The Substitution Rule:

$$\int f(g(x)).\ g'(x)\,dx = \int f(u)\ du$$

$$\int_a^b f(g(x)).\ g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\ du$$

$$u = g(x)$$
$$du = g'(x)\,dx$$

- ملاحظة:دائما نفرض $u = lnx\,, sin^{-1}x\,, tan^{-1}x \ldots\ldots\ldots$:

1- تكامل بالتعويض اذا كانت مشتقتها موجوده By Substitution: Calculus A

2- تكامل بالتعويض اذا لم تكن مشتقتها موجوده By Parts: Calculus B

- **Examples:**

1) $\int \frac{1}{1+(x+3)^2}dx$	2) $\int \frac{x^2}{1+x^6}dx$	3) $\int_1^{\sqrt{3}} \frac{\sqrt{tan^{-1}x}}{1+x^2}dx$
4) $\int_1^{\sqrt{e}} \frac{dx}{x\sqrt{1-(lnx)^1}}$	5) $\int \frac{e^x}{\sqrt{4-e^{2x}}}dx$	6) $\int \frac{e^{2x}}{\sqrt{4-e^{2x}}}dx$
7) $\int \frac{dx}{\sqrt{9-4x^2}}$	8) $\int \sqrt{2x+3}\,dx$	9) $\int x\,sec^2(x^2)\,dx$
10) $\int 4x\,tan(x^2)dx$	11) $\int \frac{sec(lnx)\,tan(lnx)}{x}dx$	12) $\int \frac{x}{\sqrt{1-x^4}}dx$
13) $\int \frac{e^{\sqrt{x-1}}}{\sqrt{x-1}}dx$	14) $\int \frac{dx}{x(lnx)^2}$	15) $\int \frac{cos(lnx)}{x}dx$
16) $\int \frac{x}{csc\,(x^2)}dx$	17) $\int \left(x\,5^{-x^2}\right)dx$	18) $\int_0^1 \left(\frac{e^z+1}{e^z+z}\right)dz$
19) $\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$	20) $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}dx$	21) $\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$

π

Answers :-

1) $\displaystyle\int \frac{1}{1+(x+3)^2}\,dx$

let $\boxed{u = x+3}$

$\boxed{du = dx}$

$$\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}(u) + C$$
$$= \tan^{-1}(x+3) + C$$

2) $\displaystyle\int \frac{x^2}{1+x^6}\,dx = \int \frac{x^2}{1+(x^3)^2}\,dx$

let $\boxed{u = x^3}$

$du = 3x^2\,dx \Rightarrow \boxed{dx = \frac{du}{3x^2}}$

$$\int \frac{x^2}{1+u^2}\cdot\frac{du}{3x^2} = \frac{1}{3}\int \frac{1}{1+u^2}\,dx$$
$$= \frac{1}{3}\tan^{-1}u + C$$
$$= \frac{1}{3}\tan^{-1}(x^3) + C$$

3) $\displaystyle\int_1^{\sqrt{3}} \frac{\sqrt{\tan^{-1}x}}{1+x^2}\,dx$

$\boxed{u = \tan^{-1}x}$

$du = \frac{1}{1+x^2}\,dx \Rightarrow \boxed{dx = (1+x^2)\,du}$

Lower Integral $= \tan^{-1}(1) = \frac{\pi}{4}$

Upper Integral $= \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$$\int_{\pi/4}^{\pi/3} \frac{\sqrt{u}}{1+x^2}\cdot(1+x^2)\,du = \int_{\pi/4}^{\pi/3} u^{\frac{1}{2}}\,du$$

$$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{2}{3} u^{\frac{3}{2}}\Bigg]_{\frac{\pi}{4}}^{3} = \frac{2}{3}\left[\left(\frac{\pi}{3}\right)^{\frac{3}{2}} - \left(\frac{\pi}{4}\right)^{3/2}\right]$$

$$= \frac{2}{3}\left[\sqrt[2]{\left(\frac{\pi}{3}\right)^3} - \sqrt[2]{\left(\frac{\pi}{4}\right)^3}\right]$$

