University of Anbar

College of Engineering

Department of Electrical Engineering

Fundamentals of
Electric Circuits

Second Course

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Chapter 6

Capacitors and inductors

Capacitors and inductors

6.1 Introduction

So far we have limited our study to resistive circuits. In this chapter, we shall introduce two new and important passive linear circuit elements: the capacitor and the inductor. Unlike resistors, which dissipate energy, capacitors and inductors do not dissipate but store energy, which can be retrieved at a later time. For this reason, capacitors and inductors are called *storage* elements

6.2 Capacitors

A capacitor is a passive element designed to store energy in its electric field. Besides resistors, capacitors are the most common electrical Components. Capacitors are used extensively in electronics, communications, computers, and power systems. For example, they are used in the tuning circuits of radio receivers and as dynamic memory elements in computer systems.

> A **capacitor** consists of two conducting plates separated by an insulator (or dielectric).

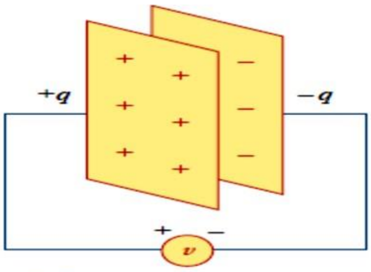

Figure 6.2
A capacitor with applied voltage v.

In many practical applications, the plates may be aluminum foil while the dielectric may be air, ceramic, paper, or mica. When a voltage source is connected to the capacitor, as in Fig. 6.2, the source deposits a positive charge q on one plate and a negative charge on the other. The capacitor is said to store the electric charge. The amount of charge stored, represented by q, is directly proportional to the applied voltage so that

$$q = Cv \tag{6.1}$$

where$C$, the constant of proportionality, is known as the *capacitance* of the capacitor. The unit of capacitance is the farad (F),

Capacitance is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates, measured in farads (F).

Note from Eq. (6.1) that 1 farad = 1 coulomb/volt.

Although the capacitance ***C*** of a capacitor is the ratio of the charge q per plate to the applied voltage it does not depend on ***q*** or ***v*** It depends on the physical dimensions of the capacitor. For example, for the parallel-plate capacitor shown in Fig. 6.1, the capacitance is given by

$$C = \frac{\epsilon A}{d} \tag{6.2}$$

where A is the surface area of each plate, d is the distance between the plates, and is Ɛ the permittivity of the dielectric material between the plates. Although Eq. (6.2) applies to only parallel-plate capacitors, we may infer from it that, in general, three factors determine the value of the capacitance:

1. The surface area of the plates—the larger the area, the greater the capacitance.

2. The spacing between the plates—the smaller the spacing, the greater the capacitance.
3. The permittivity of the material—the higher the permittivity, the greater the capacitance.
Capacitors are commercially available in different values and types. Typically, capacitors have values in the Pico farad (pF) to microfarad (μF)range. They are described by the dielectric material they are made of and by whether they are of fixed or variable type. Figure 6.3 shows the circuit symbols for fixed and variable capacitors. Note that according to the passive sign convention, if $v > 0$ and $i > 0$ or $v < 0$ if and $i < 0$ the capacitor is being charged, and if $v \cdot i < 0$,the capacitor is discharging.

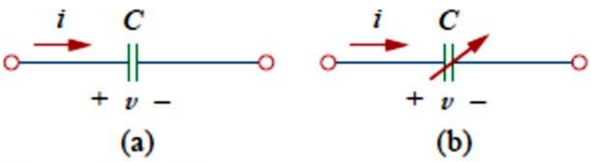

Figure 6.3
Circuit symbols for capacitors: (a) fixed capacitor, (b) variable capacitor.

capacitors are used to block dc, pass ac, shift phase, store energy, start motors, and suppress noise . Variable capacitors are used in radio receivers allowing one to tune to various stations.
To obtain the current-voltage relationship of the capacitor, we take the derivative of both sides of Eq. (6.1). Since

$$q = Cv \qquad (6.1)$$

$$i \triangleq \frac{dq}{dt}$$

differentiating both sides of Eq. (6.1) gives

$$i = C\frac{dv}{dt} \qquad (6.4)$$

The voltage-current relation of the capacitor can be obtained by integrating both sides of Eq. (6.4). We get

$$v = \frac{1}{C}\int_{-\infty}^{t} i\,dt \qquad (6.5)$$

or

$$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0) \qquad (6.6)$$

where $v(t_0) = q(t_0)/C$ is the voltage across the capacitor at time t_0. Equation (6.6) shows that capacitor voltage depends on the past history of the capacitor current. Hence, the capacitor has memory—a property that is often exploited.

The instantaneous power delivered to the capacitor is

$$p = vi = Cv\frac{dv}{dt} \qquad (6.7)$$

The energy stored in the capacitor is therefore

$$w = \int_{-\infty}^{t} p\,dt = C\int_{-\infty}^{t} v\frac{dv}{dt}dt = C\int_{v(-\infty)}^{v(t)} v\,dv = \frac{1}{2}Cv^2\Big|_{v(-\infty)}^{v(t)} \qquad (6.8)$$

We note that $v(-\infty) = 0$, because the capacitor was uncharged at $t = -\infty$. Thus,

$$w = \frac{1}{2}Cv^2 \qquad (6.9)$$

Using Eq. (6.1), we may rewrite Eq. (6.9) as

$$w = \frac{q^2}{2C} \qquad (6.10)$$

Equation (6.9) or (6.10) represents the energy stored in the electric field that exists between the plates of the capacitor. This energy can be retrieved, since an ideal capacitor cannot dissipate energy. In fact, the word *capacitor* is derived from this element's capacity to store energy in an electric field. We should note the following important properties of a capacitor:

1. Note from Eq. (6.4) that when the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero. Thus

A capacitor is an open circuit to dc.

However, if a battery (dc voltage) is connected across a capacitor, the capacitor charges.

2. The voltage on the capacitor must be continuous

The voltage on a capacitor cannot change abruptly.

The capacitor resists an abrupt change in the voltage across it. According to Eq. (6.4), a discontinuous change in voltage requires an infinite current, which is physically impossible. For example, the voltage across a capacitor may take the form shown in Fig. 6.7(a), whereas it is not physically possible for the capacitor voltage to take the form shown in Fig. 6.7(b) because of the abrupt changes. Conversely, the current through a capacitor can change instantaneously

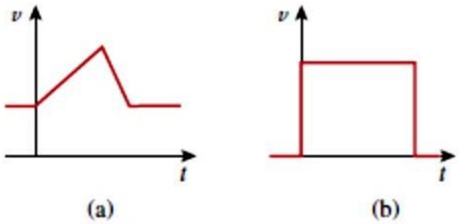

Figure 6.7
Voltage across a capacitor: (a) allowed, (b) not allowable; an abrupt change is not possible.

3. The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit

4. A real, nonideal capacitor has a parallel-model leakage resistance, as shown in Fig. 6.8. The leakage resistance may be as high as **100 MΩ** and can be neglected for most practical applications. For this reason, we will assume ideal capacitors .

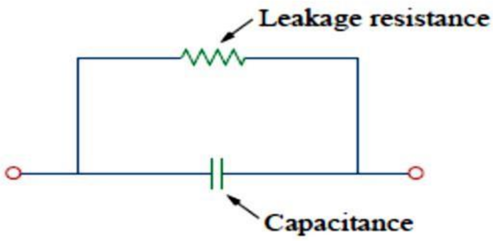

Figure 6.8
Circuit model of a nonideal capacitor.

Example

(a) Calculate the charge stored on a 3-pF capacitor with 20 V across it.
(b) Find the energy stored in the capacitor.

Solution:

(a) Since $q = Cv$,

$$q = 3 \times 10^{-12} \times 20 = 60 \text{ pC}$$

(b) The energy stored is

$$w = \frac{1}{2}Cv^2 = \frac{1}{2} \times 3 \times 10^{-12} \times 400 = 600 \text{ pJ}$$

Practice Problem

What is the voltage across a 4.5 µF capacitor if the charge on one plate is 0.12 mC? How much energy is stored

Answer: 26.67 A, 1.6 mJ.

Example

The voltage across a 5 µF capacitor is $v(t) = 10 \cos 6000t$ V Calculate the current through it.

Solution:

By definition, the current is

$$i(t) = C\frac{dv}{dt} = 5 \times 10^{-6}\frac{d}{dt}(10 \cos 6000t)$$
$$= -5 \times 10^{-6} \times 6000 \times 10 \sin 6000t = -0.3 \sin 6000t \text{ A}$$

Practice Problem

If a 10-μF capacitor is connected to a voltage source with

$$v(t) = 75 \sin 2000t \text{ V}$$

determine the current through the capacitor.

Answer: 1.5 cos 2000t A.

Example

Determine the voltage across a 2-μF capacitor if the current through it is

$$i(t) = 6e^{-3000t} \text{ mA}$$

Assume that the initial capacitor voltage is zero.

Solution:

Since $v = \frac{1}{C}\int_0^t i\,dt + v(0)$ and $v(0) = 0$,

$$v = \frac{1}{2 \times 10^{-6}} \int_0^t 6e^{-3000t}\,dt \cdot 10^{-3}$$

$$= \frac{3 \times 10^3}{-3000} e^{-3000t} \Big|_0^t = (1 - e^{-3000t}) \text{ V}$$

Practice Problem

The current through a 100-μF capacitor is $i(t) = 50 \sin 120\pi t$ mA. Calculate the voltage across it at $t = 1$ ms and $t = 5$ ms. Take $v(0) = 0$.

Answer: 93.14 mV, 1.736 V.

Example

Determine the current through a 200 µF capacitor whose voltage is shown in Fig.

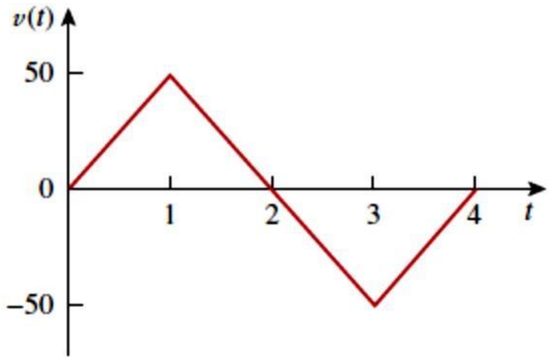

Figure 6.9

Solution:

The voltage waveform can be described mathematically as

$$v(t) = \begin{cases} 50t \text{ V} & 0 < t < 1 \\ 100 - 50t \text{ V} & 1 < t < 3 \\ -200 + 50t \text{ V} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

Since $i = C\,dv/dt$ and $C = 200\ \mu\text{F}$, we take the derivative of v to obtain

$$i(t) = 200 \times 10^{-6} \times \begin{cases} 50 & 0 < t < 1 \\ -50 & 1 < t < 3 \\ 50 & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 10 \text{ mA} & 0 < t < 1 \\ -10 \text{ mA} & 1 < t < 3 \\ 10 \text{ mA} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

Thus the current waveform is as shown in Fig. 6.10.

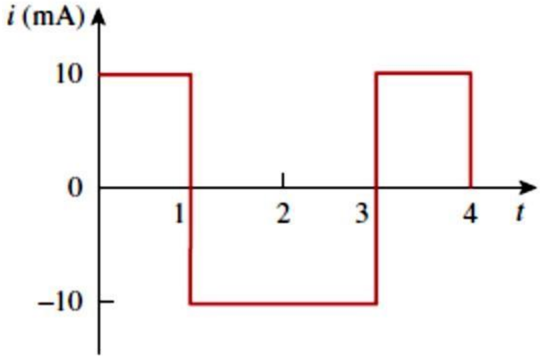

Figure 6.10

Practice Problem

An initially uncharged 1-mF capacitor has the current shown in Fig. across it. Calculate the voltage across it at t=2ms and t=5 ms

Answer: 100 mV, 400 mV.

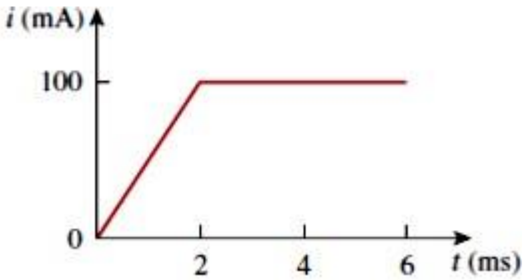

Figure 6.11

Example

Obtain the energy stored in each capacitor in Fig (a) under dc conditions

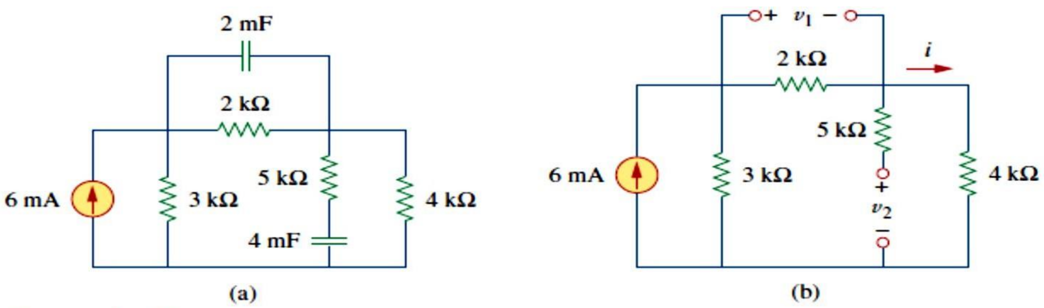

Figure 6.12

Solution:

Under dc conditions, we replace each capacitor with an open circuit, as shown in Fig. (b). The current through the series combination of the 2KΩ and 5KΩ resistors is obtained by current division as

$$i = \frac{3}{3 + 2 + 4}(6 \text{ mA}) = 2 \text{ mA}$$

Hence, the voltages v_1 and v_2 across the capacitors are

$$v_1 = 2000i = 4 \text{ V} \qquad v_2 = 4000i = 8 \text{ V}$$

and the energies stored in them are

$$w_1 = \frac{1}{2}C_1v_1^2 = \frac{1}{2}(2 \times 10^{-3})(4)^2 = 16 \text{ mJ}$$

$$w_2 = \frac{1}{2}C_2v_2^2 = \frac{1}{2}(4 \times 10^{-3})(8)^2 = 128 \text{ mJ}$$

Practice Problem

Under dc conditions, find the energy stored in the capacitors in Fig.

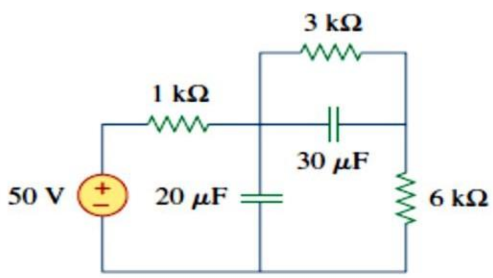

Figure 6.13

Answer: 810 μJ, 135 μJ.

6.3 Series and Parallel Capacitors

We know from resistive circuits that the series-parallel combination is a powerful tool for reducing circuits. This technique can be extended to series-parallel connections of capacitors, which are sometimes encountered. We desire to replace these capacitors by a single equivalent capacitor ***Ceq***. In order to obtain the equivalent capacitor of *N* capacitors in parallel, consider the circuit in Fig. 6.14(a). The equivalent circuit is in Fig. 6.14(b).

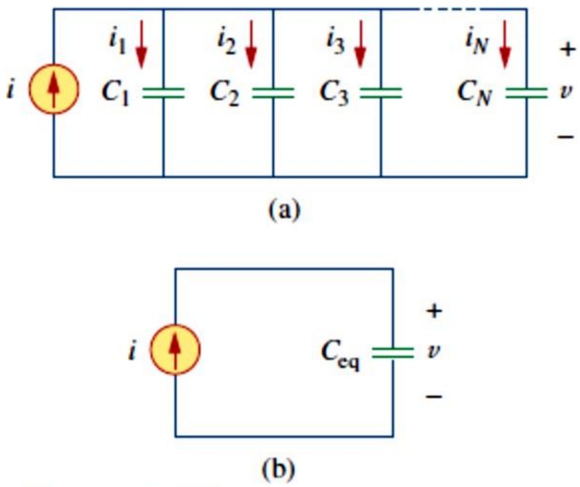

Figure 6.14
(a) Parallel-connected *N* capacitors, (b) equivalent circuit for the parallel

Note that the capacitors have the same voltage across them. Applying KCL to Fig. 6.14(a),

$$i = i_1 + i_2 + i_3 + \cdots + i_N \qquad \textbf{(6.11)}$$

But $i_k = C_k\, dv/dt$. Hence,

$$\begin{aligned} i &= C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + C_3\frac{dv}{dt} + \cdots + C_N\frac{dv}{dt} \\ &= \left(\sum_{k=1}^{N} C_k\right)\frac{dv}{dt} = C_{eq}\frac{dv}{dt} \end{aligned} \qquad \textbf{(6.12)}$$

where

$$C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N \tag{6.13}$$

The **equivalent capacitance** of N parallel-connected capacitors is the sum of the individual capacitances.

We observe that capacitors in parallel combine in the same manner as resistors in series. We now obtain ***Ceq*** of ***N*** capacitors connected in series by comparing the circuit in Fig. 6.15(a) with the equivalent circuit in Fig. 6.15(b). Note that the same current *i* flows (and consequently the same charge) through the capacitors

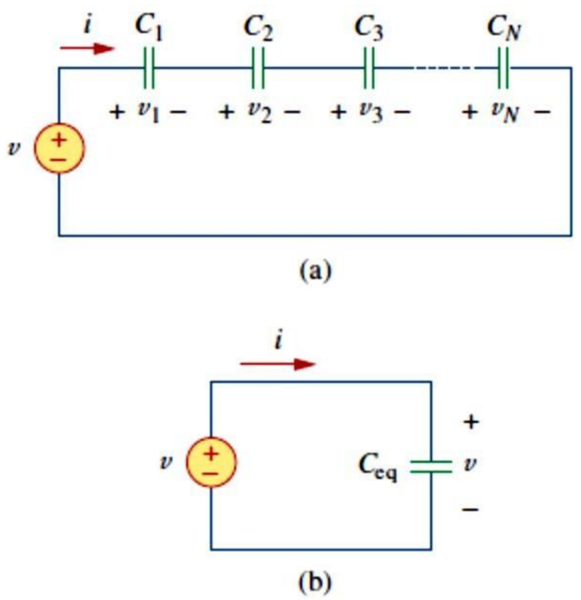

Figure 6.15
(a) Series-connected N capacitors, (b) equivalent circuit for the series capacitor.

Applying KVL to the loop in Fig. 6.15(a),

$$v = v_1 + v_2 + v_3 + \cdots + v_N \tag{6.14}$$

But $v_k = \dfrac{1}{C_k}\displaystyle\int_{t_0}^{t} i(\tau)d\tau + v_k(t_0)$. Therefore,

$$\begin{aligned} v &= \frac{1}{C_1}\int_{t_0}^{t} i(\tau)d\tau + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i(\tau)d\tau + v_2(t_0) \\ &\qquad + \cdots + \frac{1}{C_N}\int_{t_0}^{t} i(\tau)d\tau + v_N(t_0) \\ &= \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)\int_{t_0}^{t} i(\tau)d\tau + v_1(t_0) + v_2(t_0) \\ &\qquad + \cdots + v_N(t_0) \\ &= \frac{1}{C_{eq}}\int_{t_0}^{t} i(\tau)d\tau + v(t_0) \end{aligned} \tag{6.15}$$

where

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N}} \tag{6.16}$$

The initial voltage $v(t_0)$ across C_{eq} is required by KVL to be the sum of the capacitor voltages at t_0. Or according to Eq. (6.15),

$$v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)$$

Thus, according to Eq. (6.16),

The **equivalent capacitance** of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

Note that capacitors in series combine in the same manner as resistors in parallel. For $N = 2$ (i.e., two capacitors in series), Eq. (6.16) becomes

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

or

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2} \qquad (6.17)$$

Example

Find the equivalent capacitance seen between terminals *a* and *b* of the circuit in Fig. 6.16.

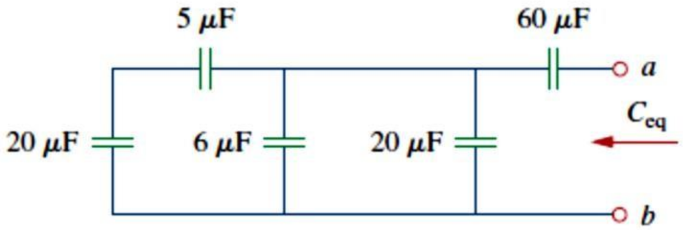

Figure 6.16

Solution:

The 20-μF and 5-μF capacitors are in series; their equivalent capacitance is

$$\frac{20 \times 5}{20 + 5} = 4\ \mu\text{F}$$

This 4-μF capacitor is in parallel with the 6-μF and 20-μF capacitors; their combined capacitance is

$$4 + 6 + 20 = 30\ \mu\text{F}$$

This 30-μF capacitor is in series with the 60-μF capacitor. Hence, the equivalent capacitance for the entire circuit is

$$C_{eq} = \frac{30 \times 60}{30 + 60} = 20\ \mu\text{F}$$

Practice Problem

Find the equivalent capacitance seen at the terminals of the circuit in Fig. 6.17.

Answer: 40 mF.

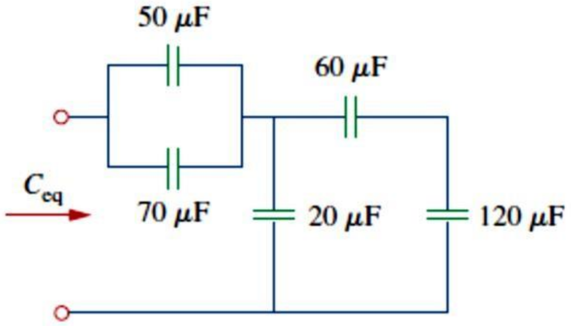

Figure 6.17

Example

For the circuit in Fig. 6.18, find the voltage across each capacitor.

