power series :-

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

↳ is Called a power series in x about c

* Interval of Convergence "I.C"

* Radius of Convergence "R.C"

$$r = \frac{b-a}{2}$$

To Solve a power series $\xrightarrow{\text{use}}$ Ratio test or Root test

$$\left.\begin{array}{l} \lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| \\[2ex] \lim\limits_{n\to\infty} \left| (a_n)^{\frac{1}{n}} \right| \end{array}\right. \begin{cases} >1 \to \text{div} \\ <1 \to \text{Conv} \end{cases}$$

Ex :- Find the interval of Convergence and the radius of Convergence of the Series ?

$\boxed{1}$ $\sum_{n=0}^{\infty} \frac{4^n}{n!}(x-1)^n$

use Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{4^{n+1}}{(n+1)!} * (x-1)^{n+1} * \frac{n!}{4^n\cdot(x-1)^n}\right|$

$\Rightarrow \lim_{n\to\infty}\left|\frac{\cancel{4^n}\cdot 4\,\cancel{(x-1)^n}\cdot(x-1)^1}{(n+1)\,\cancel{n!}} * \frac{\cancel{n!}}{\cancel{4^n}\cdot\cancel{(x-1)^n}}\right|$

$\Rightarrow \lim_{n\to\infty}\left|\frac{4(x-1)}{n+1}\right| = |x-1| \underbrace{\lim_{n\to\infty}\frac{4}{n+1}}$

$|x-1| * 0 < 1$

I. C $\Rightarrow (-\infty, \infty)$

R. C $\Rightarrow r = \infty$

$\boxed{2}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^{n+1}} \cdot (x)^n$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{n+1}{3^{n+2}} \cdot (x)^{n+1} * \frac{3^{n+1}}{n \cdot (x)^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{n+1}{\cancel{3^n} \cdot 3^{\cancel{2}}} \cancel{(x)^n} \cdot (x) * \frac{\cancel{3^n} \cdot \cancel{3}}{n \cdot \cancel{(x)^n}} \right|$$

$$\left|\frac{x}{3}\right| \underbrace{\lim_{n\to\infty} \frac{n+1}{n}}_{=\boxed{1}} \Rightarrow \frac{|x|}{3} < 1 \Rightarrow |x| < 3$$

$$-3 < x < 3$$

If $x = 3 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{n}{3^{n+1}} \cdot (3)^n = \sum_{n=1}^{\infty} \frac{n}{\cancel{3^n} \cdot 3} \cdot \cancel{3^n}$

$\displaystyle\sum_{n=1}^{\infty} \frac{n}{3} \rightarrow \lim_{n\to\infty} \frac{n}{3} = \infty \neq 0$ div by D.T

If $x = -3 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n \cdot 3} \cdot (-3)^n = \sum_{n=1}^{\infty} \frac{n \cdot (-1)^n \cdot \cancel{3^n}}{\cancel{3^n} \cdot 3}$

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{3} \rightarrow \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{3} = \infty \neq 0$
div by D.T

So I.C $\Rightarrow (-3, 3)$ R.C $\Rightarrow r = \dfrac{3-(-3)}{2} = \dfrac{6}{2} = \boxed{3}$

$\boxed{3}$ $$\sum_{n=1}^{\infty} \frac{(x)^n}{n \cdot 5^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x)^{n+1}}{(n+1)\,5^{n+1}} * \frac{n\,5^n}{(x)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{\cancel{(x)^n}\cdot(x)}{(n+1)\,\cancel{5^n}\cdot 5} * \frac{n\cdot\cancel{5^n}}{\cancel{(x)^n}}\right|$$

$$\frac{|x|}{5}\underbrace{\lim_{n\to\infty}\frac{n}{n+1}}_{=\boxed{1}} = \frac{|x|}{5} < 1 \to |x| < 5$$

$$-5 < x < 5$$

if $x=5 \to \sum_{n=1}^{\infty}\frac{(5)^n}{n\,(5)^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ $\quad p=1 \hookrightarrow$ div by p-T

if $x=-5 \to \sum_{n=1}^{\infty}\frac{(-5)^n}{n\,5^n} = \sum_{n=1}^{\infty}\frac{(-1)^n\,(5)^n}{n\,5^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