4) $$\int_1^{\sqrt{e}} \frac{dx}{x\sqrt{1-(\ln(x))^2}}$$

Let $\boxed{u = \ln x}$

$du = \frac{1}{x} dx \Rightarrow \boxed{dx = x\,du}$

upper limit: $\ln e^{\frac{1}{2}} = \frac{1}{2}\ln e = \frac{1}{2}$

lower limit: $\ln(1) = 0$

$$\int_0^{\frac{1}{2}} \frac{1}{x\sqrt{1-u^2}} x\,du = \sin^{-1} u\Big]_0^{\frac{1}{2}}$$

$$= \left[\sin^{-1}\left(\tfrac{1}{2}\right) - \sin^{-1}(0)\right]$$

$$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

5) $$\int \frac{e^x}{\sqrt{4-e^{2x}}} dx = \int \frac{e^x}{\sqrt{4-(e^x)^2}} dx$$

Let $\boxed{u = e^x}$

$du = e^x dx \Rightarrow \boxed{dx = \frac{du}{e^x}}$

$$\int \frac{e^x}{\sqrt{4-u^2}} \cdot \frac{du}{e^x} = \int \frac{1}{\sqrt{4-u^2}} du = \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= \sin^{-1}\left(\frac{e^x}{2}\right) + C$$

⑥ $\int \frac{e^{2x}}{\sqrt{4-e^{2x}}} dx$

let $\boxed{u = 4-e^{2x}}$

$du = -e^{2x}(2)\,dx \Rightarrow \boxed{dx = \frac{-du}{2e^{2x}}}$

$\int \frac{e^{2x}}{\sqrt{u}} \frac{du}{-2e^{2x}} = \frac{-1}{2} \int u^{-\frac{1}{2}} du = \frac{-1}{2} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -u^{\frac{1}{2}} + c = -\sqrt{4-e^{2x}} + c$

⑦ $\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{1}{\sqrt{9-(2x)^2}} dx$

$\boxed{u = 2x}$

$du = 2dx$

$\boxed{dx = \frac{1}{2} du}$

$\int \frac{1}{9-(u)^2} \frac{du}{2} = \frac{1}{2}\left[\sin^{-1}\left(\frac{u}{3}\right)\right] + c$

$= \frac{1}{2} \sin^{-1}\left(\frac{2x}{3}\right) + c$

⑧ $\int \sqrt{2x+3}\, dx$

$u = 2x+3 \Rightarrow du = 2dx \Rightarrow dx = \frac{du}{dz}$

$\int \sqrt{u} \frac{du}{2} = \frac{1}{2} \int u^{\frac{1}{2}} du = \frac{1}{2}\left[\frac{u^{3/2}}{\frac{3}{2}}\right] + c$

$= \frac{1}{3} u^{3/2} + c$

$= \frac{1}{3} \sqrt{(2x+3)^3} + c$

9) $\int x \cdot \sec^2 x \, dx$

$u = x^2 \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$

$\int x \cdot \sec^2 x \, dx = \int x \sec^2(u) \frac{du}{2x}$

$= \frac{1}{2} \tan u + c$

$= \frac{1}{2} \tan(x^2) + c$

10) $\int 4x \tan(x^2) \, dx$

$\boxed{u = x^2} \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$

$\int 4x \tan(u) \frac{du}{2x} = 2\ln|\sec u| + c$

$= 2\ln|\sec^2(x^2)| + c$

11) $\int \frac{\sec(\ln x) \tan(\ln x)}{x} \, dx$

$\boxed{u = \ln x} \Rightarrow du = \frac{1}{x} dx \Rightarrow \boxed{dx = x\,du}$