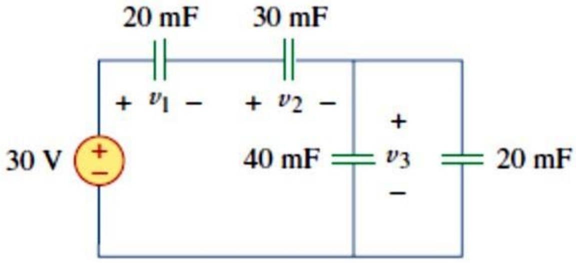

Figure 6.18

Solution:

We first find the equivalent capacitance C_{eq}, shown in Fig. 6.19. The two parallel capacitors in Fig. 6.18 can be combined to get 40 + 20 = 60 mF. This 60-mF capacitor is in series with the 20-mF and 30-mF capacitors. Thus,

$$C_{eq} = \frac{1}{\frac{1}{60} + \frac{1}{30} + \frac{1}{20}} \text{ mF} = 10 \text{ mF}$$

The total charge is

$$q = C_{eq}v = 10 \times 10^{-3} \times 30 = 0.3 \text{ C}$$

This is the charge on the 20-mF and 30-mF capacitors, because they are in series with the 30-V source. (A crude way to see this is to imagine that charge acts like current, since $i = dq/dt$.) Therefore,

$$v_1 = \frac{q}{C_1} = \frac{0.3}{20 \times 10^{-3}} = 15 \text{ V} \qquad v_2 = \frac{q}{C_2} = \frac{0.3}{30 \times 10^{-3}} = 10 \text{ V}$$

Having determined v_1 and v_2, we now use KVL to determine v_3 by

$$v_3 = 30 - v_1 - v_2 = 5 \text{ V}$$

Alternatively, since the 40-mF and 20-mF capacitors are in parallel, they have the same voltage v_3 and their combined capacitance is 40 + 20 = 60 mF. This combined capacitance is in series with the 20-mF and 30-mF capacitors and consequently has the same charge on it. Hence,

$$v_3 = \frac{q}{60 \text{ mF}} = \frac{0.3}{60 \times 10^{-3}} = 5 \text{ V}$$

Practice Problem

Find the voltage across each of the capacitors in Fig. 6.20.

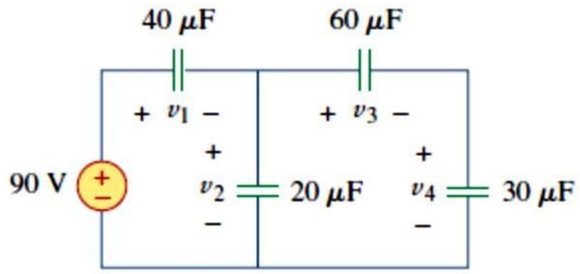

Figure 6.20

Answer: $v_1 = 45$ V, $v_2 = 45$ V, $v_3 = 15$ V, $v_4 = 30$ V.

6.4 Inductors

An inductor is a passive element designed to store energy in its magnetic field. Inductors find numerous applications in electronic and power systems. They are used in power supplies, transformers, radios, TVs, radars, and electric motors. Any conductor of electric current has inductive properties and may be regarded as an inductor. But in order to enhance the inductive effect, a practical inductor is usually formed into a cylindrical coil with many turns of conducting wire, as shown in Fig. 6.21.,

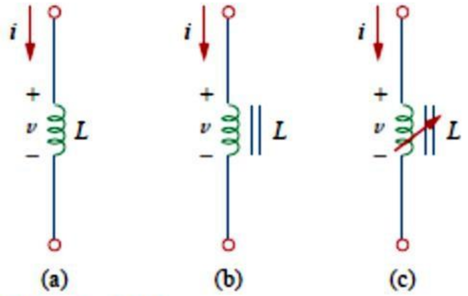

Figure 6.23
Circuit symbols for inductors: (a) air-core, (b) iron-core, (c) variable iron-core.

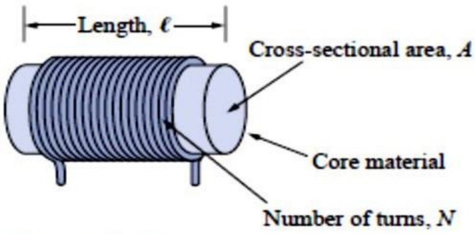

Figure 6.21
Typical form of an inductor.

An **inductor** consists of a coil of conducting wire.

If current is allowed to pass through an inductor, it is found that the voltage across the inductor is directly proportional to the time rate of change of the current. Using the passive sign convention

$$v = L\frac{di}{dt} \qquad (6.18)$$

Where ***L*** is the constant of proportionality called the *inductance* of the inductor. The unit of inductance is the henry (H),

Inductance is the property whereby an inductor exhibits opposition to the change of current flowing through it, measured in henrys (H).

The inductance of an inductor depends on its physical dimension and construction. Formulas for calculating the inductance of inductors of different shapes

$$L = \frac{N^2 \mu A}{\ell} \qquad (6.19)$$

Where ***N*** is the number of turns, is the length, ***A*** is the cross-sectional area, and is the permeability of the core. We can see from Eq. (6.19) that inductance can be increased by increasing the number of turns of coil, using material with higher permeability as the core, increasing the cross-sectional area, or reducing the length of the coil. Like capacitors, commercially available inductors come in different values and types. Typical practical inductors have inductance values ranging from a few microhenrys (μH), as in communication systems, to tens of henrys (H) as in power systems. Inductors may be fixed or variable. The core may be made of iron, steel, plastic, or air

The circuit symbols for inductors are shown in Fig. 6.23, following the passive sign convention

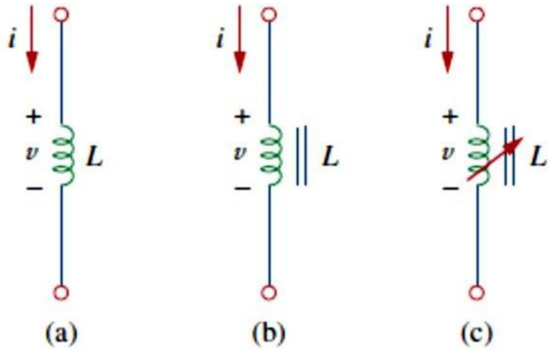

Figure 6.23
Circuit symbols for inductors: (a) air-core, (b) iron-core, (c) variable iron-core.

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The current-voltage relationship is obtained from Eq. (6.18) as

$$di = \frac{1}{L} v\, dt$$

Integrating gives

$$i = \frac{1}{L} \int_{-\infty}^{t} v(t)\, dt \qquad \textbf{(6.20)}$$

or

$$\boxed{i = \frac{1}{L} \int_{t_0}^{t} v(t)\, dt + i(t_0)} \qquad \textbf{(6.21)}$$

where $i(t_0)$ is the total current for $-\infty < t < t_0$ and $i(-\infty) = 0$. The idea of making $i(-\infty) = 0$ is practical and reasonable, because there must be a time in the past when there was no current in the inductor.

The inductor is designed to store energy in its magnetic field. The energy stored can be obtained from Eq. (6.18). The power delivered to the inductor is

$$p = vi = \left(L\frac{di}{dt}\right)i \tag{6.22}$$

The energy stored is

$$\begin{aligned} w &= \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t} \left(L\frac{di}{dt}\right)i\,dt \\ &= L\int_{-\infty}^{t} i\,di = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(-\infty) \end{aligned} \tag{6.23}$$

Since $i(-\infty) = 0$,

$$\boxed{w = \frac{1}{2}Li^2} \tag{6.24}$$

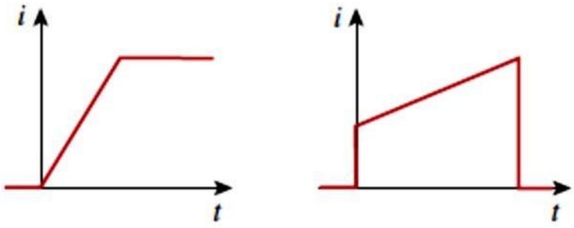
1. Note from Eq. (6.18) that the voltage across an inductor is zero when the current is constant. Thus,

   An inductor acts like a short circuit to dc.

2. An important property of the inductor is its opposition to change in current flowing through it

   The current through an inductor cannot change instantaneously.

According to Eq. (6.18), a discontinuous change in the current through an inductor requires an infinite voltage, which is not physically possible. Thus, an inductor opposes an abrupt change in the current through it. For example, the current through an inductor may take the form shown in Fig. 6.25(a), whereas the inductor current cannot take the form shown in Fig. 6.25(b) in real-life situations due to the discontinuities. However, the voltage across an inductor can change abruptly



(a) (b)

Figure 6.25
Current through an inductor: (a) allowed, (b) not allowable; an abrupt change is not possible.

3.Like the ideal capacitor, capacitor, the ideal inductor does not dissipate energy. The energy stored in it can be retrieved at a later time. The inductor takes power from the circuit when storing energy and delivers power to the circuit when returning previously stored energy.

4. A practical, nonideal inductor has a significant resistive component, as shown in Fig. 6.26. This is due to the fact that the inductor is made of a conducting material such as copper, which has some resistance. This resistance is called the *winding resistance* , ***R***ω and it appears in series with the inductance of the inductor. The presence of ***R***ω makes it both an energy storage device and an energy dissipation device. Since ***R***ω is usually very small, it is ignored in most cases. The nonideal inductor also has a *winding capacitance* due to the capacitive coupling between the conducting coils. is very small and can be ignored in most cases, except at high frequencies. We will assume ideal inductors in this book.

Since an inductor is often made of a highly conducting wire, it has a very small resistance.

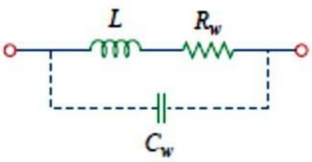


Figure 6.26
Circuit model for a practical inductor.

Example

The current through a 0.1-H inductor is ***i*(*t*) =10t**e^{-5t}**A**. Find the voltage across the inductor and the energy stored in it

Solution:

Since $v = L\,di/dt$ and $L = 0.1$ H,

$$v = 0.1\frac{d}{dt}(10te^{-5t}) = e^{-5t} + t(-5)e^{-5t} = e^{-5t}(1 - 5t)\text{ V}$$

.

The energy stored is

$$w = \frac{1}{2}Li^2 = \frac{1}{2}(0.1)100t^2e^{-10t} = 5t^2e^{-10t}\text{ J}$$

Practice Problem 6.8

If the current through a 1-mH inductor is ***i*(*t*) = 60 cos 100*t*** mA, find the terminal voltage and the energy stored

Answer: _6 sin 100*t* mV, 1.8 cos2 (100*t*) mJ.

Example

Find the current through a 5-H inductor if the voltage across it is

$$v(t) = \begin{cases} 30t^2, & t > 0 \\ 0, & t < 0 \end{cases}$$

Also, find the energy stored at $t = 5$ s. Assume $i(v) > 0$.

Solution:

Since $i = \frac{1}{L}\int_{t_0}^{t} v(t)\,dt + i(t_0)$ and $L = 5$ H,

$$i = \frac{1}{5}\int_0^t 30t^2\,dt + 0 = 6 \times \frac{t^3}{3} = 2t^3 \text{ A}$$

The power $p = vi = 60t^5$, and the energy stored is then

$$w = \int p\,dt = \int_0^5 60t^5\,dt = 60\frac{t^6}{6}\Big|_0^5 = 156.25 \text{ kJ}$$

Alternatively, we can obtain the energy stored using Eq. (6.24), by writing

$$w\big|_0^5 = \frac{1}{2}Li^2(5) - \frac{1}{2}Li(0) = \frac{1}{2}(5)(2 \times 5^3)^2 - 0 = 156.25 \text{ kJ}$$

as obtained before.

Practice Problem

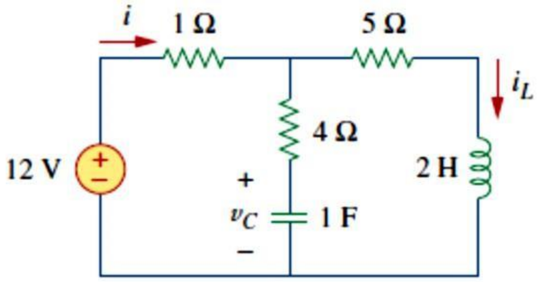
The terminal voltage of a 2-H inductor is ***v* =10(1 - *t*) V**. Find the current flowing through it at t = 4 s and the energy stored in it at t = 4 s
Assume *i*(0) = 2 A.

Answer: - 18 A, 320 J.

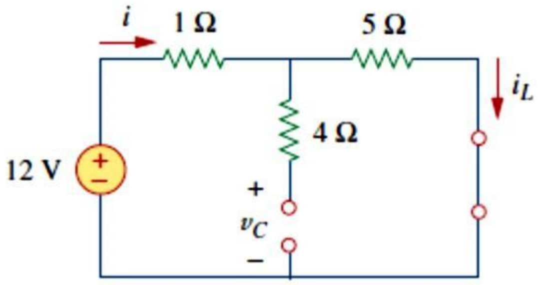
.

Example

Consider the circuit in Fig. 6.27(a). Under dc conditions, find: (a) i, v_c and, i_L, (b) the energy stored in the capacitor and inductor.



(a)



(b)

Figure 6.27

Solution:

(a) Under dc conditions, we replace the capacitor with an open circuit and the inductor with a short circuit, as in Fig. 6.27(b). It is evident from Fig. 6.27(b) that

$$i = i_L = \frac{12}{1+5} = 2\text{ A}$$

The voltage v_C is the same as the voltage across the 5-Ω resistor. Hence,

$$v_C = 5i = 10\text{ V}$$

(b) The energy in the capacitor is

$$w_C = \frac{1}{2}Cv_C^2 = \frac{1}{2}(1)(10^2) = 50\text{ J}$$

and that in the inductor is

$$w_L = \frac{1}{2}Li_L^2 = \frac{1}{2}(2)(2^2) = 4\text{ J}$$

Practice Problem

Determine v_C, i_L, and the energy stored in the capacitor and inductor in the circuit of Fig. 6.28 under dc conditions

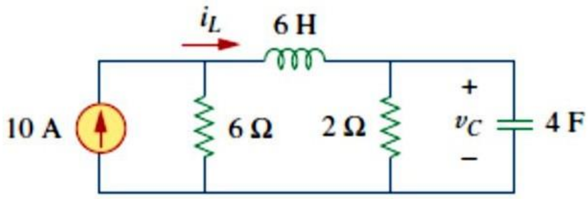


Figure 6.28

Answer: 15 V, 7.5 A, 450 J, 168.75 J.

6.5 Series and Parallel Inductors

Now that the inductor has been added to our list of passive elements, it is necessary to extend the powerful tool of series-parallel combination. We need to know how to find the equivalent inductance of a series-connected or parallel-connected set of inductors found in practical circuits.

Consider a series connection of N inductors, as shown in Fig. 6.29(a), with the equivalent circuit shown in Fig. 6.29(b). The inductors have the same current through them. Applying KVL to the loop,

$$v = v_1 + v_2 + v_3 + \cdots + v_N \tag{6.25}$$

Substituting $v_k = L_k\, di/dt$ results in

$$\begin{aligned} v &= L_1\frac{di}{dt} + L_2\frac{di}{dt} + L_3\frac{di}{dt} + \cdots + L_N\frac{di}{dt} \\ &= (L_1 + L_2 + L_3 + \cdots + L_N)\frac{di}{dt} \\ &= \left(\sum_{k=1}^{N} L_k\right)\frac{di}{dt} = L_{\text{eq}}\frac{di}{dt} \end{aligned} \tag{6.26}$$

where

$$L_{eq} = L_1 + L_2 + L_3 + \cdots + L_N \tag{6.27}$$

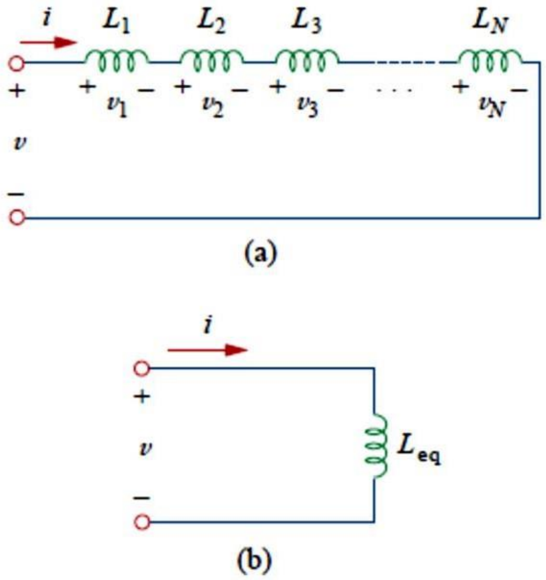


Figure 6.29
(a) A series connection of N inductors, (b) equivalent circuit for the series inductors.

Thus,

> The **equivalent inductance** of series-connected inductors is the sum of the individual inductances.

Inductors in series are combined in exactly the same way as resistors in series.

We now consider a parallel connection of N inductors, as shown in Fig. 6.30(a), with the equivalent circuit in Fig. 6.30(b). The inductors have the same voltage across them. Using KCL,

$$i = i_1 + i_2 + i_3 + \cdots + i_N \tag{6.28}$$

But $i_k = \frac{1}{L_k}\int_{t_0}^{t} v\,dt + i_k(t_0)$; hence,

$$\begin{aligned} i &= \frac{1}{L_1}\int_{t_0}^{t} v\,dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,dt + i_2(t_0) \\ &\quad + \cdots + \frac{1}{L_N}\int_{t_0}^{t} v\,dt + i_N(t_0) \\ &= \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right)\int_{t_0}^{t} v\,dt + i_1(t_0) + i_2(t_0) \\ &\quad + \cdots + i_N(t_0) \\ &= \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_0}^{t} v\,dt + \sum_{k=1}^{N} i_k(t_0) = \frac{1}{L_{eq}}\int_{t_0}^{t} v\,dt + i(t_0) \end{aligned} \tag{6.29}$$

where

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N} \tag{6.30}$$

The initial current $i(t_0)$ through L_{eq} at $t = t_0$ is expected by KCL to be the sum of the inductor currents at t_0. Thus, according to Eq. (6.29),

$$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

According to Eq. (6.30),

> The **equivalent inductance** of parallel inductors is the reciprocal of the sum of the reciprocals of the individual inductances.

Note that the inductors in parallel are combined in the same way as resistors in parallel.

For two inductors in parallel ($N = 2$), Eq. (6.30) becomes

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} \quad \text{or} \quad L_{eq} = \frac{L_1 L_2}{L_1 + L_2} \tag{6.31}$$

As long as all the elements are of the same type, the Δ-Y transformations for resistors discussed in Section 2.7 can be extended to capacitors and inductors.

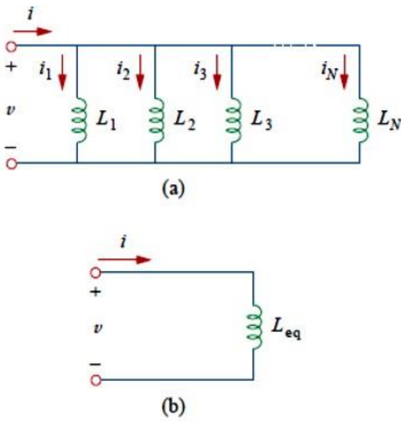


Figure 6.30
(a) A parallel connection of N inductors, (b) equivalent circuit for the parallel inductors.

TABLE 6.1

Important characteristics of the basic elements.†

Relation	Resistor (*R*)	Capacitor (*C*)	Inductor (*L*)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v\,dt + i(t_0)$
p or *w*:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	*v*	*i*

† *Passive sign convention is assumed.*

The wye-delta transformation discussed in Section 2 for resistors can be extended to capacitors and inductors

Example

Find the equivalent inductance of the circuit shown in Fig. 6.31.

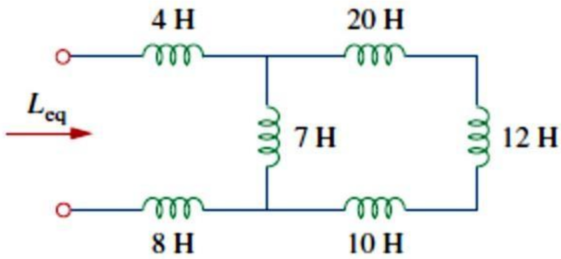


Figure 6.31

Solution:

The 10-H, 12-H, and 20-H inductors are in series; thus, combining them gives a 42-H inductance. This 42-H inductor is in parallel with the 7-H inductor so that they are combined, to give

$$\frac{7 \times 42}{7 + 42} = 6\ \text{H}$$

This 6-H inductor is in series with the 4-H and 8-H inductors. Hence,

$$L_{eq} = 4 + 6 + 8 = 18\ \text{H}$$

Practice Problem

Calculate the equivalent inductance for the inductive ladder network in Fig. 6.32

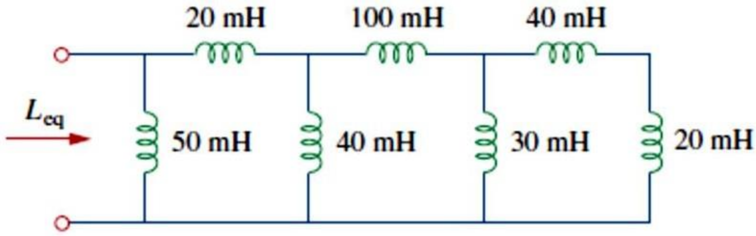


Figure 6.32

Answer: 25 mH.

Example

For the circuit in Fig. 6.33, $i(t) = 4(2 - e^{-10t})$ mA. If $i_2(0) = -1$ mA, find: (a) $i_1(0)$; (b) $v(t)$, $v_1(t)$, and $v_2(t)$; (c) $i_1(t)$ and $i_2(t)$.