A.S.T

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{n} = \frac{1}{\infty} = \boxed{0} \checkmark$$

a_n is decreasing $\quad a_1 > a_2 \quad 1 > \frac{1}{2} \checkmark$

$\Rightarrow$ Conv by A.S.T

$\hookrightarrow -5 \le x < 5$

I.C $\Rightarrow [-5, 5)$ $\quad$ R.C $\Rightarrow r=5$

$\boxed{4}$ $\displaystyle\sum_{n=0}^{\infty} n!\,(x-3)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|(n+1)!\,(x-3)^{n+1} * \frac{1}{n!\,(x-3)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(n+1)\cancel{n!}\,\cancel{(x-3)^n}(x-3)}{\cancel{n!}\,\cancel{(x-3)^n}}\right|$$

$$= \lim_{n\to\infty}\left|(n+1)(x-3)\right| = |x-3| \underbrace{\lim_{n\to\infty}(n+1)}$$

$$|x-3| * \infty < 1$$

$$\boxed{x = 3}$$

I. C $\Rightarrow$ $\{3\}$ R. C $\Rightarrow$ Zero $\Rightarrow$ $\boxed{r = 0}$

$\boxed{5}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2}$ H.W

I. C $\Rightarrow$

R. C $\Rightarrow$

$\boxed{6}$ $$\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n^2\ 3^n}$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} * (x+2)^{n+1}}{(n+1)^2\ 3^{n+1}} * \frac{n^2 \cdot 3^n}{(-1)^n \cdot (x+2)^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^n} \cdot (-1) \cdot \cancel{(x+2)^n} \cdot (x+2)}{(n+1)^2\ \cancel{3^n} \cdot 3} * \frac{n^2 \cdot \cancel{3^n}}{\cancel{(-1)^n}\ \cancel{(x+2)^n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(-1)(x+2) \cdot n^2}{3\ (n+1)^2} \right| = \frac{|x+2|}{3} \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$

$\frac{\infty}{\infty}$!! CR

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$$= \frac{|x+2|}{3} < 1 \to |x+2| < 3$$

$$-3 < x+2 < 3$$
$$-2 \quad\quad -2 \quad\quad -2$$

$$-5 < x < 1$$

if $x=1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n \cdot (3)^n}{n^2 \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \xrightarrow{A.S.T}$ Conv

if $x=-5 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n (-3)^n}{n^2 \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot (-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$

$= \sum_{n=1}^{\infty} \frac{1}{n^2} \to p=2>1$ Conv by p-T

I.C $\Rightarrow [-5, 1]$ R.C $\Rightarrow r = \frac{1-(-5)}{2} = \boxed{3}$ ✓

Representation of functions as power series

$$\sum_{n=0}^{\infty} (r)^n = \frac{1}{1-r} \qquad |r| < 1$$

$$\Rightarrow \sum_{n=0}^{\infty} (x)^n = \frac{1}{1-x} \qquad \text{I.c} \Rightarrow |x| < 1$$

$$\Rightarrow \frac{1}{1-(x)} = \sum_{n=0}^{\infty} (x)^n$$

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Ex : Express the given functions as a power series

$\boxed{1}$ $f(x) = \dfrac{1}{1+x}$

$\hookrightarrow \dfrac{1}{1-(x)} = \sum_{n=0}^{\infty} (x)^n$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (x)^n$$

I. C $\Rightarrow$ $|x| < 1$ $\Rightarrow$ $-1 < x < 1$

$\Rightarrow$ $(-1, 1)$ ✓

$\boxed{2}$ $f(x) = \dfrac{1}{1+x^2}$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$\sum_{n=0}^{\infty} (-1)^n (x)^{2n} \quad ✓$$