$\int \frac{\sec(u) \tan(u)}{x} x\,du = \int \sec(u) \tan(u)\,du$

$= \sec(u) + c = \sec(\ln x) + c$

12) $\int \frac{x}{\sqrt{1-x^4}}\,dx$

$$= \int \frac{x}{\sqrt{1-(x^2)^2}}\,dx$$

$\boxed{u = x^2} \Rightarrow du = 2x\,dx \Rightarrow \boxed{dx = \frac{du}{2x}}$

$$\int \frac{x}{\sqrt{1-u^2}} \cdot \frac{du}{2x} = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}\sin^{-1}u + c$$

$$= \frac{1}{2}\sin^{-1}(x^2) + c$$

13) $\int \frac{e^{\sqrt{x-1}}}{\sqrt{x-1}}\,dx$

$\boxed{u = \sqrt{x-1}} \Rightarrow du = \frac{1}{2\sqrt{x-1}}\,dx \Rightarrow \boxed{dx = 2\sqrt{x-1}\cdot du}$

$$\int \frac{e^u}{\sqrt{x-1}} \cdot 2\sqrt{x-1}\cdot du = 2\int e^u\,du = 2e^u + c$$

$$= 2e^{\sqrt{x-1}} + c$$

14) $\int \frac{dx}{x\cdot(\ln x)^2}$

$\boxed{u = \ln x} \Rightarrow du = \frac{1}{x}\,dx \Rightarrow \boxed{dx = x\,du}$

$$\int \frac{1}{x(u)^2}\cdot x\cdot du = \int \frac{1}{u^2}\,du = \int u^{-2}\,du$$

$$= \frac{u^{-1}}{-1} + c = -\frac{1}{u} + c = \frac{1}{\ln(x)} + c$$

15) $\int \frac{\cos(\ln x)}{x}\, dx$

$\boxed{u=\ln x} \Rightarrow du = \frac{1}{x} dx \Rightarrow \boxed{dx = x\, du}$

$\int \frac{\cos u}{x} \cdot x \cdot du \Rightarrow \int \cos u\, du = \sin u + c$

$= \sin(\ln x) + c$

16) $\int \frac{x}{\csc(x^2)}\, dx$

$\boxed{u = x^2} \Rightarrow du = 2x\, dx \Rightarrow \boxed{dx = \frac{du}{2x}}$

$\int \frac{x}{\csc(u)} \cdot \frac{du}{2x} = \frac{1}{2} \int \frac{1}{\csc(u)}\, du$

$= \frac{1}{2} \int \sin(u)\, du = -\frac{1}{2} \cos u + c$

$= -\frac{1}{2} \cos(x^2) + c$

17) $\int (x\, 5^{-x^2})\, dx$

$\boxed{u = -x^2} \Rightarrow du = -2x\, dx \Rightarrow \boxed{dx = \frac{du}{-2x}}$

$\int (x 5^u) \frac{du}{-2x} = \frac{-1}{2} \int 5^u du = \frac{-1}{2} \left(\frac{5^u}{\ln(5)} \right) + c$

$= \frac{-5^{-x^2}}{2 \ln 5} + c$

18) $\int_0^1 \frac{e^z+1}{e^z+z}\, dz$

$$= \ln|e^z+z| \Big]_0^1$$

$$= \ln(e^1+1) - \ln(e^0+0)$$

$$= \ln(e+1) - \ln(1) = \ln(e+1)$$

19) $\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$

$\boxed{u = 1+2x} \Rightarrow du = 2dx \Rightarrow \boxed{dx = \frac{du}{2}}$

To find the new integration limits:

lower limit: $u = 1+2(0) = 1$

upper limit: $u = 1+2(13) = 27$

$$\int_1^{27} \frac{1}{\sqrt[3]{u^2}} \cdot \frac{du}{2} = \frac{1}{2}\int_1^{27} u^{-\frac{2}{3}} du = \frac{1}{2} \frac{u^{\frac{1}{3}}}{\frac{1}{3}} \Big]_1^{27}$$

$$= \frac{3}{2} u^{\frac{1}{3}} \Big]_1^{27} = \frac{3}{2} \sqrt[3]{u} \Big]_1^{27} = \frac{3}{2}\left(\sqrt[3]{27} - \sqrt[3]{1}\right)$$

$$= \frac{3}{2}(3-1) = 3$$

20) $\displaystyle\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$

$\boxed{u = \frac{1}{x}} \Rightarrow du = \frac{0(x) - 1(1)}{x^2}\,dx \Rightarrow \boxed{dx = -x^2 du}$