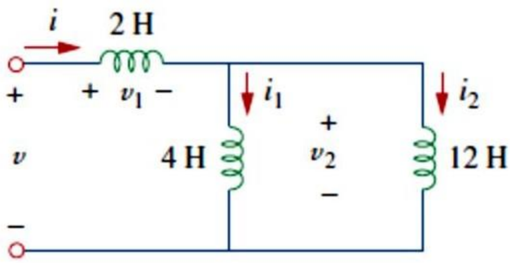


Figure 6.33

Solution:

(a) From $i(t) = 4(2 - e^{-10t})$ mA, $i(0) = 4(2 - 1) = 4$ mA. Since $i = i_1 + i_2$,

$$i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5 \text{ mA}$$

(b) The equivalent inductance is

$$L_{eq} = 2 + 4 \parallel 12 = 2 + 3 = 5 \text{ H}$$

Thus,

$$v(t) = L_{eq}\frac{di}{dt} = 5(4)(-1)(-10)e^{-10t} \text{ mV} = 200e^{-10t} \text{ mV}$$

and

$$v_1(t) = 2\frac{di}{dt} = 2(-4)(-10)e^{-10t} \text{ mV} = 80e^{-10t} \text{ mV}$$

Since $v = v_1 + v_2$,

$$v_2(t) = v(t) - v_1(t) = 120e^{-10t} \text{ mV}$$

(c) The current i_1 is obtained as

$$i_1(t) = \frac{1}{4}\int_0^t v_2 \, dt + i_1(0) = \frac{120}{4}\int_0^t e^{-10t} \, dt + 5 \text{ mA}$$
$$= -3e^{-10t}\Big|_0^t + 5 \text{ mA} = -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t} \text{ mA}$$

Similarly,

$$i_2(t) = \frac{1}{12}\int_0^t v_2 \, dt + i_2(0) = \frac{120}{12}\int_0^t e^{-10t} \, dt - 1 \text{ mA}$$
$$= -e^{-10t}\Big|_0^t - 1 \text{ mA} = -e^{-10t} + 1 - 1 = -e^{-10t} \text{ mA}$$

Note that $i_1(t) + i_2(t) = i(t)$.

Practice Problem

In the circuit of Fig. 6.34, $i_1(t) = 0.6e^{-2t}$ A. If $i(0) = 1.4$ A, find: (a) $i_2(0)$; (b) $i_2(t)$ and $i(t)$; (c) $v_1(t)$, $v_2(t)$, and $v(t)$.

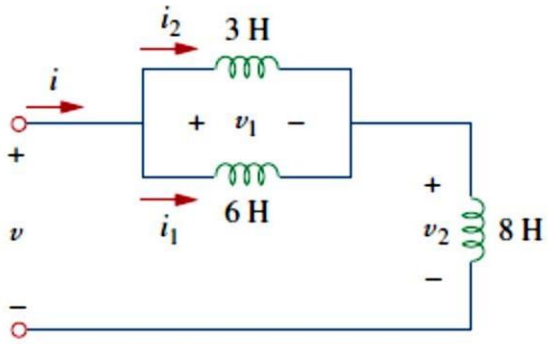


Figure 6.34

Answer: (a) 0.8 A, (b) $(-0.4 + 1.2e^{-2t})$ A, $(-0.4 + 1.8e^{-2t})$ A, (c) $-36e^{-2t}$ V, $-7.2e^{-2t}$ V, $-28.8e^{-2t}$ V.

Capacitors and inductors possess the following three special properties that make them very useful in electric circuits:

1. The capacity to store energy makes them useful as temporary voltage or current sources. Thus, they can be used for generating a large amount of current or voltage for a short period of time.

2. Capacitors oppose any abrupt change in voltage, while inductors oppose any abrupt change in current. This property makes inductors useful for spark or arc suppression and for converting pulsating dc voltage into relatively smooth dc voltage
.

3. Capacitors and inductors are frequency sensitive. This property makes them useful for frequency discrimination.

The first two properties are put to use in dc circuits, while the third one is taken advantage of in ac circuits.

Summary

1. The current through a capacitor is directly proportional to the time rate of change of the voltage across it.

$$i = C\frac{dv}{dt}$$

The current through a capacitor is zero unless the voltage is changing. Thus, a capacitor acts like an open circuit to a dc source.

2. The voltage across a capacitor is directly proportional to the time integral of the current through it.

$$v = \frac{1}{C}\int_{-\infty}^{t} i\,dt = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$$

The voltage across a capacitor cannot change instantly.

3. Capacitors in series and in parallel are combined in the same way as conductances.
4. The voltage across an inductor is directly proportional to the time rate of change of the current through it.

$$v = L\frac{di}{dt}$$

The voltage across the inductor is zero unless the current is changing. Thus, an inductor acts like a short circuit to a dc source.

5. The current through an inductor is directly proportional to the time integral of the voltage across it.

$$i = \frac{1}{L}\int_{-\infty}^{t} v\,dt = \frac{1}{L}\int_{t_0}^{t} v\,dt + i(t_0)$$

The current through an inductor cannot change instantly.

6. Inductors in series and in parallel are combined in the same way resistors in series and in parallel are combined.
7. At any given time t, the energy stored in a capacitor is $\frac{1}{2}Cv^2$, while the energy stored in an inductor is $\frac{1}{2}Li^2$.
8. Three application circuits, the integrator, the differentiator, and the analog computer, can be realized using resistors, capacitors, and op amps.

Chapter 9

Sinusoids and Phasors

Sinusoids and Phasors

9.1 Introduction

We now begin the analysis of circuits in which the source voltage or current is time-varying. In this chapter, we are particularly interested in sinusoidally time-varying excitation, or simply, excitation by a *sinusoid*.

A **sinusoid** is a signal that has the form of the sine or cosine function.

A sinusoidal current is usually referred to as *alternating current* (*ac*). Such a current reverses at regular time intervals and has alternately positiveand negative values. Circuits driven by sinusoidal current or voltage sources are called *ac circuits*

9.2 Sinusoids

Consider the sinusoidal voltage

$$v(t) = V_m \sin \omega t \tag{9.1}$$

where

V_m = the *amplitude* of the sinusoid
ω = the *angular frequency* in radians/s
ωt = the *argument* of the sinusoid

The sinusoid is shown in Fig. 9.1(a) as a function of its argument and in Fig. 9.1(b) as a function of time. It is evident that the sinusoid repeats itself every T seconds; thus, T is called the *period* of the sinusoid. From the two plots in Fig. 9.1, we observe that $\omega T = 2\pi$,

$$T = \frac{2\pi}{\omega} \tag{9.2}$$

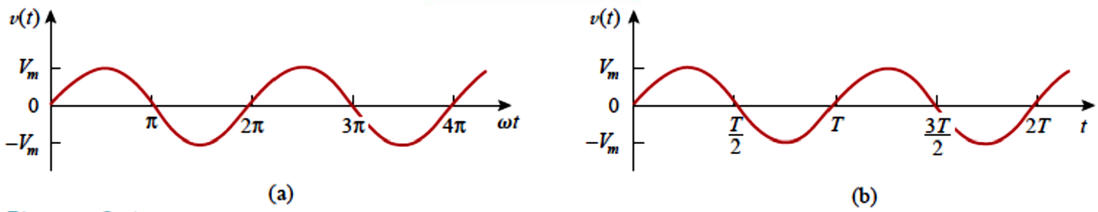


Figure 9.1
A sketch of $V_m \sin \omega t$: (a) as a function of ωt, (b) as a function of t.

The fact that $v(t)$ repeats itself every T seconds is shown by replacing t by $t + T$ in Eq. (9.1). We get

$$\begin{aligned} v(t + T) &= V_m \sin \omega(t + T) = V_m \sin \omega\left(t + \frac{2\pi}{\omega}\right) \\ &= V_m \sin(\omega t + 2\pi) = V_m \sin \omega t = v(t) \end{aligned} \tag{9.3}$$

Hence,

$$v(t + T) = v(t) \tag{9.4}$$

that is, v has the same value at $t + T$ as it does at t and $v(t)$ is said to be *periodic*. In general, A **periodic function** is one that satisfies $f(t) = f(t + nT)$, for all t and for all integers n.

As mentioned, the ***period T*** of the periodic function is the time of one complete cycle or the number of seconds per cycle. The reciprocal of this quantity is the number of cycles per second, known as the *cyclic frequency f* of the sinusoid. Thus,

$$f = \frac{1}{T} \tag{9.5}$$

From Eqs. (9.2) and (9.5), it is clear that

$$\omega = 2\pi f \tag{9.6}$$

While ω is in radians per second (rad/s), f is in hertz (Hz).

Let us now consider a more general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \phi) \tag{9.7}$$

where $(\omega t + \phi)$ is the argument and ϕ is the *phase*. Both argument and phase can be in radians or degrees.

Let us examine the two sinusoids

$$v_1(t) = V_m \sin \omega t \quad \text{and} \quad v_2(t) = V_m \sin(\omega t + \phi) \tag{9.8}$$

shown in Fig. 9.2. The starting point of v_2 in Fig. 9.2 occurs first in time. Therefore, we say that v_2 *leads* v_1 by ϕ or that v_1 *lags* v_2 by ϕ. If $\phi \neq 0$, we also say that v_1 and v_2 are *out of phase*. If $\phi = 0$, then v_1 and v_2 are said to be *in phase*; they reach their minima and maxima at exactly the same time. We can compare v_1 and v_2 in this manner because they operate at the same frequency; they do not need to have the same amplitude.

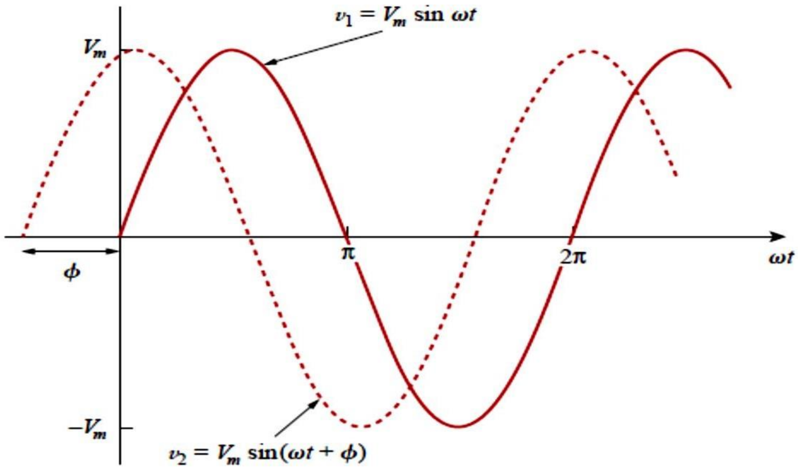


Figure 9.2
Two sinusoids with different phases.

A sinusoid can be expressed in either sine or cosine form. When comparing two sinusoids, it is expedient to express both as either sine or cosine with positive amplitudes. This is achieved by using the following trigonometric identities

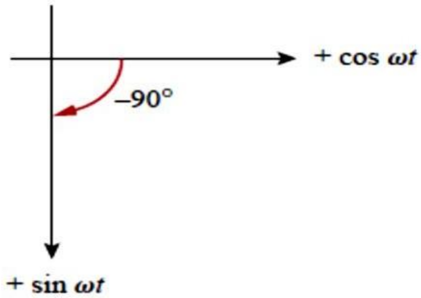
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \qquad \textbf{(9.9)}$$

With these identities, it is easy to show that

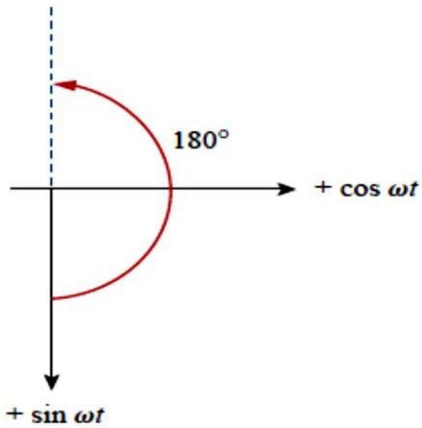
$$\sin(\omega t \pm 180^\circ) = -\sin \omega t$$
$$\cos(\omega t \pm 180^\circ) = -\cos \omega t$$
$$\sin(\omega t \pm 90^\circ) = \pm \cos \omega t \qquad \textbf{(9.10)}$$
$$\cos(\omega t \pm 90^\circ) = \mp \sin \omega t$$

Using these relationships, we can transform a sinusoid from sine form to cosine form or vice versa.

A graphical approach may be used to relate or compare sinusoids as an alternative to using the trigonometric identities in Eqs. (9.9) and (9.10). Consider the set of axes shown in Fig. 9.3(a). The horizontal axis represents the magnitude of cosine, while the vertical axis (pointing down) denotes the magnitude of sine. Angles are measured positively counterclockwise from the horizontal, as usual in polar coordinates. This graphical technique can be used to relate two sinusoids. For example, we see in Fig. 9.3(a) that subtracting 90° from the argument of $\cos\omega t$ gives $\sin\omega t$, or $\cos(\omega t - 90°) = \sin\omega t$. Similarly, adding 180° to the argument of $\sin\omega t$ gives $-\sin\omega t$, or $\sin(\omega t + 180°) = -\sin\omega t$, as shown in Fig. 9.3(b).



(a)



(b)

Figure 9.3
A graphical means of relating cosine and sine: (a) $\cos(\omega t - 90°) = \sin\omega t$, (b) $\sin(\omega t + 180°) = -\sin\omega t$.

The graphical technique can also be used to add two sinusoids of the same frequency when one is in sine form and the other is in cosine form. To add $A\cos\omega t$ and $B\sin\omega t$, we note that A is the magnitude of $\cos\omega t$ while B is the magnitude of $\sin\omega t$, as shown in Fig. 9.4(a). The magnitude and argument of the resultant sinusoid in cosine form is readily obtained from the triangle. Thus,

$$A\cos\omega t + B\sin\omega t = C\cos(\omega t - \theta) \tag{9.11}$$

where

$$C = \sqrt{A^2 + B^2}, \qquad \theta = \tan^{-1}\frac{B}{A} \tag{9.12}$$

For example, we may add $3\cos\omega t$ and $-4\sin\omega t$ as shown in Fig. 9.4(b) and obtain

$$3\cos\omega t - 4\sin\omega t = 5\cos(\omega t + 53.1^\circ) \tag{9.13}$$

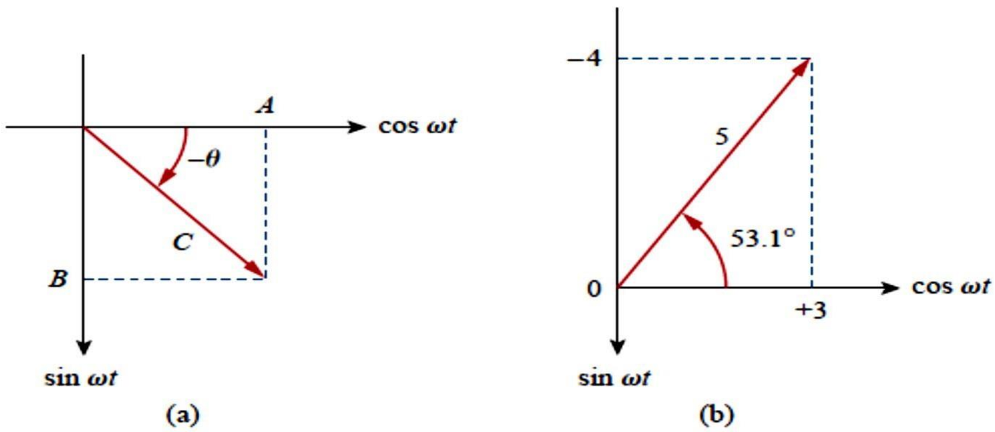


Figure 9.4
(a) Adding $A\cos\omega t$ and $B\sin\omega t$, (b) adding $3\cos\omega t$ and $-4\sin\omega t$.

Example

Find the amplitude, phase, period, and frequency of the sinusoid

$$v(t) = 12\cos(50t + 10^\circ)$$

Solution:

The amplitude is $V_m = 12$ V.
The phase is $\phi = 10^\circ$.
The angular frequency is $\omega = 50$ rad/s.
The period $T = \frac{2\pi}{\omega} = \frac{2\pi}{50} = 0.1257$ s.
The frequency is $f = \frac{1}{T} = 7.958$ Hz.

Practice Problem

Given the sinusoid $5\sin(4\pi t - 60^\circ)$, calculate its amplitude, phase, angular frequency, period, and frequency.

Answer: 5, -60°, 12.57 rad/s, 0.5 s, 2 Hz.

Example

Calculate the phase angle between $v_1 = -10\cos(\omega t + 50^\circ)$ and $v_2 = 12\sin(\omega t - 10^\circ)$. State which sinusoid is leading.

Solution:
Let us calculate the phase in three ways. The first two methods use trigonometric identities, while the third method uses the graphical approach.

■ **METHOD 1** In order to compare v_1 and v_2, we must express them in the same form. If we express them in cosine form with positive amplitudes,

$$v_1 = -10\cos(\omega t + 50^\circ) = 10\cos(\omega t + 50^\circ - 180^\circ)$$
$$v_1 = 10\cos(\omega t - 130^\circ) \quad \text{or} \quad v_1 = 10\cos(\omega t + 230^\circ) \qquad \textbf{(9.2.1)}$$

and

$$v_2 = 12\sin(\omega t - 10^\circ) = 12\cos(\omega t - 10^\circ - 90^\circ)$$
$$v_2 = 12\cos(\omega t - 100^\circ) \qquad \textbf{(9.2.2)}$$

It can be deduced from Eqs. (9.2.1) and (9.2.2) that the phase difference between v_1 and v_2 is 30°. We can write v_2 as

$$v_2 = 12\cos(\omega t - 130^\circ + 30^\circ) \quad \text{or} \quad v_2 = 12\cos(\omega t + 260^\circ) \qquad \textbf{(9.2.3)}$$

Comparing Eqs. (9.2.1) and (9.2.3) shows clearly that v_2 leads v_1 by 30°.

■ **METHOD 2** Alternatively, we may express v_1 in sine form:

$$v_1 = -10\cos(\omega t + 50^\circ) = 10\sin(\omega t + 50^\circ - 90^\circ)$$
$$= 10\sin(\omega t - 40^\circ) = 10\sin(\omega t - 10^\circ - 30^\circ)$$

But $v_2 = 12\sin(\omega t - 10°)$. Comparing the two shows that v_1 lags v_2 by 30°. This is the same as saying that v_2 leads v_1 by 30°.

METHOD 3 We may regard v_1 as simply $-10\cos\omega t$ with a phase shift of $+50°$. Hence, v_1 is as shown in Fig. 9.5. Similarly, v_2 is $12\sin\omega t$ with a phase shift of $-10°$, as shown in Fig. 9.5. It is easy to see from Fig. 9.5 that v_2 leads v_1 by 30°, that is, $90° - 50° - 10°$.

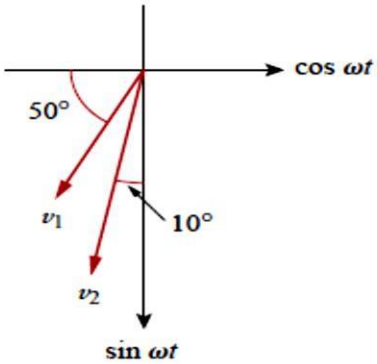


Figure 9.5
For Example 9.2.

Practice Problem

Find the phase angle between

$$i_1 = -4\sin(377t + 25°) \quad \text{and} \quad i_2 = 5\cos(377t - 40°)$$

Does i_1 lead or lag i_2?

Answer: 155°, i_1 leads i_2.

9.3 Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions

A **phasor** is a complex number that represents the amplitude and phase of a sinusoid.

Before we completely define phasors and apply them to circuit analysis, we need to be thoroughly familiar with complex numbers A complex number *z* can be written in rectangular form as .A complex number *z* can be written in rectangular form as

$$z = x + jy \tag{9.14a}$$

where $j = \sqrt{-1}$; x is the real part of z; y is the imaginary part of z. In this context, the variables x and y do not represent a location as in two-dimensional vector analysis but rather the real and imaginary parts of z in the complex plane. Nevertheless, we note that there are some resemblances between manipulating complex numbers and manipulating two-dimensional vectors.

The complex number z can also be written in polar or exponential form as

$$z = r\underline{/\phi} = re^{j\phi} \tag{9.14b}$$

where r is the magnitude of z, and ϕ is the phase of z. We notice that z can be represented in three ways:

$$\begin{aligned} z &= x + jy && \text{Rectangular form} \\ z &= r\underline{/\phi} && \text{Polar form} \\ z &= re^{j\phi} && \text{Exponential form} \end{aligned} \tag{9.15}$$

.

The relationship between the rectangular form and the polar form is shown in Fig. 9.6, where the x axis represents the real part and the y axis represents the imaginary part of a complex number. Given x and y, we can get r and ϕ as

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x} \tag{9.16a}$$

On the other hand, if we know r and ϕ, we can obtain x and y as

$$x = r\cos\phi, \qquad y = r\sin\phi \tag{9.16b}$$

Thus, z may be written as

$$z = x + jy = r\angle\phi = r(\cos\phi + j\sin\phi) \tag{9.17}$$

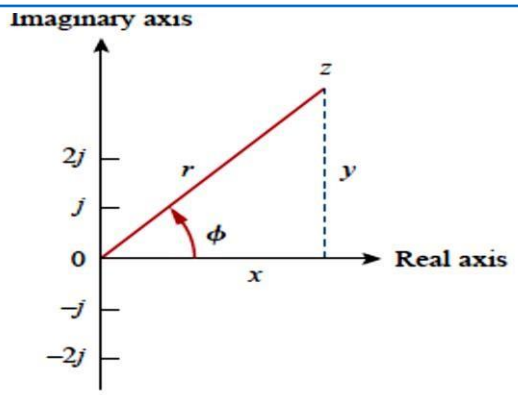


Figure 9.6
Representation of a complex number $z = x + jy = r\angle\phi$.

Addition and subtraction of complex numbers are better performed in rectangular form; multiplication and division are better done in polar form. Given the complex numbers

$$z = x + jy = r\angle\phi, \qquad z_1 = x_1 + jy_1 = r_1\angle\phi_1$$
$$z_2 = x_2 + jy_2 = r_2\angle\phi_2$$

the following operations are important.

