Taylor and Maclaurin Series :-

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!}$$

$$+ \frac{f'''(a)(x-a)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{f^{n}(a)(x-a)^n}{n!}$$

If $a=0$ $\rightsquigarrow$ "Maclaurin Series"

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)(x)^2}{2!}$$

$$+ \frac{f'''(0)(x)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{f^{n}(0)(x)^n}{n!}$$

Ex :- Find the Maclaurin series for the function $f(x) = e^x$?

$f(x) = e^x \rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \rightarrow f'(0) = 1$

$f''(x) = e^x \rightarrow f''(0) = 1$

$f'''(x) = e^x \rightarrow f'''(0) = 1$

$\vdots \qquad\qquad \vdots$

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)\,(x)^2}{2!} + \cdots$$

$$e^x = 1 + (1)(x) + \frac{(1)\,(x)^2}{2!} + \frac{(1)\,(x)^3}{3!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{n!} \checkmark$$

CamScanner

* Maclaurin Series for Some Functions

$$\frac{1}{1-x} \Rightarrow \sum_{n=0}^{\infty} (x)^n$$

$$e^x \Rightarrow \sum_{n=0}^{\infty} \frac{(x)^n}{n!}$$

$$\sin(x) \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!}$$

$$\cos(x) \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!}$$

$$\tan^{-1}(x) \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{2n+1}$$

$$\ln(1+x) \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (x)^n}{n}$$

Ex : Find the Maclaurin Series for the following functions

1) $f(x) = e^{x^2+1}$ $\rightarrow$ $e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{n!}$

$$f(x) = e^{x^2} \cdot e^1$$

$$\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} \cdot e = e \sum_{n=0}^{\infty} \frac{(x)^{2n}}{n!}$$

2) $f(x) = x^2 \cdot \cos(x^3)$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!}$$

$$= x^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!}$$

$$= x^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{6n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{6n+2}}{(2n)!}$$

$\boxed{3}$ $f(x) = \dfrac{x^6}{x+2}$ "past paper 2021"

$$= x^6 \cdot \frac{1}{2+x}$$

$$= x^6 \cdot \frac{1}{2\left(1+\frac{x}{2}\right)}$$

$$= \frac{x^6}{2} \cdot \frac{1}{\underbrace{\left(1-\left(\frac{-x}{2}\right)\right)}}$$

$$= \frac{x^6}{2} \sum_{n=0}^{\infty} \left(\frac{-x}{2}\right)^n$$

$$= \frac{x^6}{2} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (x)^n}{(2)^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{n+6}}{2^{n+1}} \quad \checkmark$$

EX : Find the Sum for the power series

"Maclaurin series"

$\boxed{1}$ $$\sum_{n=0}^{\infty} \frac{(5)^n}{n!} \qquad e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{n!}$$

Us $= \boxed{e^5}$ ✓

$\boxed{2}$ $$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+2}}{(2n+1)!} \sim \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1} \cdot \left(\frac{\pi}{3}\right)^1}{(2n+1)!}$$

$$\begin{array}{c|ccc} & \frac{\pi}{6} & \frac{\pi}{4} & \frac{\pi}{3} \\ \hline \sin & \sqrt{1} & \sqrt{2} & \sqrt{3} \\ \cos & \sqrt{3} & \sqrt{2} & \sqrt{1} \\ \hline & & 2 & \end{array}$$

$$= \frac{\pi}{3} * \sin\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{3} * \frac{\sqrt{3}}{2} = \frac{\sqrt{3}\,\pi}{6}$$ ✓

$\boxed{3}$ $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^{n+1}(n!)}$

"past paper 2021"

مهم جداً

Find the sum .. ?