New lower limit: $u = \frac{1}{1} = 1$

New upper limit: $u = \frac{1}{2}$

$$\int_1^{\frac{1}{2}} \frac{e^u}{x^2}(-x^2)\,du = -\int_1^{\frac{1}{2}} e^u\,du = -\left[e^u\right]_1^{\frac{1}{2}}$$

$$= -\left[e^{\frac{1}{2}} - e^1\right] = e^1 - e^{\frac{1}{2}}$$

21) $\displaystyle\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$

$\boxed{u = 1+\sqrt{x}} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow \boxed{dx = 2\sqrt{x}\,du}$

New lower limit: $1+\sqrt{0} = 1$

New upper limit: $1+\sqrt{1} = 2$

$$\int_1^2 \frac{1}{u^4}\cdot 2\sqrt{x}\,du$$

$u = 1+\sqrt{x}$

$\sqrt{x} = u-1$

$2\sqrt{x} = 2(u-1)$

$2\sqrt{x} = 2u-2$

$$\int_1^2 u^{-4}(2u-2)\,du$$

$$= \int_1^2 (2u^{-3} - 2u^{-4})\,du$$

$$= 2\int_1^2 (u^{-3} - u^{-4})\,du$$

$$= 2\left[\frac{u^{-2}}{-2} - \frac{u^{-3}}{-3}\right]_1^2$$

$$= 2\left[\frac{-1}{2u^2} + \frac{1}{3u^3}\right]_1^2$$

$$= 2\left[\left(\frac{-1}{2(2)^2}+\frac{1}{3(2)^3}\right)-\left(\frac{-1}{2(1)^2}+\frac{1}{3(1)^3}\right)\right]$$

$$= 2\left[\left(\frac{-1}{8}+\frac{1}{24}\right)-\left(\frac{-1}{2}+\frac{1}{3}\right)\right]$$

$$= \left[\left(\frac{-1}{4}+\frac{1}{12}\right)-\left(\frac{-2}{2}+\frac{2}{3}\right)\right]$$

$$= \left(\frac{-3+1}{12}\right)-\left(\frac{-6+4}{6}\right) = \frac{-2}{12}+\frac{2}{6}$$

$$= \frac{-2+4}{12} = \frac{2}{12} = \frac{1}{6}$$

Homework:

1) $\int_{1}^{2}\left(x\sqrt{x^2-1}\right)dx$

2) $\int\left(1+\frac{1}{x^2}\right)sec^2\left(x-\frac{1}{x}\right)\,dx$

20-a

- ***Integrals of Symmetric Functions:***

f is continuous on [-a , a]:

f is even if : [f (-x) = f (x)] $\Rightarrow$ $\int_{-a}^{a} f(x) = 2\int_{0}^{a} f(x)\, dx$
f is odd if : [f (-x) = - f (x)] $\Rightarrow$ $\int_{-a}^{a} f(x) = 0$

- تستخدم هذه النظرية عندما تكون حدود التكامل لعدد وسالب نفس العدد

Examples

1) $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} (x^3 + x^4 \tan x)dx$	2) $\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} x^4 \sin x\, dx$

1) $\int_{-\frac{\pi}{4}}^{\pi/4} (x^3 + x^4 \tan x)\, dx$

$f(x) = x^3 + x^4 \tan x$

$f(-x) = (-x)^3 + (-x)^4 \tan(-x)$

$= -x^3 + x^4 (-\tan x)$

$= -x^3 - x^4 \tan x = -(x^3 + x^4 \tan x)$

$= -f(x)$

$\therefore f(x) = -f(x)$

$\therefore f(x)$ is odd.