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \tag{9.18a}$$

Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \tag{9.18b}$$

Multiplication:

$$z_1 z_2 = r_1 r_2 \underline{/\phi_1 + \phi_2} \tag{9.18c}$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \underline{/\phi_1 - \phi_2} \tag{9.18d}$$

Reciprocal:

$$\frac{1}{z} = \frac{1}{r} \underline{/-\phi} \tag{9.18e}$$

Square Root:

$$\sqrt{z} = \sqrt{r} \underline{/\phi/2} \tag{9.18f}$$

Complex Conjugate:

$$z^* = x - jy = r\underline{/-\phi} = re^{-j\phi} \tag{9.18g}$$

Note that from Eq. (9.18e),

$$\frac{1}{j} = -j \tag{9.18h}$$

The idea of phasor representation is based on Euler's identity. In general,

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi \tag{9.19}$$

which shows that we may regard $\cos\phi$ and $\sin\phi$ as the real and imaginary parts of $e^{j\phi}$; we may write

$$\cos\phi = \text{Re}(e^{j\phi}) \tag{9.20a}$$

$$\sin\phi = \text{Im}(e^{j\phi}) \tag{9.20b}$$

where Re and Im stand for the *real part of* and the *imaginary part of*. Given a sinusoid $v(t) = V_m \cos(\omega t + \phi)$, we use Eq. (9.20a) to express $v(t)$ as

$$v(t) = V_m \cos(\omega t + \phi) = \mathrm{Re}(V_m e^{j(\omega t+\phi)}) \tag{9.21}$$

or

$$v(t) = \mathrm{Re}(V_m e^{j\phi} e^{j\omega t}) \tag{9.22}$$

Thus,

$$\boxed{v(t) = \mathrm{Re}(\mathbf{V} e^{j\omega t})} \tag{9.23}$$

where

$$\mathbf{V} = V_m e^{j\phi} = V_m \underline{/\phi} \tag{9.24}$$

$\mathbf{V}$ is thus the *phasor representation* of the sinusoid $v(t)$, as we said earlier. In other words, a phasor is a complex representation of the magnitude and phase of a sinusoid. Either Eq. (9.20a) or Eq. (9.20b) can be used to develop the phasor, but the standard convention is to use Eq. (9.20a).

One way of looking at Eqs. (9.23) and (9.24) is to consider the plot of the *sinor* $\mathbf{V}e^{j\omega t} = V_m e^{j(\omega t+\phi)}$ on the complex plane. As time increases, the sinor rotates on a circle of radius V_m at an angular velocity ω in the counterclockwise direction, as shown in Fig. 9.7(a). We may regard $v(t)$ as the projection of the sinor $\mathbf{V}e^{j\omega t}$ on the real axis, as shown in Fig. 9.7(b). The value of the sinor at time $t = 0$ is the phasor $\mathbf{V}$ of the sinusoid $v(t)$. The sinor may be regarded as a rotating phasor. Thus, whenever a sinusoid is expressed as a phasor, the term $e^{j\omega t}$ is implicitly present. It is therefore important, when dealing with phasors, to keep in mind the frequency ω of the phasor; otherwise we can make serious mistakes.

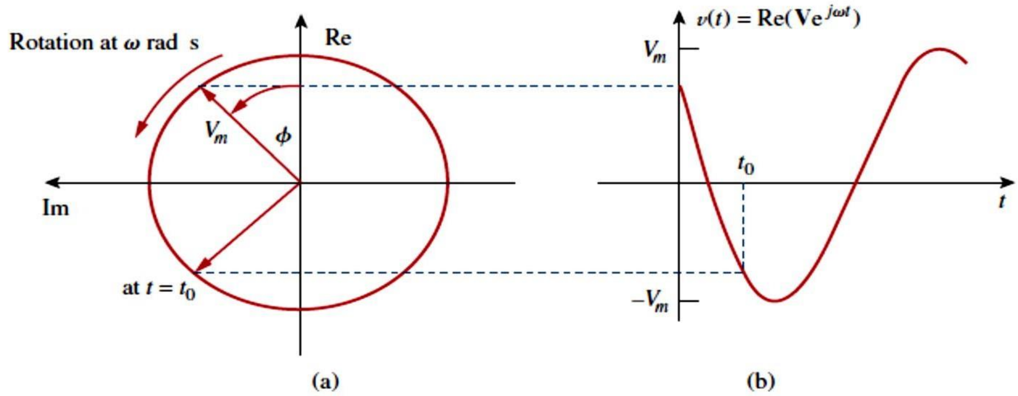


Figure 9.7
Representation of $\mathbf{V}e^{j\omega t}$: (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

Equation (9.23) states that to obtain the sinusoid corresponding to a given phasor $\mathbf{V}$, multiply the phasor by the time factor $e^{j\omega t}$ and take the real part. As a complex quantity, a phasor may be expressed in rectangular form, polar form, or exponential form. Since a phasor has magnitude and phase ("direction"), it behaves as a vector and is printed in boldface. For example, phasors $\mathbf{V} = V_m\underline{/\phi}$ and $\mathbf{I} = I_m\underline{/-\theta}$ are graphically represented in Fig. 9.8. Such a graphical representation of phasors is known as a *phasor diagram*.

Equations (9.21) through (9.23) reveal that to get the phasor corresponding to a sinusoid, we first express the sinusoid in the cosine form so that the sinusoid can be written as the real part of a complex number. Then we take out the time factor $e^{j\omega t}$, and whatever is left is the phasor corresponding to the sinusoid. By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$v(t) = V_m\cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m\underline{/\phi} \qquad (9.25)$$

(Time-domain representation) (Phasor-domain representation)

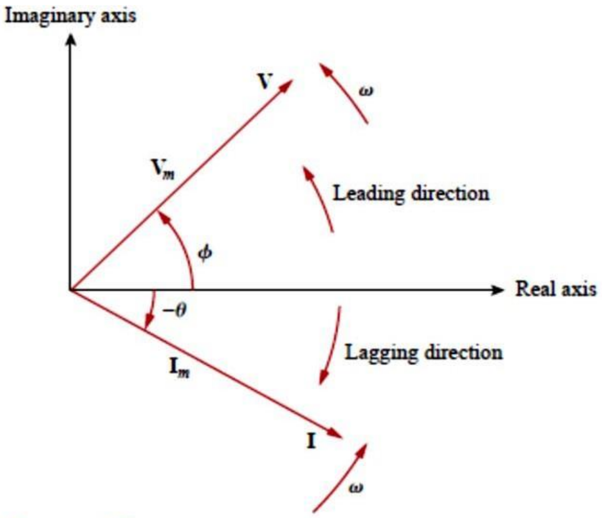


Figure 9.8
A phasor diagram showing $\mathbf{V} = V_m \angle \phi$ and $\mathbf{I} = I_m \angle -\theta$.

Given a sinusoid $v(t) = V_m \cos(\omega t + \phi)$, we obtain the corresponding phasor as $\mathbf{V} = V_m \angle \phi$. Equation (9.25) is also demonstrated in Table 9.1, where the sine function is considered in addition to the cosine function. From Eq. (9.25), we see that to get the phasor representation of a sinusoid, we express it in cosine form and take the magnitude and phase. Given a phasor, we obtain the time domain representation as the cosine function with the same magnitude as the phasor and the argument as ωt plus the phase of the phasor. The idea of expressing information in alternate domains is fundamental to all areas of engineering.

TABLE 9.1

Sinusoid-phasor transformation.

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90^\circ$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90^\circ$

Note that in Eq. (9.25) the frequency (or time) factor $e^{j\omega t}$ is suppressed, and the frequency is not explicitly shown in the phasor domain representation because ω is constant. However, the response depends on ω. For this reason, the phasor domain is also known as the *frequency domain*.

From Eqs. (9.23) and (9.24), $v(t) = \text{Re}(\mathbf{V}e^{j\omega t}) = V_m \cos(\omega t + \phi)$, so that

$$\begin{aligned}\frac{dv}{dt} &= -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90^\circ) \\ &= \text{Re}(\omega V_m e^{j\omega t} e^{j\phi} e^{j90^\circ}) = \text{Re}(j\omega \mathbf{V} e^{j\omega t})\end{aligned} \tag{9.26}$$

This shows that the derivative $v(t)$ is transformed to the phasor domain as $j\omega\mathbf{V}$

$$\underset{\text{(Time domain)}}{\frac{dv}{dt}} \quad \Leftrightarrow \quad \underset{\text{(Phasor domain)}}{j\omega\mathbf{V}} \tag{9.27}$$

Similarly, the integral of $v(t)$ is transformed to the phasor domain as $\mathbf{V}/j\omega$

$$\underset{\text{(Time domain)}}{\int v\, dt} \quad \Leftrightarrow \quad \underset{\text{(Phasor domain)}}{\frac{\mathbf{V}}{j\omega}} \tag{9.28}$$

Equation (9.27) allows the replacement of a derivative with respect to time with multiplication of $j\omega$ in the phasor domain, whereas Eq. (9.28) allows the replacement of an integral with respect to time with division by $j\omega$ in the phasor domain. Equations (9.27) and (9.28) are useful in finding the steady-state solution, which does not require knowing the initial values of the variable involved. This is one of the important applications of phasors.

Besides time differentiation and integration, another important use of phasors is found in summing sinusoids of the same frequency. This is best illustrated with an example, and Example 9.6 provides one.

The differences between $v(t)$ and $\mathbf{V}$ should be emphasized:

1. $v(t)$ is the *instantaneous or time domain* representation, while $\mathbf{V}$ is the *frequency or phasor domain* representation.
2. $v(t)$ is time dependent, while $\mathbf{V}$ is not. (This fact is often forgotten by students.)
3. $v(t)$ is always real with no complex term, while $\mathbf{V}$ is generally complex.

Finally, we should bear in mind that phasor analysis applies only when frequency is constant; it applies in manipulating two or more sinusoidal signals only if they are of the same frequency.

Example

Evaluate these complex numbers:

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

Solution:

(a) Using polar to rectangular transformation,

$$40\angle 50^\circ = 40(\cos 50^\circ + j\sin 50^\circ) = 25.71 + j30.64$$

$$20\angle -30^\circ = 20[\cos(-30^\circ) + j\sin(-30^\circ)] = 17.32 - j10$$

Adding them up gives

$$40\angle 50^\circ + 20\angle -30^\circ = 43.03 + j20.64 = 47.72\angle 25.63^\circ$$

Taking the square root of this,

$$(40\angle 50^\circ + 20\angle -30^\circ)^{1/2} = 6.91\angle 12.81^\circ$$

(b) Using polar-rectangular transformation, addition, multiplication, and division,

$$\frac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*} = \frac{8.66 - j5 + (3 - j4)}{(2 + j4)(3 + j5)}$$

$$= \frac{11.66 - j9}{-14 + j22} = \frac{14.73\angle -37.66^\circ}{26.08\angle 122.47^\circ}$$

$$= 0.565\angle -160.13^\circ$$

Practice Problem

Evaluate the following complex numbers:

(a) $[(5 + j2)(-1 + j4) - 5\angle 60^\circ]^*$

(b) $\dfrac{10 + j5 + 3\angle 40^\circ}{-3 + j4} + 10\angle 30^\circ + j5$

Answer: (a) $-15.5 - j13.67$, (b) $8.293 + j7.2$.

Example

Transform these sinusoids to phasors:

(a) $i = 6\cos(50t - 40^\circ)$ A
(b) $v = -4\sin(30t + 50^\circ)$ V

Solution:

(a) $i = 6\cos(50t - 40^\circ)$ has the phasor

$$\mathbf{I} = 6\angle{-40^\circ} \text{ A}$$

(b) Since $-\sin A = \cos(A + 90^\circ)$,

$$v = -4\sin(30t + 50^\circ) = 4\cos(30t + 50^\circ + 90^\circ)$$
$$= 4\cos(30t + 140^\circ) \text{ V}$$

The phasor form of v is

$$\mathbf{V} = 4\angle 140^\circ \text{ V}$$

Practice Problem

Express these sinusoids as phasors:

(a) $v = 7\cos(2t + 40^\circ)$ V
(b) $i = -4\sin(10t + 10^\circ)$ A

Answer: (a) $\mathbf{V} = 7\angle 40^\circ$ V, (b) $\mathbf{I} = 4\angle 100^\circ$ A.

Example

Find the sinusoids represented by these phasors:

(a) $\mathbf{I} = -3 + j4$ A
(b) $\mathbf{V} = j8e^{-j20^\circ}$ V

Solution:

(a) $\mathbf{I} = -3 + j4 = 5\angle 126.87^\circ$. Transforming this to the time domain gives

$$i(t) = 5\cos(\omega t + 126.87^\circ) \text{ A}$$

(b) Since $j = 1\angle 90^\circ$,

$$\mathbf{V} = j8\angle{-20^\circ} = (1\angle 90^\circ)(8\angle{-20^\circ})$$
$$= 8\angle 90^\circ - 20^\circ = 8\angle 70^\circ \text{ V}$$

Converting this to the time domain gives

$$v(t) = 8\cos(\omega t + 70^\circ) \text{ V}$$

Practice Problem

Find the sinusoids corresponding to these phasors:

(a) $\mathbf{V} = -10\underline{/30^\circ}$ V
(b) $\mathbf{I} = j(5 - j12)$ A

Answer: (a) $v(t) = 10\cos(\omega t + 210^\circ)$ V or $10\cos(\omega t - 150^\circ)$ V,
(b) $i(t) = 13\cos(\omega t + 22.62^\circ)$ A.

Example

Given $i_1(t) = 4\cos(\omega t + 30^\circ)$ A and $i_2(t) = 5\sin(\omega t - 20^\circ)$ A, find their sum.

Solution:
Here is an important use of phasors—for summing sinusoids of the same frequency. Current $i_1(t)$ is in the standard form. Its phasor is

$$\mathbf{I}_1 = 4\underline{/30^\circ}$$

We need to express $i_2(t)$ in cosine form. The rule for converting sine to cosine is to subtract 90°. Hence,

$$i_2 = 5\cos(\omega t - 20^\circ - 90^\circ) = 5\cos(\omega t - 110^\circ)$$

and its phasor is

$$\mathbf{I}_2 = 5\underline{/-110^\circ}$$

If we let $i = i_1 + i_2$, then

$$\begin{aligned}\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 &= 4\underline{/30^\circ} + 5\underline{/-110^\circ} \\ &= 3.464 + j2 - 1.71 - j4.698 = 1.754 - j2.698 \\ &= 3.218\underline{/-56.97^\circ} \text{ A}\end{aligned}$$

Transforming this to the time domain, we get

$$i(t) = 3.218\cos(\omega t - 56.97^\circ) \text{ A}$$

Of course, we can find $i_1 + i_2$ using Eq. (9.9), but that is the hard way.

Practice Problem

If $v_1 = -10\sin(\omega t - 30^\circ)$ V and $v_2 = 20\cos(\omega t + 45^\circ)$ V, find $v = v_1 + v_2$.

Answer: $v(t) = 29.77\cos(\omega t + 49.98^\circ)$ V.

Example

Using the phasor approach, determine the current $i(t)$ in a circuit described by the integrodifferential equation

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75^\circ)$$

Solution:
We transform each term in the equation from time domain to phasor domain. Keeping Eqs. (9.27) and (9.28) in mind, we obtain the phasor form of the given equation as

$$4\mathbf{I} + \frac{8\mathbf{I}}{j\omega} - 3j\omega\mathbf{I} = 50\angle 75^\circ$$

But $\omega = 2$, so

$$\mathbf{I}(4 - j4 - j6) = 50\angle 75^\circ$$

$$\mathbf{I} = \frac{50\angle 75^\circ}{4 - j10} = \frac{50\angle 75^\circ}{10.77\angle -68.2^\circ} = 4.642\angle 143.2^\circ \text{ A}$$

Converting this to the time domain,

$$i(t) = 4.642\cos(2t + 143.2^\circ) \text{ A}$$

Keep in mind that this is only the steady-state solution, and it does not require knowing the initial values.

Practice Problem

Find the voltage $v(t)$ in a circuit described by the integrodifferential equation

$$2\frac{dv}{dt} + 5v + 10\int v\,dt = 50\cos(5t - 30^\circ)$$

using the phasor approach.

Answer: $v(t) = 5.3\cos(5t - 88^\circ)$ V.

9.4 Phasor Relationships for Circuit Elements

We begin with the resistor. If the current through a resistor R is $i = I_m \cos(\omega t + \phi)$, the voltage across it is given by Ohm's law as

$$v = iR = RI_m \cos(\omega t + \phi) \tag{9.29}$$

The phasor form of this voltage is

$$\mathbf{V} = RI_m \underline{/\phi} \tag{9.30}$$

But the phasor representation of the current is $\mathbf{I} = I_m \underline{/\phi}$. Hence,

$$\mathbf{V} = R\mathbf{I} \tag{9.31}$$

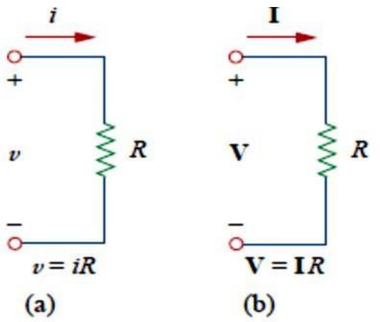


Figure 9.9
Voltage-current relations for a resistor in the: (a) time domain, (b) frequency domain.

showing that the voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain. Figure 9.9 illustrates the voltage-current relations of a resistor. We should note from Eq. (9.31) that voltage and current are in phase, as illustrated in the phasor diagram in Fig. 9.10.

For the inductor L, assume the current through it is $i = I_m \cos(\omega t + \phi)$. The voltage across the inductor is

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi) \qquad \textbf{(9.32)}$$

Recall from Eq. (9.10) that $-\sin A = \cos(A + 90°)$. We can write the voltage as

$$v = \omega L I_m \cos(\omega t + \phi + 90°) \qquad \textbf{(9.33)}$$

which transforms to the phasor

$$\mathbf{V} = \omega L I_m e^{j(\phi + 90°)} = \omega L I_m e^{j\phi} e^{j90°} = \omega L I_m \underline{/\phi + 90°} \qquad \textbf{(9.34)}$$

But $I_m \underline{/\phi} = \mathbf{I}$, and from Eq. (9.19), $e^{j90°} = j$. Thus,

$$\mathbf{V} = j\omega L\mathbf{I} \qquad \textbf{(9.35)}$$

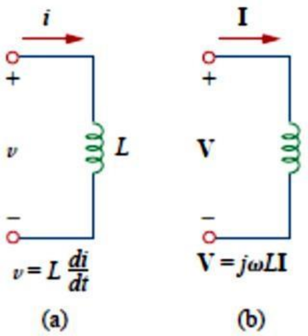


Figure 9.11
Voltage-current relations for an inductor in the: (a) time domain, (b) frequency domain.

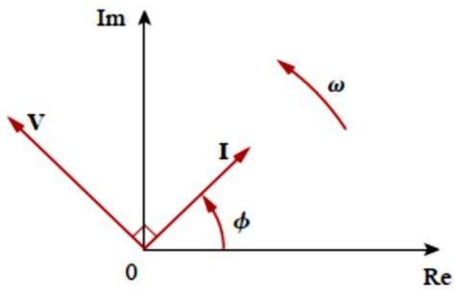


Figure 9.12
Phasor diagram for the inductor; **I** lags **V**.

showing that the voltage has a magnitude of $\omega L I_m$ and a phase of $\phi + 90°$. The voltage and current are 90° out of phase. Specifically, the current lags the voltage by 90°. Figure 9.11 shows the voltage-current relations for the inductor. Figure 9.12 shows the phasor diagram.

For the capacitor C, assume the voltage across it is $v = V_m \cos(\omega t + \phi)$. The current through the capacitor is

$$i = C\frac{dv}{dt} \qquad \textbf{(9.36)}$$

By following the same steps as we took for the inductor or by applying Eq. (9.27) on Eq. (9.36), we obtain

$$\mathbf{I} = j\omega C\mathbf{V} \quad \Rightarrow \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \qquad \textbf{(9.37)}$$

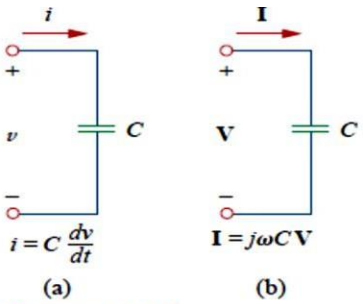


Figure 9.13
Voltage-current relations for a capacitor in the: (a) time domain, (b) frequency domain.

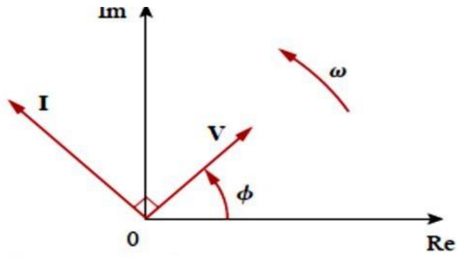


Figure 9.14
Phasor diagram for the capacitor; **I** leads **V**.

showing that the current and voltage are out of phase. To be specific, the current leads the voltage by Figure 9.13 shows the voltage current relations for the capacitor; Fig. 9.14 gives the phasor diagram. Table 9.2 summarizes the time domain and phasor domain representations of the circuit elements

TABLE 9.2

Summary of voltage-current relationships.

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\frac{dv}{dt}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$

Example

The voltage $v = 12\cos(60t + 45°)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

Solution:
For the inductor, $\mathbf{V} = j\omega L\mathbf{I}$, where $\omega = 60$ rad/s and $\mathbf{V} = 12\underline{/45°}$ V. Hence,

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\underline{/45°}}{j60 \times 0.1} = \frac{12\underline{/45°}}{6\underline{/90°}} = 2\underline{/-45°} \text{ A}$$

Converting this to the time domain,

$$i(t) = 2\cos(60t - 45°) \text{ A}$$

Practice Problem

If voltage $v = 10\cos(100t + 30°)$ is applied to a 50 μF capacitor, calculate the current through the capacitor.