$$e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{n!}$$

$$\sum_{n=1}^{\infty} \frac{3^n}{5^n \cdot 5\,(n!)}$$

$$\frac{1}{5} \underbrace{\sum_{n=1}^{\infty} \frac{\left(\frac{3}{5}\right)^n}{n!}}$$

$$= \frac{1}{5} e^{\frac{3}{5}} - a_0$$

$$= \frac{1}{5} e^{\frac{3}{5}} - \frac{1}{5}$$ ✓

Note :- $\displaystyle\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots$

Find the sum $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_n - a_0$

Ex: If $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (x)^{2n}}{(2n)!}$

then $\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n+1}}{3^{2n+1} (2n)!}$ equal ?

$$= \sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1}}{(2n)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n)!} \cdot \frac{\pi}{3}$$

$$= \cos\left(\frac{\pi}{3}\right) \cdot \frac{\pi}{3} - a_0$$

$$= \frac{1}{2} \cdot \frac{\pi}{3} - \frac{\pi}{3}$$

$$= \frac{\pi}{6} - \frac{\pi}{3} = \boxed{\frac{-\pi}{6}}$$

Ex: Express the integral $\int e^{-x^3} dx$ as a power series ?

$$e^{x} = \sum_{n=0}^{\infty} \frac{(x)^n}{n!}$$

$$e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{3n}}{n!}$$

$$\int e^{-x^3} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{3n}}{n!} \, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} * \frac{x^{3n+1}}{(3n+1)} + C$$

$$= \overset{a_0}{(x+c)} + \overset{a_1}{\frac{(-1)(x)^4}{4}} + \overset{a_2}{\frac{(x)^7}{2! * 7}} ---$$

$$= c + x - \frac{x^4}{4} + \frac{x^7}{7.2!} - \frac{x^{10}}{10.3!} + ---.$$

EX : Express the integral $\int \sin(x^2)\,dx$ as a power series ?

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$\int \sin(x^2)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{4n+2}}{(2n+1)!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{4n+3}}{(2n+1)! * (4n+3)} + c$$

$$= \frac{x^3}{3} + c + \frac{(-1)(x)^7}{3! * 7} + \frac{(x)^{11}}{5! * 11} + -$$

$$= c + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + -$$

EX: Express the integral $\int \frac{1}{1+x^5}\,dx$ as a power series?

H.W

A) $C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \cdots\cdots$

B) $C + x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} + \cdots\cdots$

C) $C + x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots$

D) $C + x - \frac{x^6}{6} + \frac{x^{11}}{11} - \frac{x^{16}}{16} + \cdots\cdots$

E) $C + x - \frac{x^4}{4} + \frac{x^7}{7 \cdot 2!} - \frac{x^{10}}{10 \cdot 3!} + \cdots\cdots$

* Taylor series centered at a

<u>Ex:</u> Find the Taylor series for the function

$$f(x) = e^x \quad \text{at} \quad a = 1 \quad ?$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots$$

$$f(x) = e^x \rightarrow f(1) = e^1 = e$$

$$f'(x) = e^x \rightarrow f'(1) = e^1 = e$$

$$f''(x) = e^x \rightarrow f''(1) = e$$

$$\vdots \qquad \qquad \vdots$$

$$f(x) = e + e(x-1) + \frac{e(x-1)^2}{2!} + \frac{e(x-1)^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!} = e \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} \checkmark$$

<u>or</u> $f(x) = e^{x-1+1}$

$$= e^{x-1} \cdot e^1 = \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} \cdot e \checkmark$$

Ex: Expand the function $f(x) = \frac{1}{x}$ about $a = 3$ as indicated ?

$$f(x) = \frac{1}{(x-3)+3} = \frac{1}{3+(x-3)}$$

$$= \frac{1}{3} \cdot \frac{1}{1+\left(\frac{x-3}{3}\right)}$$

$$= \frac{1}{3} \cdot \frac{1}{1-\left(\frac{-(x-3)}{3}\right)} \curvearrowright \frac{1}{1-\square}$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{-(x-3)}{3}\right)^n$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{3^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{3^{n+1}} \checkmark$$