$\therefore \int_{-\frac{\pi}{4}}^{\pi/4} (x^3 + x^4 \tan x)\, dx = 0$

2) $\int_{-\frac{\pi}{3}}^{\pi/3} x^4 \sin x \, dx$

$f(x) = x^4 \sin x$

$f(-x) = (-x)^4 \sin(-x)$

$= x^4 (-\sin x) = -x^4 \sin x = -f(x)$

$\therefore$ f is odd

$\therefore \int_{-\frac{\pi}{3}}^{\pi/3} x^4 \sin x \, dx = 0$

The Fundamental Theorem of Calculus:

1) اشتقاق تكامل غير محدد:

$\frac{d}{dx}\int f(x)\,dx = f(x)$	**or**	$\int \frac{d}{dx} f(x)\,dx = f(x) + c$

2) اشتقاق تكامل محدد بقيم ثابته:

$$\frac{d}{dx}\int_a^b F(t)\,dt = 0$$

3) اشتقاق تكامل محدد بقيم متغيره:

$$\frac{d}{dx}\int_{g(x)}^{f(x)} F(t)\,dt = F\big(f(x)\big).[f'(x)] - F\big(g(x)\big).[g'(x)]$$

- ***Examples:***

1) Find

$$\frac{d}{dx}\left[\int_{e^x}^{\sqrt{x}} \frac{1}{1-t^2}\,dt + \int_1^{\ln x}\sqrt{\sin t}\,dt - \int_e^{\pi}\log_3(\tan x)\,dt\right]$$

Solution:

$$= \left[\frac{1}{1-(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}} - \frac{1}{1-e^{2x}}\cdot e^x\right] + \left[\sqrt{\sin(\ln x)}\cdot\frac{1}{x} - \sqrt{\sin(1)}\cdot(0)\right]$$

$$-[0]$$

$$= \left[\frac{1}{1-x}\cdot\frac{1}{2\sqrt{x}}\right] - \frac{1}{1-e^{2x}}\,e^x + \sqrt{\sin(\ln x)}\cdot\frac{1}{x}$$

② Find $f'(x)$ if $f(x) = x^3 \int_1^{\tan^{-1}(x)} \ln(2+\cos t)\,dt$.

solution:

Note: $\frac{d}{dx}[\tan^{-1}(u)] = \frac{1}{1+u^2} \cdot u'$

$$f'(x) = 3x^2 \int_1^{\tan^{-1}(x)} \ln(2+\cos t)\,dt + x^3 \left[\ln(2+\cos(\tan^{-1}x)) \cdot \frac{1}{1+x^2} - 0\right]$$

③ Find an equation of the normal line to graph of:

$$f(x) = \int_2^{3x-x^2} \frac{1}{t^2+4}\,dt \quad \text{at } x=1$$

Solution:

Note: Normal line is the same as tangent line

$$y-y_1 = m(x-x_1), \quad m = y' = f'(x)$$

To find y_1 use x_1 in $f(x)$.

$$f(1) = \int_2^{3(1)-(1)^2} \frac{1}{t^2+4}\,dt = \int_2^2 \frac{1}{t^2+4}\,dt = 0$$

$$m = f'(x) = \frac{1}{(3x-x^2)^2+4}(3-2x) - 0$$

at $x=1$

$$m = f'(1) = \frac{1}{(3(1)-(1)^2+4}(3-2(1)) = \frac{1}{8}$$

∴ equation of normal line:

$$y-y_1 = m(x-x_1)$$

$$y-0 = \frac{1}{8}(x-1)$$

$$\boxed{y = \frac{1}{8}(x-1)}$$

Integration Techniques

① Integration by parts

$$\int u \cdot dv = u \cdot v - \int v\, du$$

سهل اشتقاقه سهل تكامله

تستخدم عندما يكون هناك دالتين ليسا لاحدهما مشتقة للآخر مثل $\int \sin(x) \cdot x \cdot dx$