Answer: $50\cos(100t + 120°)$ mA.

9.5 Impedance and Admittance

In the preceding section, we obtained the voltage-current relations for the three passive elements as

$$\mathbf{V} = R\mathbf{I}, \qquad \mathbf{V} = j\omega L\mathbf{I}, \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \tag{9.38}$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor current as

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C} \tag{9.39}$$

From these three expressions, we obtain Ohm's law in phasor form for any type of element as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = \mathbf{Z}\mathbf{I} \tag{9.40}$$

where **Z** is a frequency-dependent quantity known as *impedance*, measured in ohms.

> The **impedance Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in ohms (Ω).

The impedance represents the opposition that the circuit exhibits to the flow of sinusoidal current. Although the impedance is the ratio of two phasors, it is not a phasor, because it does not correspond to a sinusoidally varying quantity.

The impedances of resistors, inductors, and capacitors can be readily obtained from Eq. (9.39). Table 9.3 summarizes their impedances. From the table we notice that $\mathbf{Z}_L = j\omega L$ and $\mathbf{Z}_C = -j/\omega C$. Consider two extreme cases of angular frequency. When $\omega = 0$ (i.e., for dc sources), $\mathbf{Z}_L = 0$ and $\mathbf{Z}_C \to \infty$, confirming what we already know—that the inductor acts like a short circuit, while the capacitor acts like an open circuit. When $\omega \to \infty$ (i.e., for high frequencies), $\mathbf{Z}_L \to \infty$ and $\mathbf{Z}_C = 0$, indicating that the inductor is an open circuit to high frequencies, while the capacitor is a short circuit. Figure 9.15 illustrates this.

TABLE 9.3

Impedances and admittanc
of passive elements.

Element	Impedance	Admitt
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{\ }{j}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j$

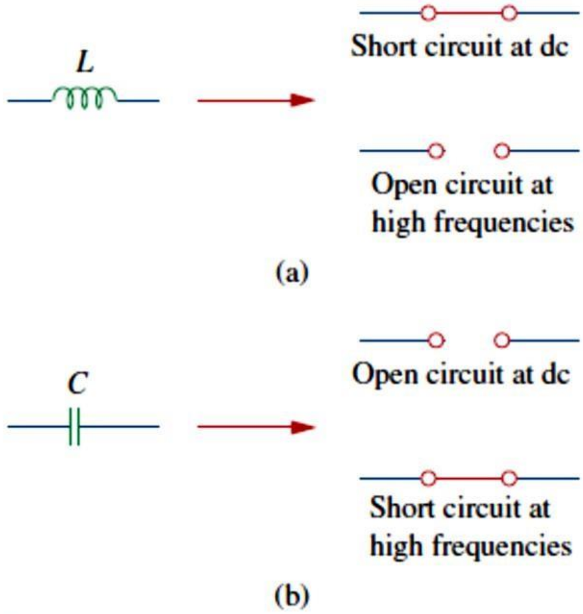


Figure 9.15
Equivalent circuits at dc and high frequencies: (a) inductor, (b) capacitor.

As a complex quantity, the impedence may be expressed in rectangular form as

$$\mathbf{Z} = R + jX \tag{9.41}$$

where $R = \text{Re}\,\mathbf{Z}$ is the *resistance* and $X = \text{Im}\,\mathbf{Z}$ is the *reactance*. The reactance X may be positive or negative. We say that the impedance is inductive when X is positive or capacitive when X is negative. Thus, impedance $\mathbf{Z} = R + jX$ is said to be *inductive* or lagging since current lags voltage, while impedance $\mathbf{Z} = R - jX$ is capacitive or leading because current leads voltage. The impedance, resistance, and reactance are all measured in ohms. The impedance may also be expressed in polar form as

$$\mathbf{Z} = |\mathbf{Z}| \angle \theta \tag{9.42}$$

Comparing Eqs. (9.41) and (9.42), we infer that

$$\mathbf{Z} = R + jX = |\mathbf{Z}|\underline{/\theta} \tag{9.43}$$

where

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{X}{R} \tag{9.44}$$

and

$$R = |\mathbf{Z}|\cos\theta, \qquad X = |\mathbf{Z}|\sin\theta \tag{9.45}$$

It is sometimes convenient to work with the reciprocal of impedance, known as *admittance*.

The **admittance Y** is the reciprocal of impedance, measured in siemens (S).

The admittance **Y** of an element (or a circuit) is the ratio of the phasor current through it to the phasor voltage across it, or

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}} \tag{9.46}$$

The admittances of resistors, inductors, and capacitors can be obtained from Eq. (9.39). They are also summarized in Table 9.3.

As a complex quantity, we may write **Y** as

$$\mathbf{Y} = G + jB \tag{9.47}$$

where $G = \text{Re}\,\mathbf{Y}$ is called the *conductance* and $B = \text{Im}\,\mathbf{Y}$ is called the *susceptance*. Admittance, conductance, and susceptance are all expressed in the unit of siemens (or mhos). From Eqs. (9.41) and (9.47),

$$G + jB = \frac{1}{R + jX} \tag{9.48}$$

By rationalization,

By rationalization,

$$G + jB = \frac{1}{R + jX} \cdot \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2} \qquad (9.49)$$

Equating the real and imaginary parts gives

$$G = \frac{R}{R^2 + X^2}, \qquad B = -\frac{X}{R^2 + X^2} \qquad (9.50)$$

showing that $G \neq 1/R$ as it is in resistive circuits. Of course, if $X = 0$, then $G = 1/R$.

Example

Find $v(t)$ and $i(t)$ in the circuit shown in Fig. 9.16.

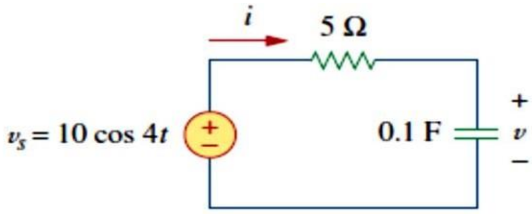


Figure 9.16

Solution:

From the voltage source 10 cos 4t, $\omega = 4$,

$$\mathbf{V}_s = 10\angle 0^\circ \text{ V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\angle 0^\circ}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2} \quad \textbf{(9.9.1)}$$

$$= 1.6 + j0.8 = 1.789\angle 26.57^\circ \text{ A}$$

The voltage across the capacitor is

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\angle 26.57^\circ}{j4 \times 0.1} \quad \textbf{(9.9.2)}$$

$$= \frac{1.789\angle 26.57^\circ}{0.4\angle 90^\circ} = 4.47\angle -63.43^\circ \text{ V}$$

Converting **I** and **V** in Eqs. (9.9.1) and (9.9.2) to the time domain, we get

$$i(t) = 1.789 \cos(4t + 26.57^\circ) \text{ A}$$

$$v(t) = 4.47 \cos(4t - 63.43^\circ) \text{ V}$$

Notice that $i(t)$ leads $v(t)$ by 90° as expected.

Practice Problem

Refer to Fig. 9.17. Determine $v(t)$ and $i(t)$.

Answer: 8.944 sin(10t + 93.43°) V, 4.472 sin(10t + 3.43°) A.

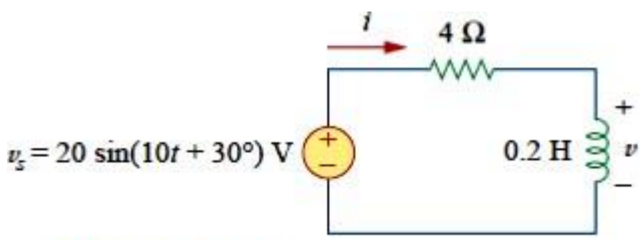


Figure 9.17

9.6 †Kirchhoff's Laws in the Frequency Domain

We cannot do circuit analysis in the frequency domain without Kirchhoff's current and voltage laws. Therefore, we need to express them in the frequency domain.

For KVL, let $v_1, v_2, \ldots, v_n$ be the voltages around a closed loop. Then

$$v_1 + v_2 + \cdots + v_n = 0 \tag{9.51}$$

In the sinusoidal steady state, each voltage may be written in cosine form, so that Eq. (9.51) becomes

$$V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) + \cdots + V_{mn}\cos(\omega t + \theta_n) = 0 \tag{9.52}$$

This can be written as

$$\text{Re}(V_{m1}e^{j\theta_1}e^{j\omega t}) + \text{Re}(V_{m2}e^{j\theta_2}e^{j\omega t}) + \cdots + \text{Re}(V_{mn}e^{j\theta_n}e^{j\omega t}) = 0$$

or

$$\text{Re}[(V_{m1}e^{j\theta_1} + V_{m2}e^{j\theta_2} + \cdots + V_{mn}e^{j\theta_n})e^{j\omega t}] = 0 \tag{9.53}$$

If we let $\mathbf{V}_k = V_{mk}e^{j\theta_k}$, then

$$\text{Re}[(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n)e^{j\omega t}] = 0 \tag{9.54}$$

Since $e^{j\omega t} \neq 0$,

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0 \tag{9.55}$$

indicating that Kirchhoff's voltage law holds for phasors.

By following a similar procedure, we can show that Kirchhoff's current law holds for phasors. If we let $i_1, i_2, \ldots, i_n$ be the current leaving or entering a closed surface in a network at time t, then

$$i_1 + i_2 + \cdots + i_n = 0 \tag{9.56}$$

If $\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n$ are the phasor forms of the sinusoids $i_1, i_2, \ldots, i_n$, then

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0 \tag{9.57}$$

which is Kirchhoff's current law in the frequency domain.

Once we have shown that both KVL and KCL hold in the frequency domain, it is easy to do many things, such as impedance combination, nodal and mesh analyses, superposition, and source transformation.

9.7 Impedance Combinations

Consider the N series-connected impedances shown in Fig. 9.18. The same current **I** flows through the impedances. Applying KVL around the loop gives

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N) \qquad (9.58)$$

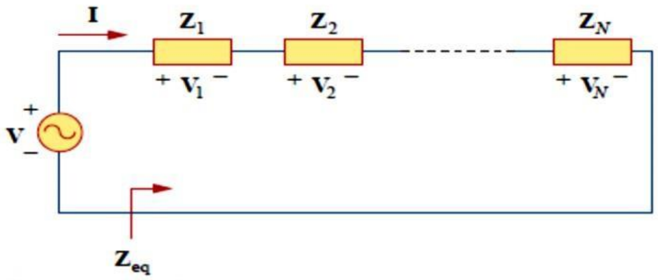


Figure 9.18
N impedances in series.

The equivalent impedance at the input terminals is

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$

or

$$\boxed{\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N} \qquad (9.59)$$

showing that the total or equivalent impedance of series-connected impedances is the sum of the individual impedances. This is similar to the series connection of resistances.

If $N = 2$, as shown in Fig. 9.19, the current through the impedances is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2} \qquad (9.60)$$

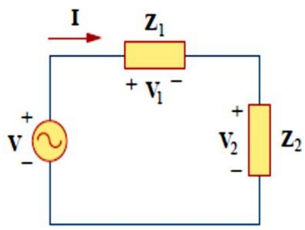


Figure 9.19
Voltage division.

Since $\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}$, then

$$\boxed{\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}} \qquad (9.61)$$

which is the *voltage-division* relationship.

we can obtain the equivalent impedance or admittance of the N parallel-connected impedances shown in Fig. 9.20. The voltage across each impedance is the same. Applying KCL at the top node

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right) \qquad (9.62)$$

,

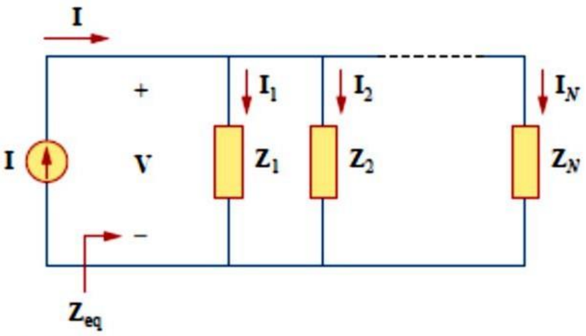


Figure 9.20
N impedances in parallel.

The equivalent impedance is

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N} \qquad (9.63)$$

and the equivalent admittance is

$$\boxed{\mathbf{Y}_{eq} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_N} \qquad (9.64)$$

This indicates that the equivalent admittance of a parallel connection of admittances is the sum of the individual admittances.

When $N = 2$, as shown in Fig. 9.21, the equivalent impedance becomes

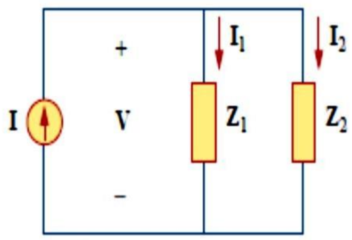


Figure 9.21
Current division.

$$\mathbf{Z}_{eq} = \frac{1}{\mathbf{Y}_{eq}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{1/\mathbf{Z}_1 + 1/\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} \qquad (9.65)$$

Also, since

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = \mathbf{I}_1\mathbf{Z}_1 = \mathbf{I}_2\mathbf{Z}_2$$

the currents in the impedances are

$$\boxed{\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}, \qquad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}} \qquad (9.66)$$

which is the *current-division* principle.

The delta-to-wye and wye-to-delta transformations that we applied to resistive circuits are also valid for impedances. With reference to Fig. 9.22, the conversion formulas are as follows.

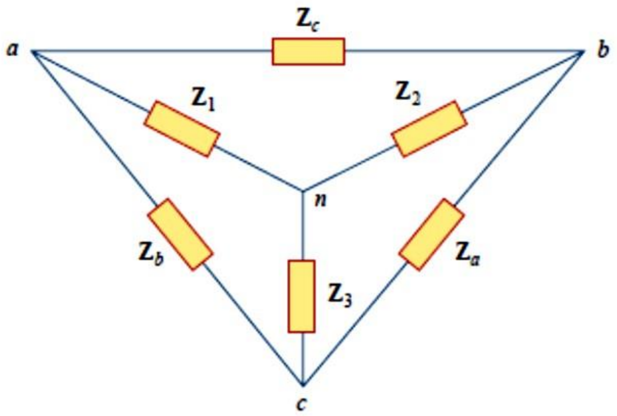


Figure 9.22
Superimposed *Y* and Δ networks.

Y-Δ Conversion:

$$\mathbf{Z}_a = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_1}$$
$$\mathbf{Z}_b = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2} \qquad (9.67)$$
$$\mathbf{Z}_c = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3}$$

Δ-Y Conversion:

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$
$$\mathbf{Z}_2 = \frac{\mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c} \qquad (9.68)$$
$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

A delta or wye circuit is said to be **balanced** if it has equal impedances in all three branches.

When a Δ-Y circuit is balanced, Eqs. (9.67) and (9.68) become

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \quad \text{or} \quad \mathbf{Z}_Y = \frac{1}{3}\mathbf{Z}_\Delta \tag{9.69}$$

where $\mathbf{Z}_Y = \mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3$ and $\mathbf{Z}_\Delta = \mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c$.

As you see in this section, the principles of voltage division, current division, circuit reduction, impedance equivalence, and Y-Δ transformation all apply to ac circuits. Chapter 10 will show that other circuit techniques—such as superposition, nodal analysis, mesh analysis, source transformation, the Thevenin theorem, and the Norton theorem—are all applied to ac circuits in a manner similar to their application in dc circuits.

Example

Find the input impedance of the circuit in Fig. 9.23. Assume that the circuit operates at ω = 50 rad/s.

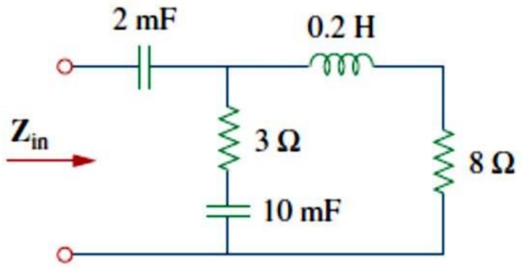


Figure 9.23

Solution:

Let

$\mathbf{Z}_1$ = Impedance of the 2-mF capacitor

$\mathbf{Z}_2$ = Impedance of the 3-Ω resistor in series with the10-mF capacitor

$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

F
F

Then

$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

The input impedance is

$$\mathbf{Z}_{in} = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07\ \Omega$$

Practice Problem

Determine the input impedance of the circuit in Fig. 9.24 at ω = 10 rad/ s.

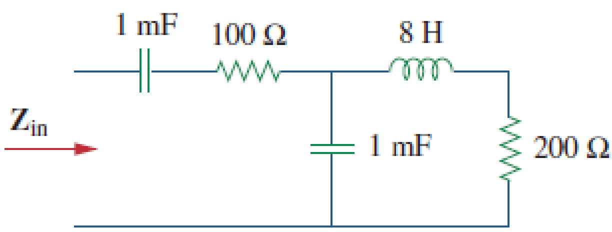


Figure 9.24

Answer: $(149.52 - j195)$

Example

Determine ***vo*** **(*t*)** in the circuit of Fig. 9.25.

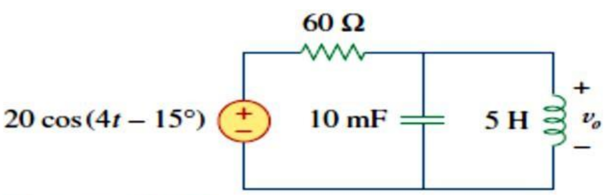


Figure 9.25

Solution:

To do the analysis in the frequency domain, we must first transform the time domain circuit in Fig. 9.25 to the phasor domain equivalent in Fig. 9.26. The transformation produces

$$v_s = 20\cos(4t - 15^\circ) \quad \Rightarrow \quad \mathbf{V}_s = 20\angle -15^\circ \text{ V}, \qquad \omega = 4$$

$$10 \text{ mF} \quad \Rightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}} = -j25\ \Omega$$

$$5 \text{ H} \quad \Rightarrow \quad j\omega L = j4 \times 5 = j20\ \Omega$$

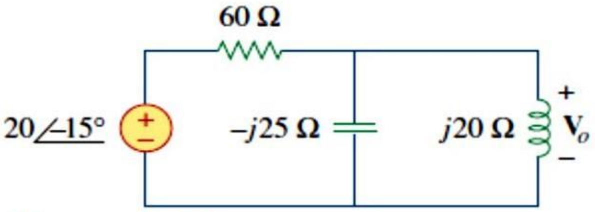


Figure 9.26
The frequency domain equivalent of the circuit in Fig. 9.25.

Let

$\mathbf{Z}_1$ = Impedance of the 60-Ω resistor

$\mathbf{Z}_2$ = Impedance of the parallel combination of the 10-mF capacitor and the 5-H inductor

Then $\mathbf{Z}_1 = 60\ \Omega$ and

$$\mathbf{Z}_2 = -j25 \parallel j20 = \frac{-j25 \times j20}{-j25 + j20} = j100\ \Omega$$

By the voltage-division principle,

$$\mathbf{V}_o = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \frac{j100}{60 + j100}(20\angle -15^\circ)$$
$$= (0.8575\angle 30.96^\circ)(20\angle -15^\circ) = 17.15\angle 15.96^\circ \text{ V}$$

We convert this to the time domain and obtain

$$v_o(t) = 17.15\cos(4t + 15.96^\circ) \text{ V}$$

Practice Problem

Calculate Vo in the circuit of Fig. 9.27

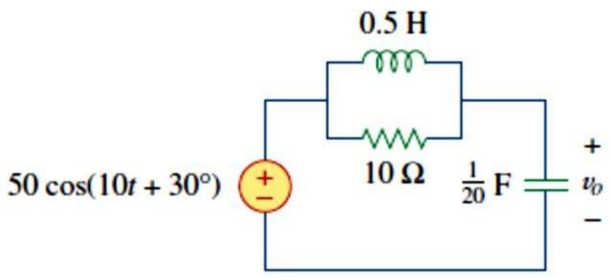


Figure 9.27

Answer: $v_o(t) = 35.36\cos(10t - 105°)$ V.

Example

Find current **I** in the circuit of Fig. 9.28.

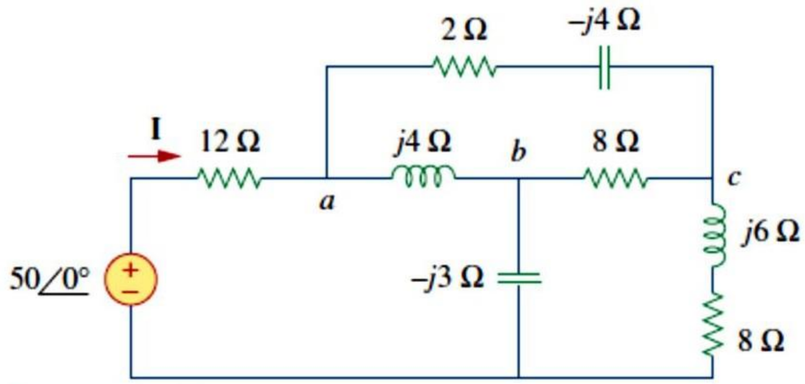


Figure 9.28

Solution:
The delta network connected to nodes *a*, *b*, and *c* can be converted to the *Y* network of Fig. 9.29. We obtain the *Y* impedances as follows using Eq. (9.68):

$$\mathbf{Z}_{an} = \frac{j4(2 - j4)}{j4 + 2 - j4 + 8} = \frac{4(4 + j2)}{10} = (1.6 + j0.8)\ \Omega$$

$$\mathbf{Z}_{bn} = \frac{j4(8)}{10} = j3.2\ \Omega, \qquad \mathbf{Z}_{cn} = \frac{8(2 - j4)}{10} = (1.6 - j3.2)\ \Omega$$

The total impedance at the source terminals is

$$\begin{aligned}\mathbf{Z} &= 12 + \mathbf{Z}_{an} + (\mathbf{Z}_{bn} - j3) \parallel (\mathbf{Z}_{cn} + j6 + 8) \\ &= 12 + 1.6 + j0.8 + (j0.2) \parallel (9.6 + j2.8) \\ &= 13.6 + j0.8 + \frac{j0.2(9.6 + j2.8)}{9.6 + j3} \\ &= 13.6 + j1 = 13.64\angle 4.204^\circ\ \Omega\end{aligned}$$

The desired current is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{50\angle 0^\circ}{13.64\angle 4.204^\circ} = 3.666\angle -4.204^\circ\ \text{A}$$

—

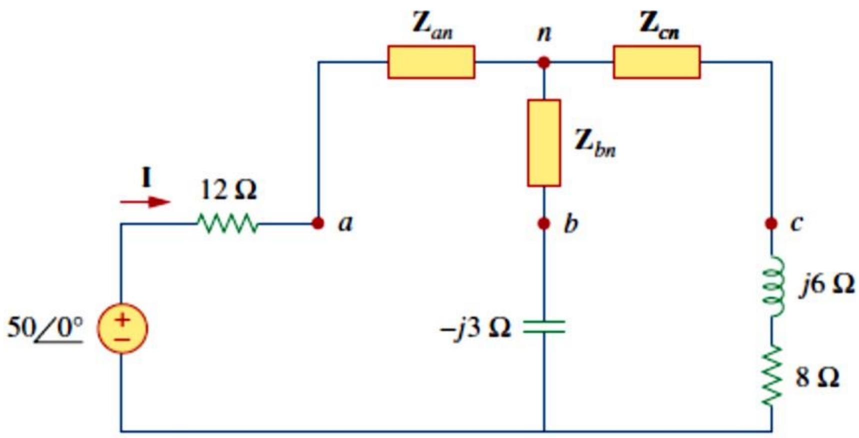


Figure 9.29
The circuit in Fig. 9.28 after delta-to-wye transformation.

