dx = \ln\lvert\sec x\rvert + c$	$\int (\cot x)dx = \ln\lvert\sin x\rvert + c$
$\int \sec x\, dx = \ln\lvert\sec x + \tan x\rvert + c$	$\int \csc x\, dx = \ln\lvert\csc x - \cot x\rvert + c$