a) $\displaystyle\int \underbrace{\text{Polynomail}}_{u} \cdot \underbrace{\text{Trigonometric}\,(\overbrace{ax+b}^{\text{Linear}})}_{dv}\, dx$

b) $\displaystyle\int \underbrace{\text{Polynomail}}_{u} \cdot (\underbrace{\text{Linear}}_{dv})^{n}\, dx$

c) $\displaystyle\int \underbrace{\text{Polynomail}}_{dv} \cdot \underbrace{\ln(\text{Linear or not linear})}_{u}\, dx$

d) $\displaystyle\int \underbrace{\ln(\ )}_{u}\, \underbrace{dx}_{dv}$

e) $\displaystyle\int \underbrace{\text{Polynomail}}_{u} \cdot \underbrace{e^{(\text{linear})}}_{dv}\, dx$

f) $\displaystyle\int \underbrace{\text{Trigonometric}\,(\text{linear})}_{u} \cdot \underbrace{e^{(\text{linear})}}_{dv}\, dx$ تكامل دوري

g) $\displaystyle\int \underbrace{\text{inverse}(\ )}_{u}\, \underbrace{dx}_{dv}$

Examples:

1) $\int x \cdot \sin x \, dx$

$$u = x \qquad dv = \sin x \, dx$$
$$du = dx \qquad v = -\cos x$$

$$\int x \sin x \, dx = -x\cos x - \int -\cos x \, dx$$
$$= -x\cos x + \int \cos x \, dx$$
$$= -x\cos x + \sin x + c$$

2) $\int x^2 \cdot \cos x \, dx$

$$u = x^2 \qquad dv = \cos x \, dx$$
$$du = 2x \, dx \qquad v = \sin x$$

$$\int x^2 \cdot \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$$
$$= x^2 \sin x - 2\int \underbrace{x}_{u} \underbrace{\sin x \, dx}_{dv}$$

$$u = x \qquad dv = \sin x \, dx$$
$$du = dx \qquad v = -\cos x$$

$$= x^2 \sin x - 2\left[-x\cos x - \int -\cos x \, dx\right]$$
$$= x^2 \sin x - 2\left[-x\cos x + \sin x\right] + c$$
$$= x^2 \sin x + 2x\cos x - 2\sin x + c$$

ملاحظة : تكامل الاجزاء في المثال اعلاه حصل مرتين بسبب أن ال Polynomail
كان x^2 ولو كان x^3 لحصل ثلاث مرات ولو كان x^4 لحصل اربع
مرات وهكذا . الحل هو استخدام طريقة Tabular method
وهي أن نشتق ال u لحين الوصول الى الصفر وتكامل ال dv
لحين الوصول الى ما يقابل الصفر . وبعدها نرسم اسهم كما في المثال
التالي . ونضع اشارة + لأول سهم ونقلب الاشارة للسهم الذي يليه

Tabular Method

ثم نجمع الحدود.

3) $\int x^5 \cdot \cos x \, dx$

u	sign	dv
x^5	$+$	$\cos x$
$5x^4$	$-$	$\sin x$
$20x^3$	$+$	$-\cos x$
$60x^2$	$-$	$-\sin x$
$120x$	$+$	$\cos x$
120	$-$	$\sin x$
0		$-\cos x$

$$\int x^5 \cdot \cos x \, dx = x^5 \cdot \sin x + 5x^4 \cos x - 20x^3 \sin x - 60x^2 \cdot \cos x + 120x \cdot \sin x + 120 \cos x + C$$

4) $\int x^2 \ln x \, dx$

$u = \ln x \qquad dv = x^2 dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^3}{3}$

$$\int x^2 \ln x \, dx = \ln x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$$

$$= \ln x \cdot \frac{x^3}{3} - \frac{1}{3} \int x^2 dx$$

$$= \ln x \cdot \frac{x^3}{3} - \frac{1}{3} \cdot \frac{x^3}{3} + C$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C$$

5) $\int \ln x \, dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$\int \ln x \, dx = \ln(x) \cdot x - \int x \cdot \frac{1}{x} dx$