Practice Problem

Find **I** in the circuit of Fig. 9.30

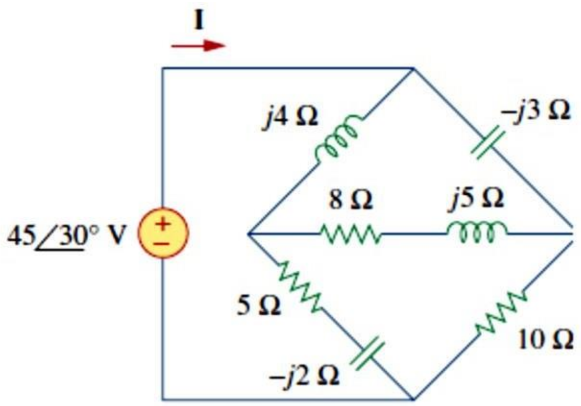


Figure 9.30

Answer: $6.364\angle 3.8^\circ$ A.

Summary

1. A sinusoid is a signal in the form of the sine or cosine function. It has the general form

$$v(t) = V_m \cos(\omega t + \phi)$$

where V_m is the amplitude, $\omega = 2\pi f$ is the angular frequency, $(\omega t + \phi)$ is the argument, and ϕ is the phase.

2. A phasor is a complex quantity that represents both the magnitude and the phase of a sinusoid. Given the sinusoid $v(t) = V_m \cos(\omega t + \phi)$, its phasor **V** is

$$\mathbf{V} = V_m \underline{/\phi}$$

3. In ac circuits, voltage and current phasors always have a fixed relation to one another at any moment of time. If $v(t) = V_m \cos(\omega t + \phi_v)$ represents the voltage through an element and $i(t) = I_m \cos(\omega t + \phi_i)$ represents the current through the element, then $\phi_i = \phi_v$ if the element is a resistor, ϕ_i leads ϕ_v by 90° if the element is a capacitor, and ϕ_i lags ϕ_v by 90° if the element is an inductor.

4. The impedance **Z** of a circuit is the ratio of the phasor voltage across it to the phasor current through it:

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = R(\omega) + jX(\omega)$$

The admittance **Y** is the reciprocal of impedance:

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = G(\omega) + jB(\omega)$$

Impedances are combined in series or in parallel the same way as resistances in series or parallel; that is, impedances in series add while admittances in parallel add.

5. For a resistor $\mathbf{Z} = R$, for an inductor $\mathbf{Z} = jX = j\omega L$, and for a capacitor $\mathbf{Z} = -jX = 1/j\omega C$.

6. Basic circuit laws (Ohm's and Kirchhoff's) apply to ac circuits in the same manner as they do for dc circuits; that is,

$$\mathbf{V} = \mathbf{ZI}$$
$$\Sigma\, \mathbf{I}_k = 0 \quad \text{(KCL)}$$
$$\Sigma\, \mathbf{V}_k = 0 \quad \text{(KVL)}$$

Chapter 10

Sinusoidal Steady State Analysis

10.1 Introduction

In Chapter 9, we learned that the forced or steady-state response of circuits to sinusoidal inputs can be obtained by using phasors. We also know that Ohm's and Kirchhoff's laws are applicable to ac circuits. In this chapter, we want to see how nodal analysis, mesh analysis, Thevenin's theorem, Norton's theorem, superposition, and source transformations are applied in analyzing ac circuits. Since these techniques were already introduced for dc circuits, our major effort here will be to illustrate with examples. Analyzing ac circuits usually requires three steps

Steps to Analyze AC Circuits:

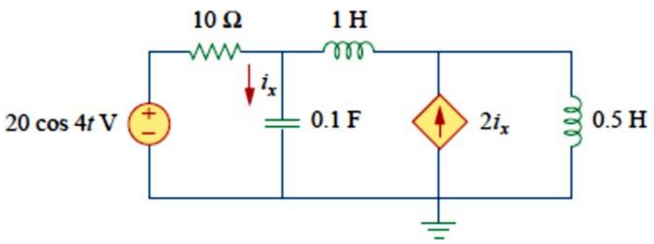
1. Transform the circuit to the phasor or frequency domain.
2. Solve the problem using circuit techniques (nodal analysis, mesh analysis, superposition, etc.).
3. Transform the resulting phasor to the time domain.

10.2 Nodal Analysis

The basis of nodal analysis is Kirchhoff's current law. Since KCL is valid for phasors, as demonstrated in Section 9.6, we can analyze ac ircuits by nodal analysis. The following examples illustrate this.

Example

Find ***ix*** in the circuit of Fig. 10.1 using nodal analysis.



Solution:

We first convert the circuit to the frequency domain:

$$20\cos 4t \quad \Rightarrow \quad 20\angle 0^\circ, \quad \omega = 4 \text{ rad/s}$$

$$1 \text{ H} \quad \Rightarrow \quad j\omega L = j4$$

$$0.5 \text{ H} \quad \Rightarrow \quad j\omega L = j2$$

$$0.1 \text{ F} \quad \Rightarrow \quad \frac{1}{j\omega C} = -j2.5$$

Thus, the frequency domain equivalent circuit is as shown in Fig. 10.2.

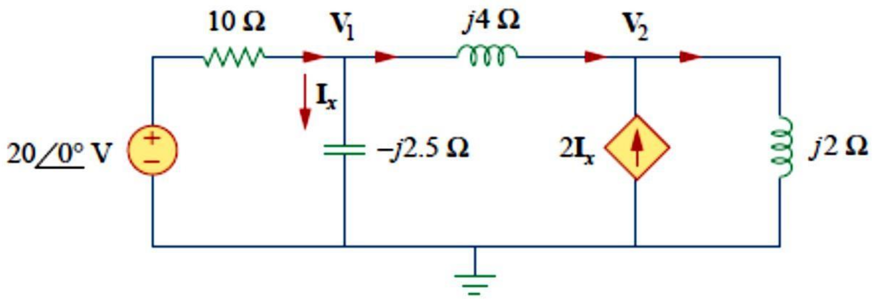


Figure 10.2
Frequency domain equivalent of the circuit in Fig. 10.1.

Applying KCL at node 1,

$$\frac{20 - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4}$$

or

$$(1 + j1.5)\mathbf{V}_1 + j2.5\mathbf{V}_2 = 20 \qquad \textbf{(10.1.1)}$$

At node 2,

$$2\mathbf{I}_x + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

But $\mathbf{I}_x = \mathbf{V}_1/-j2.5$. Substituting this gives

$$\frac{2\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

By simplifying, we get

$$11\mathbf{V}_1 + 15\mathbf{V}_2 = 0 \qquad \textbf{(10.1.2)}$$

Equations (10.1.1) and (10.1.2) can be put in matrix form as

$$\begin{bmatrix} 1 + j1.5 & j2.5 \\ 11 & 15 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

We obtain the determinants as

$$\Delta = \begin{vmatrix} 1 + j1.5 & j2.5 \\ 11 & 15 \end{vmatrix} = 15 - j5$$

$$\Delta_1 = \begin{vmatrix} 20 & j2.5 \\ 0 & 15 \end{vmatrix} = 300, \qquad \Delta_2 = \begin{vmatrix} 1 + j1.5 & 20 \\ 11 & 0 \end{vmatrix} = -220$$

$$\mathbf{V}_1 = \frac{\Delta_1}{\Delta} = \frac{300}{15 - j5} = 18.97\underline{/18.43^\circ} \text{ V}$$

$$\mathbf{V}_2 = \frac{\Delta_2}{\Delta} = \frac{-220}{15 - j5} = 13.91\underline{/198.3^\circ} \text{ V}$$

The current $\mathbf{I}_x$ is given by

$$\mathbf{I}_x = \frac{\mathbf{V}_1}{-j2.5} = \frac{18.97\angle 18.43^\circ}{2.5\angle -90^\circ} = 7.59\angle 108.4^\circ \text{ A}$$

Transforming this to the time domain,

$$i_x = 7.59 \cos(4t + 108.4^\circ) \text{ A}$$

Practice Problem

Using nodal analysis, find v_1 and v_2 in the circuit of Fig. 10.3.

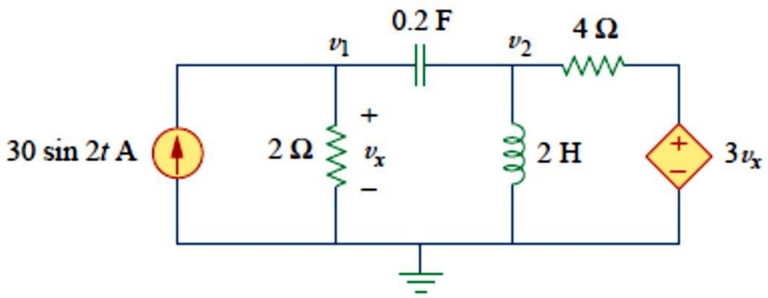


Figure 10.3

Answer: $v_1(t) = 33.96 \sin(2t + 60.01^\circ)$ V,
$v_2(t) = 99.06 \sin(2t + 57.12^\circ)$ V.

Example

Compute $\mathbf{V}_1$ and $\mathbf{V}_2$ in the circuit of Fig. 10.4.

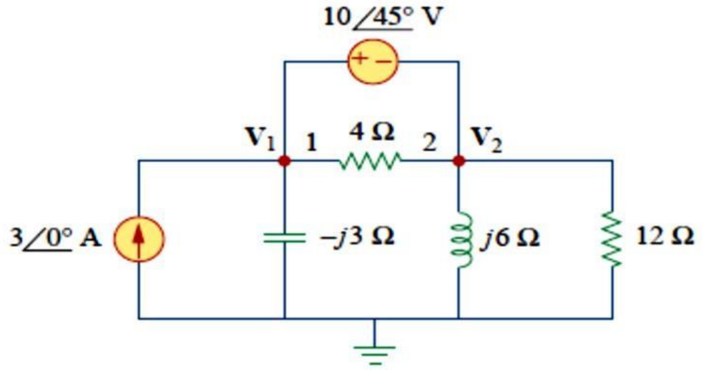


Figure 10.4

Solution:

Nodes 1 and 2 form a supernode as shown in Fig. 10.5. Applying KCL at the supernode gives

$$3 = \frac{\mathbf{V}_1}{-j3} + \frac{\mathbf{V}_2}{j6} + \frac{\mathbf{V}_2}{12}$$

or

$$36 = j4\mathbf{V}_1 + (1 - j2)\mathbf{V}_2 \qquad \textbf{(10.2.1)}$$

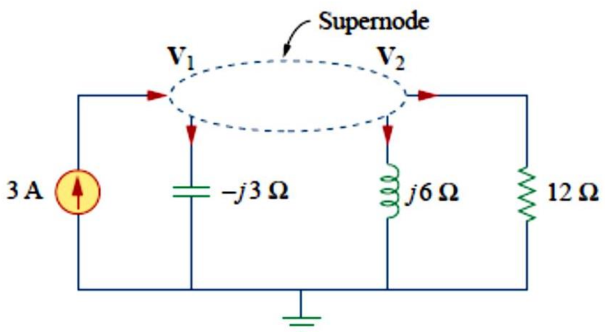


Figure 10.5
A supernode in the circuit of Fig. 10.4.

But a voltage source is connected between nodes 1 and 2, so that

$$\mathbf{V}_1 = \mathbf{V}_2 + 10\angle 45^\circ \qquad \textbf{(10.2.2)}$$

Substituting Eq. (10.2.2) in Eq. (10.2.1) results in

$$36 - 40\angle 135^\circ = (1 + j2)\mathbf{V}_2 \quad \Rightarrow \quad \mathbf{V}_2 = 31.41\angle -87.18^\circ \text{ V}$$

From Eq. (10.2.2),

$$\mathbf{V}_1 = \mathbf{V}_2 + 10\angle 45^\circ = 25.78\angle -70.48^\circ \text{ V}$$

Practice Problem

Calculate $\mathbf{V}_1$ and $\mathbf{V}_2$ in the circuit shown in Fig. 10.6.

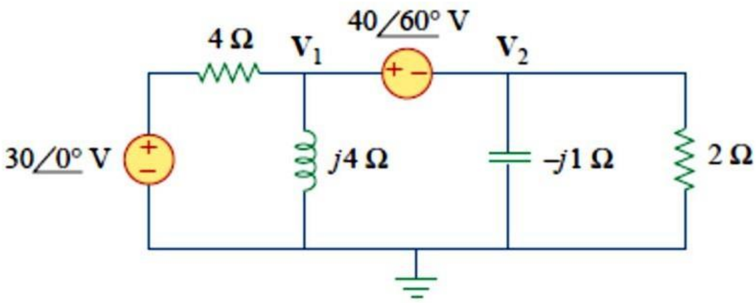


Figure 10.6

Answer: $\mathbf{V}_1 = 38.72\angle 69.67^\circ$ V, $\mathbf{V}_2 = 6.752\angle 165.7^\circ$ V.

10.3 Mesh Analysis

Kirchhoff's voltage law (KVL) forms the basis of mesh analysis. The validity of KVL for ac circuits was shown in Section 9.6 and is illustrated in the following examples. Keep in mind that the very nature of using mesh analysis is that it is to be applied to planar circuits

Example

Determine **I*o*** current in the circuit of Fig. 10.7 using mesh analysis.

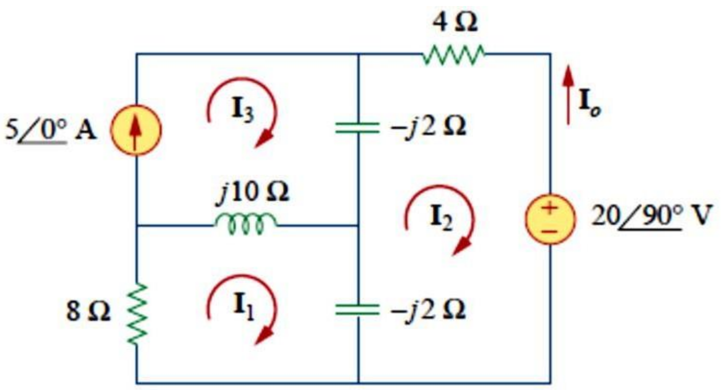


Figure 10.7

Solution:

Applying KVL to mesh 1, we obtain

$$(8 + j10 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - j10\mathbf{I}_3 = 0 \qquad \textbf{(10.3.1)}$$

For mesh 2,

$$(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\angle 90^\circ = 0 \qquad \textbf{(10.3.2)}$$

For mesh 3, $\mathbf{I}_3 = 5$. Substituting this in Eqs. (10.3.1) and (10.3.2), we get

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50 \qquad \textbf{(10.3.3)}$$

$$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j20 - j10 \qquad \textbf{(10.3.4)}$$

Equations (10.3.3) and (10.3.4) can be put in matrix form as

$$\begin{bmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} j50 \\ -j30 \end{bmatrix}$$

from which we obtain the determinants

$$\Delta = \begin{vmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{vmatrix} = 32(1 + j)(1 - j) + 4 = 68$$

$$\Delta_2 = \begin{vmatrix} 8 + j8 & j50 \\ j2 & -j30 \end{vmatrix} = 340 - j240 = 416.17\angle -35.22^\circ$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{416.17\angle -35.22^\circ}{68} = 6.12\angle -35.22^\circ \text{ A}$$

The desired current is

$$\mathbf{I}_o = -\mathbf{I}_2 = 6.12\angle 144.78^\circ \text{ A}$$

Practice Problem

Find **I*o*** in Fig. 10.8 using mesh analysis.

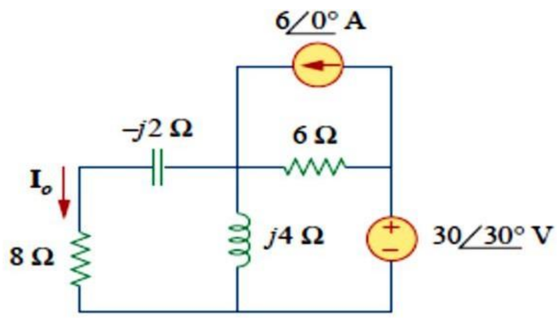


Figure 10.8

Answer: $3.582\angle 65.45^\circ$ A.

Example

Solve for **V*o*** in the circuit of Fig. 10.9 using mesh analysis.

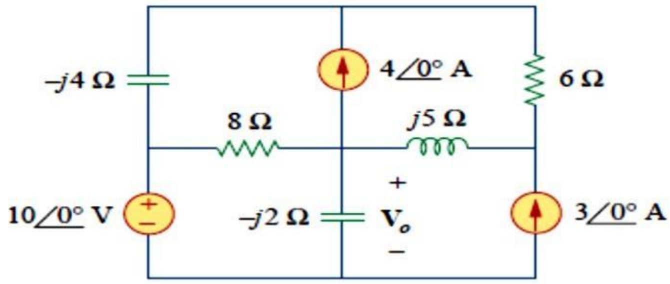


Figure 10.9

Solution:

As shown in Fig. 10.10, meshes 3 and 4 form a supermesh due to the current source between the meshes. For mesh 1, KVL gives

$$-10 + (8 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - 8\mathbf{I}_3 = 0$$

or

$$(8 - j2)\mathbf{I}_1 + j2\mathbf{I}_2 - 8\mathbf{I}_3 = 10 \qquad \textbf{(10.4.1)}$$

For mesh 2,

$$\mathbf{I}_2 = -3 \qquad \textbf{(10.4.2)}$$

For the supermesh,

$$(8 - j4)\mathbf{I}_3 - 8\mathbf{I}_1 + (6 + j5)\mathbf{I}_4 - j5\mathbf{I}_2 = 0 \qquad \textbf{(10.4.3)}$$

Due to the current source between meshes 3 and 4, at node *A*,

$$\mathbf{I}_4 = \mathbf{I}_3 + 4 \qquad \textbf{(10.4.4)}$$

■ **METHOD 1** Instead of solving the above four equations, we reduce them to two by elimination.

Combining Eqs. (10.4.1) and (10.4.2),

$$(8 - j2)\mathbf{I}_1 - 8\mathbf{I}_3 = 10 + j6 \qquad \textbf{(10.4.5)}$$

Combining Eqs. (10.4.2) to (10.4.4),

$$-8\mathbf{I}_1 + (14 + j)\mathbf{I}_3 = -24 - j35 \qquad \textbf{(10.4.6)}$$

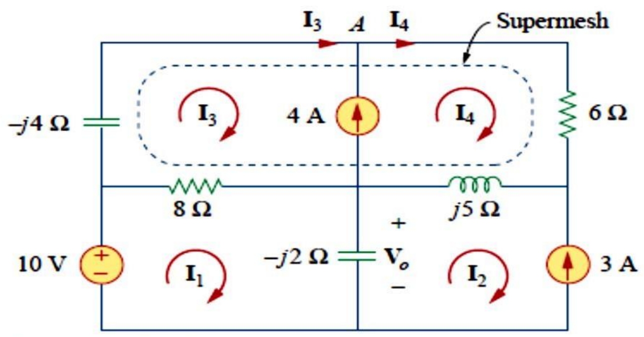


Figure 10.10
Analysis of the circuit in Fig. 10.9.

From Eqs. (10.4.5) and (10.4.6), we obtain the matrix equation

$$\begin{bmatrix} 8 - j2 & -8 \\ -8 & 14 + j \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 10 + j6 \\ -24 - j35 \end{bmatrix}$$

We obtain the following determinants

$$\Delta = \begin{vmatrix} 8 - j2 & -8 \\ -8 & 14 + j \end{vmatrix} = 112 + j8 - j28 + 2 - 64 = 50 - j20$$

$$\Delta_1 = \begin{vmatrix} 10 + j6 & -8 \\ -24 - j35 & 14 + j \end{vmatrix} = 140 + j10 + j84 - 6 - 192 - j280$$

$$= -58 - j186$$

Current $\mathbf{I}_1$ is obtained as

$$\mathbf{I}_1 = \frac{\Delta_1}{\Delta} = \frac{-58 - j186}{50 - j20} = 3.618\underline{/274.5^\circ} \text{ A}$$

The required voltage $\mathbf{V}_0$ is

$$\mathbf{V}_o = -j2(\mathbf{I}_1 - \mathbf{I}_2) = -j2(3.618\underline{/274.5^\circ} + 3)$$

$$= -7.2134 - j6.568 = 9.756\underline{/222.32^\circ} \text{ V}$$

Practice Problem

Calculate $\mathbf{I_o}$ current in the circuit of Fig. 10.11.