$\boxed{3}$ $f(x) = \dfrac{2x}{1-x^3}$

$$2x \cdot \frac{1}{1-x^3} = 2x \cdot \sum_{n=0}^{\infty} (x^3)^n$$

$$= \sum_{n=0}^{\infty} 2x^1 \cdot x^{3n}$$

$$= \sum_{n=0}^{\infty} 2(x)^{3n+1} \quad ✓$$

$\boxed{4}$ $f(x) = \dfrac{x^3}{2+x}$

$$\frac{x^3}{2+x} = x^3 \cdot \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{x^3}{2} \cdot \frac{1}{\left(1-\left(\frac{-x}{2}\right)\right)}$$

$$\frac{x^3}{2} * \sum_{n=0}^{\infty} \left(\frac{-x}{2}\right)^n$$

$$\sum_{n=0}^{\infty} \frac{x^3}{2} * \frac{(-1)^n \cdot (x)^n}{(2)^n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \, (x)^{n+3}}{2^{n+1}}$$

* Differentiation and integration of power series

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} a_n (x-c)^n\right] = \sum_{n=1}^{\infty} a_n * n(x-c)^{n-1}$$

$$\int\left[\sum_{n=0}^{\infty} a_n (x-c)^n\right] dx = \sum_{n=0}^{\infty} \frac{a_n (x-c)^{n+1}}{n+1} + C$$

Ex :- Express the following functions as a power series ?

1. $f(x) = \tan^{-1}(x)$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (x)^{2n}$$

$$\int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n (x)^{2n}\,dx$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{2n+1} + c$$

لـ كيفية حساب ثابت التكامل (c) ـ نعوّض $x=0$

$$\tan^{-1}(0) = 0 + c$$

$$\boxed{0 = c}$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{2n+1}$$

$\boxed{2}$ $f(x) = \dfrac{1}{(1+x)^2}$ $\qquad \dfrac{}{(\text{المقام})^2} \longrightarrow$ مشتقة القسمة

$$\frac{1}{1+x} \xrightarrow{\frac{d}{dx}} \frac{-1(1)}{(1+x)^2} = \frac{-1}{(1+x)^2}$$

$$\frac{-1}{1+x} \xrightarrow{\frac{d}{dx}} \frac{(+1)(1)}{(1+x)^2} = \frac{1}{(1+x)^2}$$

$$\frac{-1}{1-(-x)} = \sum_{n=0}^{\infty} (-1) * (-x)^n$$

$$\frac{d}{dx}\left[\frac{-1}{1+x} = \sum_{n=0}^{\infty} (-1)^{n+1} (x)^n \right]$$

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n+1} * n (x)^{n-1}$$

$\boxed{3}$ $f(x) = \ln(1-2x)$

$$\frac{-2}{1-2x} = -2 \cdot \frac{1}{1-2x} = \sum_{n=0}^{\infty} (-2) \cdot (2x)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (2) \cdot (2)^n (x)^n$$

$$\int \frac{-2}{1-2x} dx = \int \sum_{n=0}^{\infty} (-1)^n (2)^{n+1} (x)^n \, dx$$

$$\ln|1-2x| = \sum_{n=0}^{\infty} \frac{(-1)^n (2)^{n+1} (x)^{n+1}}{n+1} + C$$

at $x = 0$

$\hookrightarrow \ln(1-0) = 0 + C$

$\ln(1) = C \rightarrow \boxed{C=0}$

$$\ln(1-2x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2)^{n+1} (x)^{n+1}}{n+1}$$

$\boxed{4}$ $f(x) = \int \frac{1}{1+x^5}\,dx$ as a power Series ?

$$\frac{1}{1+x^5} = \frac{1}{1-(-x^5)} = \sum_{n=0}^{\infty} (-x^5)^n$$

$$\frac{1}{1+x^5} = \sum_{n=0}^{\infty} (-1)^n (x)^{5n}$$

$$\int \frac{1}{1+x^5}\,dx = \int \sum_{n=0}^{\infty} (-1)^n (x)^{5n}\,dx$$

$$\int \frac{1}{1+x^5}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{5n+1}}{5n+1} + c$$

at $x=0$

$\hookrightarrow \int 1\,dx = 0 + c$

$x = 0 + c \rightarrow c = 0$

$$\int \frac{1}{1+x^5}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{5n+1}}{5n+1}$$