$= \ln(x) \cdot x - x + c$

6) $x \cdot e^x dx$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$\int x e^x dx = x e^x - \int e^x dx$

$= x e^x - e^x + c$

7) $\int \tan^{-1}(x) \, dx$

$u = \tan^{-1} x \qquad dv = dx$

$du = \frac{1}{1+x^2} \qquad v = x$

$\int \tan^{-1}(x) \, dx = x \cdot \tan^{-1} x - \int \frac{x}{1+x^2} dx$

$= x \cdot \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$

$= x \cdot \tan^{-1} x - \frac{1}{2} \ln |1+x^2| + c$

8) $\int \sin x \cdot e^x \cdot dx$

$u = \sin x \qquad dv = e^x dx$

$du = \cos x \qquad v = e^x$

$$\int \sin x \cdot e^x \cdot dx = e^x \sin x - \int \underbrace{e^x}_{dv} \underbrace{\cos x}_{u} dx$$

$u = \cos x \qquad dv = e^x dx$

$du = -\sin x \qquad v = e^x$

$$\int \sin x\, e^x dx = e^x \sin x - [e^x \cos x - \int -e^x \sin x\, dx]$$

$$\int \sin x\, e^x dx = e^x \sin x - e^x \cos x + \int -e^x \sin x\, dx$$

$$\int \sin x\, e^x dx = e^x \sin x - e^x \cos x - \int e^x \sin x\, dx$$

هذا الحد يشبه السؤال
وهذا يسمى تكامل دوري والحل ننقله الى الجهة الثانية

$$2\int \sin x\, e^x dx = e^x \sin x - e^x \cos x$$

$$\int \sin x\, e^x dx = \frac{e^x \sin x - e^x \cos x}{2} + c$$

9) $\int \cos(\sqrt{2x+1})\, dx$

إذا كانت زاوية الدالة غير خطية
افرض الغير خطية
$z =$

$z = \sqrt{2x+1}$

$z^2 = 2x+1$

$2z\,dz = 2\,dx$

$\boxed{dx = z\,dz}$

$\int \cos(z) \cdot z \cdot dz$

$u = z \qquad dv = \cos z$

$du = dz \qquad v = \sin z$

$$\int \cos(z)\, z\, dz = z \sin z - \int \sin z\, dz$$

$$= z \sin z + \cos z + c$$

$$= \sqrt{2x+1} \cdot \sin\sqrt{2x+1} + \cos\sqrt{2x+1} + c$$

10) $\int x^3 \cdot e^{x^2} \cdot dx$

$$\boxed{z = x^2}$$

$$dz = 2x\,dx$$

$$\boxed{dx = \frac{dz}{2x}}$$

$$\int x^3 \cdot e^z \cdot \frac{dz}{2x} = \frac{1}{2}\int z e^z dz$$

$u = z \qquad dv = e^z dz$

$du = dz \qquad v = e^z$

$$\frac{1}{2}\int z e^z dz = \frac{1}{2}\left[e^z z - \int e^z dz\right]$$

$$= \frac{1}{2}\left[e^z z - e^z\right] + c$$

$$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + c$$

11) $\int \sin(2x) \cdot e^{\cos x} \cdot dx$

$$\boxed{z = \cos x} \Rightarrow dz = -\sin x\,dx \Rightarrow \boxed{dx = \frac{dz}{-\sin x}}$$

$$\int \sin(2x) e^z \frac{dz}{-\sin x} = \int 2\cancel{\sin x} \cos x\, e^z \frac{dz}{\cancel{-\sin x}}$$

$$= -2\int z e^z dz$$

$u = z \qquad dv = e^z dz$

$du = dz \qquad v = e^z$

$$= -2\left[z e^z - \int e^z dz\right] = -2\left[z e^z - e^z\right] + c$$

$$= -2z e^z + 2e^z + c$$

$$= -2\cos x\, e^{\cos x} + 2e^{\cos x} + c$$

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