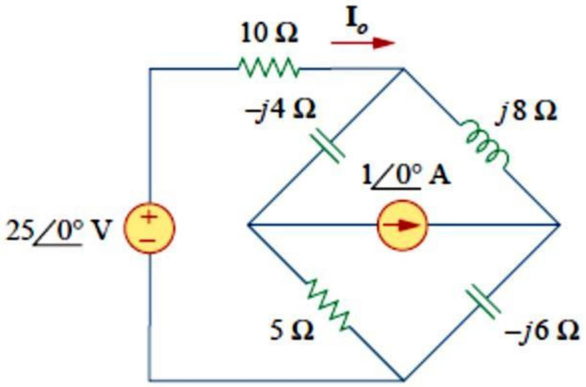


Figure 10.11

Answer: $2.538\angle 5.943°$ A.

10.4 Superposition Theorem

Since ac circuits are linear, the superposition theorem applies to AC circuits the same way it applies to dc circuits. The theorem becomes important if the circuit has sources operating at *different* frequencies. In this case, since the impedances depend on frequency, we must have a different frequency domain circuit for each frequency. The total response must be obtained by adding the individual responses in the *time* domain. It is incorrect to try to add the responses in the phasor or frequency domain. Why? Because the exponential factor $e^{j\omega t}$ is implicit in sinusoidal analysis, and that factor would change for every angular frequency It would therefore not make sense to add responses at different frequencies **ω** in the phasor domain. Thus, when a circuit has sources operating at different frequencies, one must add the responses due to the individual frequencies in the time domain

Example

Use the superposition theorem to find **I*o*** in the circuit in Fig. 10.7.

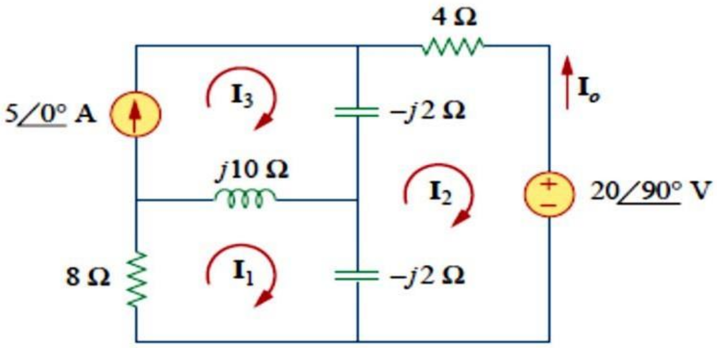


Figure 10.7

Solution:

Let

$$\mathbf{I}_o = \mathbf{I}'_o + \mathbf{I}''_o \qquad (10.5.1)$$

where $\mathbf{I}'_o$ and $\mathbf{I}''_o$ are due to the voltage and current sources, respectively. To find $\mathbf{I}'_o$, consider the circuit in Fig. 10.12(a). If we let **Z** be the parallel combination of $-j2$ and $8 + j10$, then

$$\mathbf{Z} = \frac{-j2(8 + j10)}{-2j + 8 + j10} = 0.25 - j2.25$$

and current $\mathbf{I}'_o$ is

$$\mathbf{I}'_o = \frac{j20}{4 - j2 + \mathbf{Z}} = \frac{j20}{4.25 - j4.25}$$

or

$$\mathbf{I}'_o = -2.353 + j2.353 \qquad (10.5.2)$$

To get $\mathbf{I}''_o$, consider the circuit in Fig. 10.12(b). For mesh 1,

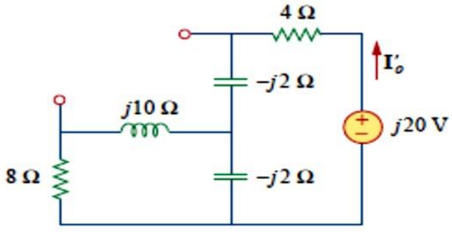
$$(8 + j8)\mathbf{I}_1 - j10\mathbf{I}_3 + j2\mathbf{I}_2 = 0 \qquad (10.5.3)$$

For mesh 2,

$$(4 - j4)\mathbf{I}_2 + j2\mathbf{I}_1 + j2\mathbf{I}_3 = 0 \qquad (10.5.4)$$

For mesh 3,

$$\mathbf{I}_3 = 5 \qquad (10.5.5)$$



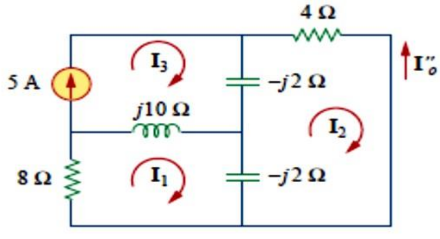


Figure 10.12
Solution of Example 10.5.

From Eqs. (10.5.4) and (10.5.5),

$$(4 - j4)\mathbf{I}_2 + j2\mathbf{I}_1 + j10 = 0$$

Expressing $\mathbf{I}_1$ in terms of $\mathbf{I}_2$ gives

$$\mathbf{I}_1 = (2 + j2)\mathbf{I}_2 - 5 \qquad (10.5.6)$$

Substituting Eqs. (10.5.5) and (10.5.6) into Eq. (10.5.3), we get

$$(8 + j8)[(2 + j2)\mathbf{I}_2 - 5] - j50 + j2\mathbf{I}_2 = 0$$

or

$$\mathbf{I}_2 = \frac{90 - j40}{34} = 2.647 - j1.176$$

Current $\mathbf{I}''_o$ is obtained as

$$\mathbf{I}''_o = -\mathbf{I}_2 = -2.647 + j1.176 \qquad (10.5.7)$$

From Eqs. (10.5.2) and (10.5.7), we write

$$\mathbf{I}_o = \mathbf{I}'_o + \mathbf{I}''_o = -5 + j3.529 = 6.12\underline{/144.78^\circ} \text{ A}$$

Practice Problem

Find current **I*o*** in the circuit of Fig. 10.8 using the superposition theorem

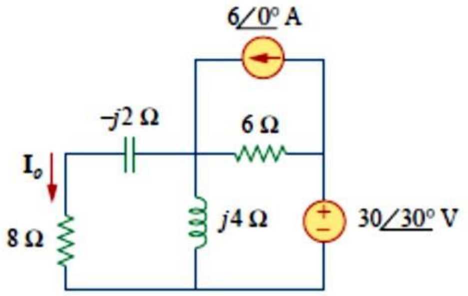


Figure 10.8

Answer: $3.582\angle 65.45°$ A.

Example

Find ***vo*** of the circuit of Fig. 10.13 using the superposition theorem.

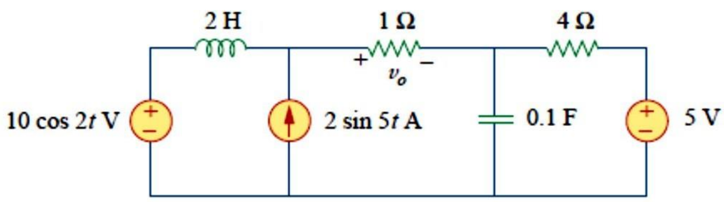


Figure 10.13
For Example 10.6.

Solution:

Since the circuit operates at three different frequencies ($\omega = 0$ for the dc voltage source), one way to obtain a solution is to use superposition, which breaks the problem into single-frequency problems. So we let

$$v_o = v_1 + v_2 + v_3 \tag{10.6.1}$$

where v_1 is due to the 5-V dc voltage source, v_2 is due to the 10 cos 2*t* V voltage source, and v_3 is due to the 2 sin 5*t* A current source.

To find v_1, we set to zero all sources except the 5-V dc source. We recall that at steady state, a capacitor is an open circuit to dc while an inductor is a short circuit to dc. There is an alternative way of looking at this. Since $\omega = 0$, $j\omega L = 0$, $1/j\omega C = \infty$. Either way, the equivalent circuit is as shown in Fig. 10.14(a). By voltage division,

$$-v_1 = \frac{1}{1+4}(5) = 1\text{ V} \qquad \textbf{(10.6.2)}$$

To find v_2, we set to zero both the 5-V source and the 2 sin 5t current source and transform the circuit to the frequency domain.

$$10\cos 2t \quad \Rightarrow \quad 10\angle 0^\circ, \qquad \omega = 2\text{ rad/s}$$
$$2\text{ H} \quad \Rightarrow \quad j\omega L = j4\ \Omega$$
$$0.1\text{ F} \quad \Rightarrow \quad \frac{1}{j\omega C} = -j5\ \Omega$$

The equivalent circuit is now as shown in Fig. 10.14(b). Let

$$\mathbf{Z} = -j5 \parallel 4 = \frac{-j5 \times 4}{4 - j5} = 2.439 - j1.951$$

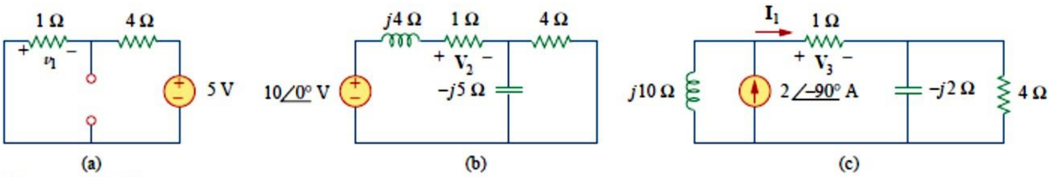


Figure 10.14
Solution of Example 10.6: (a) setting all sources to zero except the 5-V dc source, (b) setting all sources to zero except the ac voltage source, (c) setting all sources to zero except the ac current source.

By voltage division,

$$\mathbf{V}_2 = \frac{1}{1 + j4 + \mathbf{Z}}(10\angle 0^\circ) = \frac{10}{3.439 + j2.049} = 2.498\angle -30.79^\circ$$

In the time domain,

$$v_2 = 2.498\cos(2t - 30.79^\circ) \qquad \textbf{(10.6.3)}$$

To obtain v_3, we set the voltage sources to zero and transform what is left to the frequency domain.

$$2\sin 5t \quad \Rightarrow \quad 2\angle -90^\circ, \qquad \omega = 5\text{ rad/s}$$
$$2\text{ H} \quad \Rightarrow \quad j\omega L = j10\ \Omega$$
$$0.1\text{ F} \quad \Rightarrow \quad \frac{1}{j\omega C} = -j2\ \Omega$$

The equivalent circuit is in Fig. 10.14(c). Let

$$\mathbf{Z}_1 = -j2 \parallel 4 = \frac{-j2 \times 4}{4 - j2} = 0.8 - j1.6\ \Omega$$

By current division,

$$\mathbf{I}_1 = \frac{j10}{j10 + 1 + \mathbf{Z}_1}(2\angle -90^\circ)\ \text{A}$$

$$\mathbf{V}_3 = \mathbf{I}_1 \times 1 = \frac{j10}{1.8 + j8.4}(-j2) = 2.328\angle -80^\circ\ \text{V}$$

In the time domain,

$$v_3 = 2.33\cos(5t - 80^\circ) = 2.33\sin(5t + 10^\circ)\ \text{V} \qquad \textbf{(10.6.4)}$$

Substituting Eqs. (10.6.2) to (10.6.4) into Eq. (10.6.1), we have

$$v_o(t) = -1 + 2.498\cos(2t - 30.79^\circ) + 2.33\sin(5t + 10^\circ)\ \text{V}$$

Practice Problem

Calculate v_o in the circuit of Fig. 10.15 using the superposition theorem.

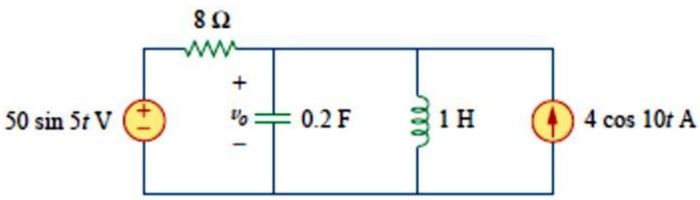


Figure 10.15

Answer: $7.718\sin(5t - 81.12^\circ) + 2.102\cos(10t - 86.24^\circ)$ V.

10.5 Source Transformation

As Fig. 10.16 shows, source transformation in the frequency domain involves transforming a voltage source in series with an impedance to a current source in parallel with an impedance, or vice versa. As we go from one source type to another, we must keep the following relationship in mind:

$$\mathbf{V}_s = \mathbf{Z}_s\mathbf{I}_s \quad \Leftrightarrow \quad \mathbf{I}_s = \frac{\mathbf{V}_s}{\mathbf{Z}_s} \tag{10.1}$$

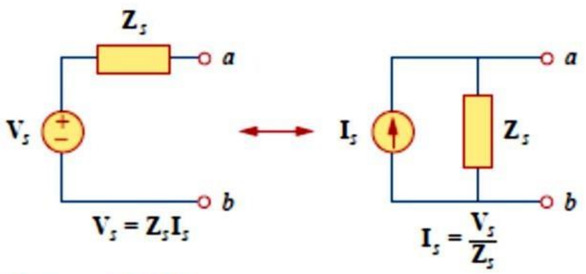


Figure 10.16

Example

Calculate in the circuit of Fig. 10.17 using the method of source transformation

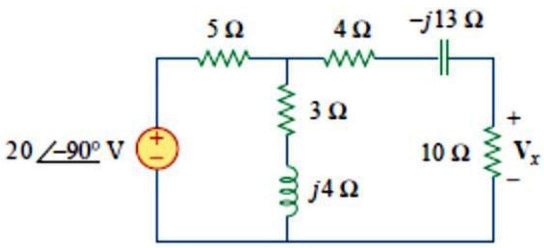


Figure 10.17

Solution:

We transform the voltage source to a current source and obtain the circuit in Fig. 10.18(a), where

$$\mathbf{I}_s = \frac{20\angle -90^\circ}{5} = 4\angle -90^\circ = -j4 \text{ A}$$

The parallel combination of 5-Ω resistance and (3 + j4) impedance gives

$$\mathbf{Z}_1 = \frac{5(3 + j4)}{8 + j4} = 2.5 + j1.25\ \Omega$$

Converting the current source to a voltage source yields the circuit in Fig. 10.18(b), where

$$\mathbf{V}_s = \mathbf{I}_s\mathbf{Z}_1 = -j4(2.5 + j1.25) = 5 - j10 \text{ V}$$

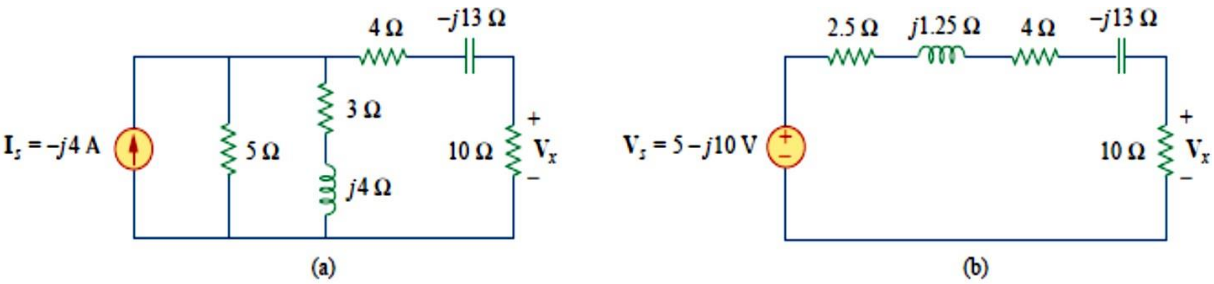


Figure 10.18
Solution of the circuit in Fig. 10.17.

By voltage division,

$$\mathbf{V}_x = \frac{10}{10 + 2.5 + j1.25 + 4 - j13}(5 - j10) = 5.519\angle -28^\circ \text{ V}$$

Practice Problem

Find $\mathbf{I}_o$ in the circuit of Fig. 10.19 using the concept of source transformation.

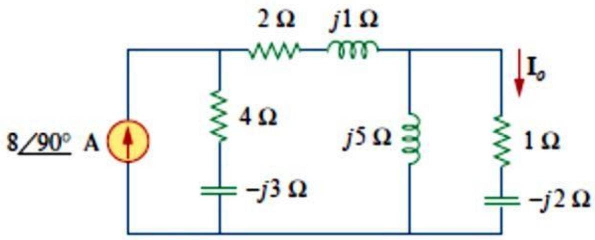


Figure 10.19

Answer: $6.576\angle 99.46^\circ$ A.

10.3 Thevenin and Norton Equivalent Circuits

Thevenin's and Norton's theorems are applied to ac circuits in the same way as they are to dc circuits. The only additional effort arises from the need to manipulate complex numbers. The frequency domain version of a Thevenin equivalent circuit is depicted in Fig. 10.20, where a linear circuit is replaced by a voltage source in series with an impedance. The Norton equivalent circuit is illustrated in Fig. 10.21, where a linear circuit is replaced by a current source in parallel with an impedance. Keep in mind that the two equivalent circuits are related as

$$\mathbf{V}_{Th} = \mathbf{Z}_N \mathbf{I}_N, \qquad \mathbf{Z}_{Th} = \mathbf{Z}_N \tag{10.2}$$

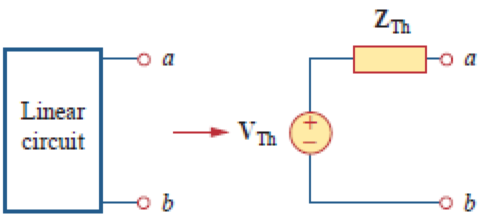


Figure 10.20
Thevenin equivalent.

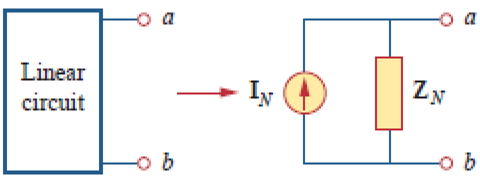


Figure 10.21
Norton equivalent.

just as in source transformation. is the open-circuit voltage while is the short-circuit current.
If the circuit has sources operating at different frequencies (see Example 10.6, for example), the Thevenin or Norton equivalent circuit must be determined at each frequency. This leads to entirely different equivalent circuits, one for each frequency, not one equivalent circuit with equivalent sources and equivalent impedances.

Example

Obtain the Thevenin equivalent at terminals ***a-b*** of the circuit in Fig. 10.22.

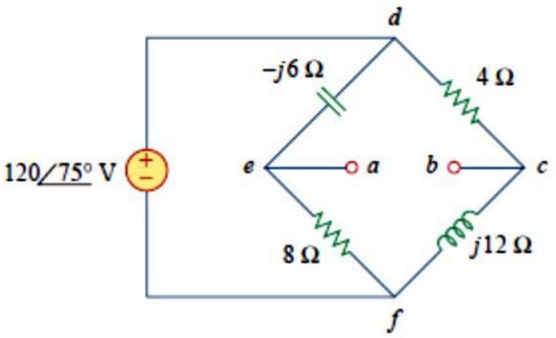


Figure 10.22

Solution:

We find $\mathbf{Z}_{Th}$ by setting the voltage source to zero. As shown in Fig. 10.23(a), the 8-Ω resistance is now in parallel with the $-j6$ reactance, so that their combination gives

$$\mathbf{Z}_1 = -j6 \parallel 8 = \frac{-j6 \times 8}{8 - j6} = 2.88 - j3.84\ \Omega$$

Similarly, the 4-Ω resistance is in parallel with the $j12$ reactance, and their combination gives

$$\mathbf{Z}_2 = 4 \parallel j12 = \frac{j12 \times 4}{4 + j12} = 3.6 + j1.2\ \Omega$$

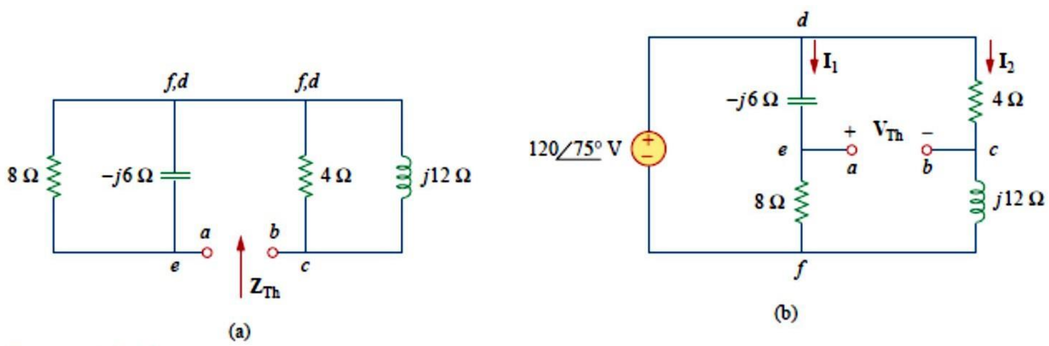


Figure 10.23
Solution of the circuit in Fig. 10.22: (a) finding $\mathbf{Z}_{Th}$, (b) finding $\mathbf{V}_{Th}$.

The Thevenin impedance is the series combination of $\mathbf{Z}_1$ and $\mathbf{Z}_2$; that is,

$$\mathbf{Z}_{Th} = \mathbf{Z}_1 + \mathbf{Z}_2 = 6.48 - j2.64\ \Omega$$

To find $\mathbf{V}_{Th}$, consider the circuit in Fig. 10.23(b). Currents $\mathbf{I}_1$ and $\mathbf{I}_2$ are obtained as

$$\mathbf{I}_1 = \frac{120\angle 75^\circ}{8 - j6}\ \text{A}, \qquad \mathbf{I}_2 = \frac{120\angle 75^\circ}{4 + j12}\ \text{A}$$

Applying KVL around loop *bcdeab* in Fig. 10.23(b) gives

$$\mathbf{V}_{Th} - 4\mathbf{I}_2 + (-j6)\mathbf{I}_1 = 0$$

or

$$\mathbf{V}_{Th} = 4\mathbf{I}_2 + j6\mathbf{I}_1 = \frac{480\angle 75^\circ}{4 + j12} + \frac{720\angle 75^\circ + 90^\circ}{8 - j6}$$
$$= 37.95\angle 3.43^\circ + 72\angle 201.87^\circ$$
$$= -28.936 - j24.55 = 37.95\angle 220.31^\circ\ \text{V}$$

Practice Problem

Find the Thevenin equivalent at terminals *a*-*b* of the circuit in Fig. 10.24.

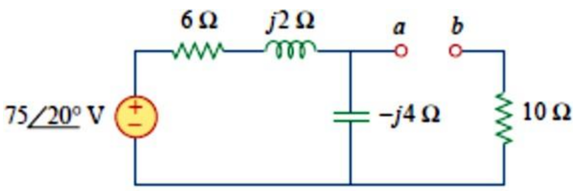


Figure 10.24

Answer: $\mathbf{Z}_{Th} = 12.4 - j3.2\ \Omega$, $\mathbf{V}_{Th} = 47.42\angle -51.57^\circ\ \text{V}$

Example

Find the Thevenin equivalent of the circuit in Fig. 10.25 as seen from terminals ***a* - *b*.**

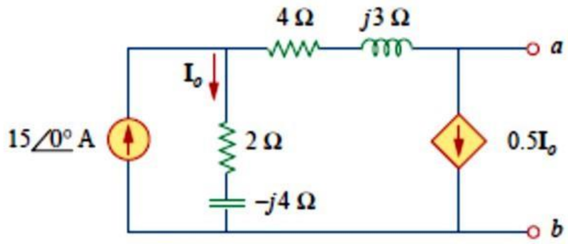


Figure 10.25

Solution:

To find $\mathbf{V}_{Th}$, we apply KCL at node 1 in Fig. 10.26(a).

$$15 = \mathbf{I}_o + 0.5\mathbf{I}_o \quad \Rightarrow \quad \mathbf{I}_o = 10 \text{ A}$$

Applying KVL to the loop on the right-hand side in Fig. 10.26(a), we obtain

$$-\mathbf{I}_o(2 - j4) + 0.5\mathbf{I}_o(4 + j3) + \mathbf{V}_{Th} = 0$$

or

$$\mathbf{V}_{Th} = 10(2 - j4) - 5(4 + j3) = -j55$$

Thus, the Thevenin voltage is

$$\mathbf{V}_{Th} = 55\angle{-90^\circ} \text{ V}$$

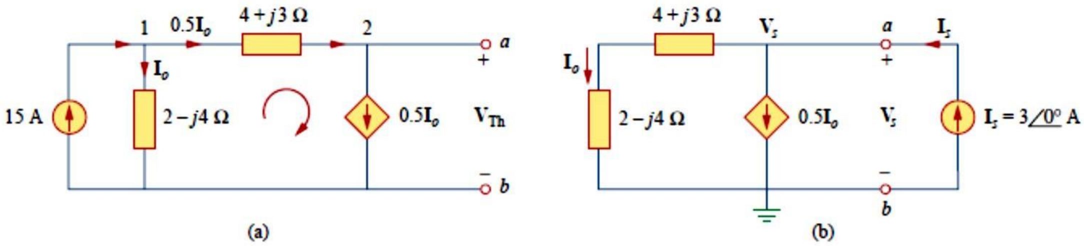


Figure 10.26
Solution of the problem in Fig. 10.25: (a) finding $\mathbf{V}_{Th}$, (b) finding $\mathbf{Z}_{Th}$.

To obtain $\mathbf{Z}_{Th}$, we remove the independent source. Due to the presence of the dependent current source, we connect a 3-A current source (3 is an arbitrary value chosen for convenience here, a number divisible by the sum of currents leaving the node) to terminals *a-b* as shown in Fig. 10.26(b). At the node, KCL gives

$$3 = \mathbf{I}_o + 0.5\mathbf{I}_o \quad \Rightarrow \quad \mathbf{I}_o = 2 \text{ A}$$

Applying KVL to the outer loop in Fig. 10.26(b) gives

$$\mathbf{V}_s = \mathbf{I}_o(4 + j3 + 2 - j4) = 2(6 - j)$$

The Thevenin impedance is

$$\mathbf{Z}_{Th} = \frac{\mathbf{V}_s}{\mathbf{I}_s} = \frac{2(6 - j)}{3} = 4 - j0.6667 \ \Omega$$

Practice Problem

Determine the Thevenin equivalent of the circuit in Fig. 10.27 as seen from the terminals ***a-b*.**

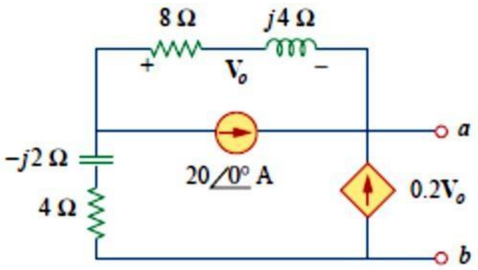


Figure 10.27

Answer: $\mathbf{Z}_{Th} = 4.473\angle{-7.64^\circ}\ \Omega$, $\mathbf{V}_{Th} = 29.4\angle{72.9^\circ}$ V.

Example

Obtain current **I***o* in Fig. 10.28 using Norton's theorem.

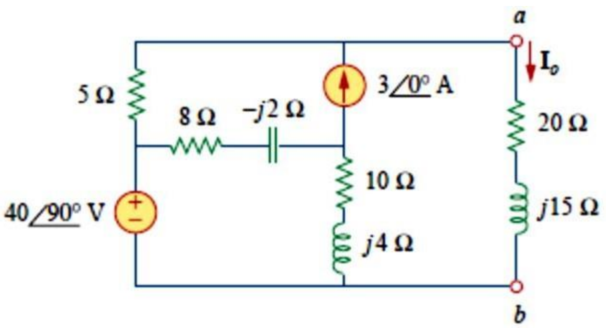


Figure 10.28

Solution:

Our first objective is to find the Norton equivalent at terminals *a-b*. $\mathbf{Z}_N$ is found in the same way as $\mathbf{Z}_{Th}$. We set the sources to zero as shown in Fig. 10.29(a). As evident from the figure, the $(8 - j2)$ and $(10 + j4)$ impedances are short-circuited, so that

$$\mathbf{Z}_N = 5\ \Omega$$

To get $\mathbf{I}_N$, we short-circuit terminals *a-b* as in Fig. 10.29(b) and apply mesh analysis. Notice that meshes 2 and 3 form a supermesh because of the current source linking them. For mesh 1,

$$-j40 + (18 + j2)\mathbf{I}_1 - (8 - j2)\mathbf{I}_2 - (10 + j4)\mathbf{I}_3 = 0 \qquad \textbf{(10.10.1)}$$

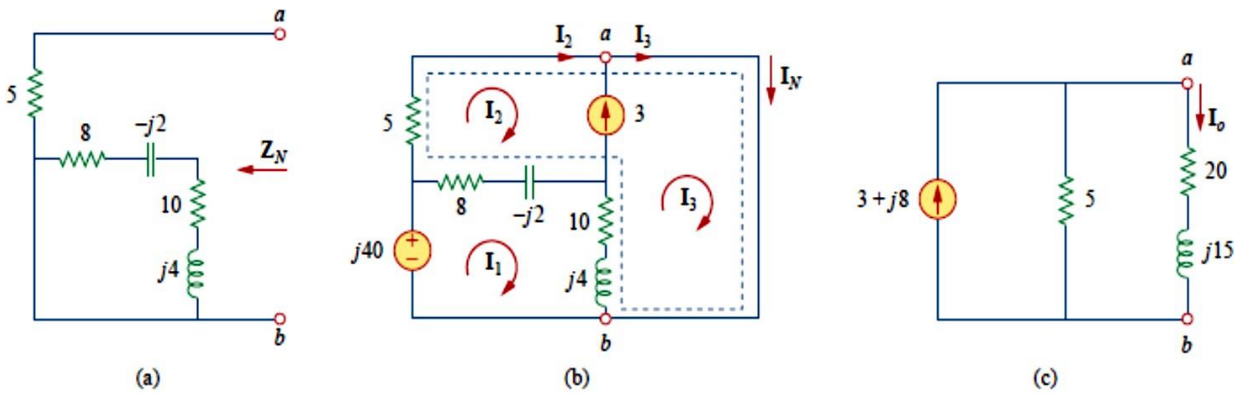


Figure 10.29
Solution of the circuit in Fig. 10.28: (a) finding $\mathbf{Z}_N$, (b) finding $\mathbf{V}_N$, (c) calculating $\mathbf{I}_o$.

For the supermesh,

$$(13 - j2)\mathbf{I}_2 + (10 + j4)\mathbf{I}_3 - (18 + j2)\mathbf{I}_1 = 0 \qquad \textbf{(10.10.2)}$$

At node *a*, due to the current source between meshes 2 and 3,

$$\mathbf{I}_3 = \mathbf{I}_2 + 3 \qquad \textbf{(10.10.3)}$$

Adding Eqs. (10.10.1) and (10.10.2) gives

$$-j40 + 5\mathbf{I}_2 = 0 \qquad \Rightarrow \qquad \mathbf{I}_2 = j8$$

From Eq. (10.10.3),

$$\mathbf{I}_3 = \mathbf{I}_2 + 3 = 3 + j8$$

The Norton current is

$$\mathbf{I}_N = \mathbf{I}_3 = (3 + j8)\text{ A}$$

Figure 10.29(c) shows the Norton equivalent circuit along with the impedance at terminals *a-b*. By current division,

$$\mathbf{I}_o = \frac{5}{5 + 20 + j15}\mathbf{I}_N = \frac{3 + j8}{5 + j3} = 1.465\underline{/38.48^\circ}\text{ A}$$

Practice Problem

Determine the Norton equivalent of the circuit in Fig. 10.30 as seen from terminals *a*-*b*. Use the equivalent to find $\mathbf{I}_o$.

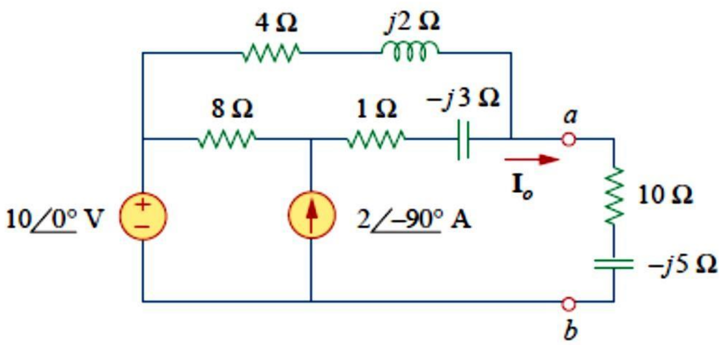


Figure 10.30

Answer: $\mathbf{Z}_N = 3.176 + j0.706\ \Omega$, $\mathbf{I}_N = 4.198\underline{/-32.68^\circ}$ A, $\mathbf{I}_o = 985.5\underline{/-2.101^\circ}$ mA.

Summary

1. We apply nodal and mesh analysis to ac circuits by applying KCL and KVL to the phasor form of the circuits.
2. In solving for the steady-state response of a circuit that has independent
sources with different frequencies, each independent source must be considered separately. The most natural approach to analyzing such circuits is to apply the superposition theorem. A separate phasor circuit for each frequency must be solved independently, and the corresponding response should be obtained in the time domain. The overall response is the sum of the time domain responses of all the individual phasor circuits.
3. The concept of source transformation is also applicable in the frequency domain.
4. The Thevenin equivalent of an ac circuit consists of a voltage source in series with the Thevenin impedance
5. The Norton equivalent of an ac circuit consists of a current source in parallel with the Norton impedance
6. PSpice is a simple and powerful tool for solving ac circuit problems. It relieves us of the tedious task of working with the complex numbers involved in steady-state analysis.
7. The capacitance multiplier and the ac oscillator provide two typical applications for the concepts presented in this chapter. A capacitance multiplier is an op amp circuit used in producing a multiple of a physical capacitance. An oscillator is a device that uses a dc input to generate an ac output.

Chapter 11

AC Power Analysis

AC Power Analysis

11.1 Introduction

Power is the most important quantity in electric utilities, electronic, and communication systems, because such systems involve transmission of power from one point to another. Also, every industrial and household electrical device—every fan, motor, lamp, pressing iron, TV, personal computer has a power rating that indicates how much power the equipment requires; exceeding the power rating can do permanent damage to an appliance. The most common form of electric power is 50- or 60-Hz ac power. The choice of ac over dc allowed high-voltage power transmission from the power generating plant to the consumer. We will begin by defining and deriving *instantaneous power* and *average power*. We will then introduce other power concepts. As practical applications of these concepts, we will discuss how power is measured and reconsider how electric utility companies charge their customers.

11.2 Instantaneous and Average Power

the *instantaneous power* $p(t)$ absorbed by an element is the product of the instantaneous voltage $v(t)$ across the element and the instantaneous current $i(t)$ through it. Assuming the passive sign convention

$$p(t) = v(t)i(t) \tag{11.1}$$

The **instantaneous power** (in watts) is the power at any instant of time.

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It is the rate at which an element absorbs energy. Consider the general case of instantaneous power absorbed by an arbitrary combination of circuit elements under sinusoidal excitation, as shown in Fig. 11.1. Let the voltage and current at the terminals of the circuit be

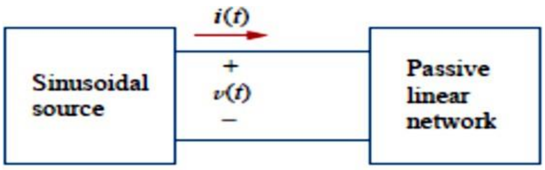


Figure 11.1
Sinusoidal source and passive linear circuit.

$$v(t) = V_m \cos(\omega t + \theta_v) \tag{11.2a}$$

$$i(t) = I_m \cos(\omega t + \theta_i) \tag{11.2b}$$

where V_m and I_m are the amplitudes (or peak values), and θ_v and θ_i are the phase angles of the voltage and current, respectively. The instantaneous power absorbed by the circuit is

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i) \tag{11.3}$$

We apply the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)] \tag{11.4}$$

and express Eq. (11.3) as

$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i) \quad \textbf{(11.5)}$$

This shows us that the instantaneous power has two parts. The first part is constant or time independent. Its value depends on the phase difference between the voltage and the current. The second part is a sinusoidal function whose frequency is 2ω, which is twice the angular frequency of the voltage or current.

A sketch of $p(t)$ in Eq. (11.5) is shown in Fig. 11.2, where $T = 2\pi/\omega$ is the period of voltage or current. We observe that $p(t)$ is periodic, $p(t) = p(t + T_0)$, and has a period of $T_0 = T/2$, since its frequency is twice that of voltage or current. We also observe that $p(t)$ is positive for some part of each cycle and negative for the rest of the cycle. When $p(t)$ is positive, power is absorbed by the circuit. When $p(t)$ is negative, power is absorbed by the source; that is, power is transferred from the circuit to the source. This is possible because of the storage elements (capacitors and inductors) in the circuit.

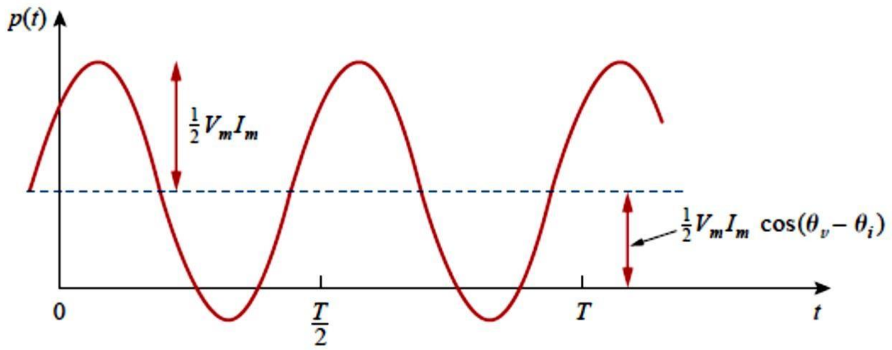


Figure 11.2
The instantaneous power $p(t)$ entering a circuit.

The instantaneous power changes with time and is therefore difficult to measure. The *average* power is more convenient to measure. In fact, the wattmeter, the instrument for measuring power, responds to average power

The **average power**, in watts, is the average of the instantaneous power over one period.

Although Eq. (11.6) shows the averaging done over T, we would get the same result if we performed the integration over the actual period

of $p(t)$ which is $T_0 = T/2$.

Substituting $p(t)$ in Eq. (11.5) into Eq. (11.6) gives

$$\begin{aligned} P &= \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\, dt \\ &\quad + \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)\, dt \\ &= \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\frac{1}{T}\int_0^T dt \\ &\quad + \frac{1}{2}V_m I_m \frac{1}{T}\int_0^T \cos(2\omega t + \theta_v + \theta_i)\, dt \end{aligned} \qquad \textbf{(11.7)}$$

The first integrand is constant, and the average of a constant is the same constant. The second integrand is a sinusoid. We know that the average of a sinusoid over its period is zero because the area under the sinusoid during a positive half-cycle is canceled by the area under it during the following negative half-cycle. Thus, the second term in Eq. (11.7) vanishes and the average power becomes

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) \qquad \textbf{(11.8)}$$

Since $\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$, what is important is the difference in the phases of the voltage and current.

Note that $p(t)$ is time-varying while P does not depend on time. To find the instantaneous power, we must necessarily have $v(t)$ and $i(t)$ in the time domain. But we can find the average power when voltage and current are expressed in the time domain, as in Eq. (11.8), or when they are expressed in the frequency domain. The phasor forms of $v(t)$ and $i(t)$ in Eq. (11.2) are $\mathbf{V} = V_m\underline{/\theta_v}$ and $\mathbf{I} = I_m\underline{/\theta_i}$, respectively. P is calculated using Eq. (11.8) or using phasors $\mathbf{V}$ and $\mathbf{I}$. To use phasors, we notice that

$$\begin{aligned} \frac{1}{2}\mathbf{V}\mathbf{I}^* &= \frac{1}{2}V_m I_m \underline{/\theta_v - \theta_i} \\ &= \frac{1}{2}V_m I_m[\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)] \end{aligned} \qquad \textbf{(11.9)}$$

We recognize the real part of this expression as the average power P according to Eq. (11.8). Thus,

$$P = \frac{1}{2}\text{Re}[\mathbf{VI}^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) \tag{11.10}$$

Consider two special cases of Eq. (11.10). When $\theta_v = \theta_i$, the voltage and current are in phase. This implies a purely resistive circuit or resistive load R, and

$$P = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2 R = \frac{1}{2}|\mathbf{I}|^2 R \tag{11.11}$$

where $|\mathbf{I}|^2 = \mathbf{I} \times \mathbf{I}^*$. Equation (11.11) shows that a purely resistive circuit absorbs power at all times. When $\theta_v - \theta_i = \pm 90°$, we have a purely reactive circuit, and

$$P = \frac{1}{2}V_m I_m \cos 90° = 0 \tag{11.12}$$

showing that a purely reactive circuit absorbs no average power. In summary,

A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power.

Example

Given that

$$v(t) = 120\cos(377t + 45^\circ)\ \text{V} \quad \text{and} \quad i(t) = 10\cos(377t - 10^\circ)\ \text{A}$$

find the instantaneous power and the average power absorbed by the passive linear network of Fig. 11.1.

Solution:

The instantaneous power is given by

$$p = vi = 1200\cos(377t + 45^\circ)\cos(377t - 10^\circ)$$

Applying the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

gives

$$p = 600[\cos(754t + 35^\circ) + \cos 55^\circ]$$

or

$$p(t) = 344.2 + 600\cos(754t + 35^\circ)\ \text{W}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}120(10)\cos[45^\circ - (-10^\circ)]$$

$$= 600\cos 55^\circ = 344.2\ \text{W}$$

which is the constant part of $p(t)$ above.

Practice Problem

Calculate the instantaneous power and average power absorbed by the passive linear network of Fig. 11.1 if

$$v(t) = 165\cos(10t + 20^\circ)\ \text{V} \quad \text{and} \quad i(t) = 20\sin(10t + 60^\circ)\ \text{A}$$

Answer: $1.0606 + 1.65\cos(20t - 10^\circ)$ kW, 1.0606 kW.

Example

Calculate the average power absorbed by an impedance $\mathbf{Z} = 30 - j70\ \Omega$ when a voltage $\mathbf{V} = 120\angle 0^\circ$ is applied across it.

Solution:

The current through the impedance is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{120\angle 0^\circ}{30 - j70} = \frac{120\angle 0^\circ}{76.16\angle -66.8^\circ} = 1.576\angle 66.8^\circ \text{ A}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}(120)(1.576)\cos(0 - 66.8^\circ) = 37.24 \text{ W}$$

Practice Problem

A current $\mathbf{I} = 20\angle 30^\circ$ A flows through an impedance $\mathbf{Z} = 40\angle -22^\circ\ \Omega$. Find the average power delivered to the impedance.

Answer: 3.709 kW.

Example

For the circuit shown in Fig. 11.3, find the average power supplied by the source and the average power absorbed by the resistor.

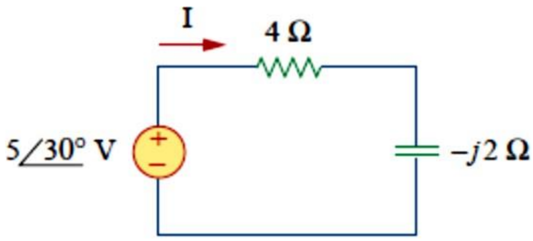


Figure 11.3

Solution:

The current **I** is given by

$$\mathbf{I} = \frac{5\angle 30^\circ}{4 - j2} = \frac{5\angle 30^\circ}{4.472\angle -26.57^\circ} = 1.118\angle 56.57^\circ \text{ A}$$

The average power supplied by the voltage source is

$$P = \frac{1}{2}(5)(1.118)\cos(30^\circ - 56.57^\circ) = 2.5 \text{ W}$$

The current through the resistor is

$$\mathbf{I}_R = \mathbf{I} = 1.118\angle 56.57^\circ \text{ A}$$

and the voltage across it is

$$\mathbf{V}_R = 4\mathbf{I}_R = 4.472\angle 56.57^\circ \text{ V}$$

The average power absorbed by the resistor is

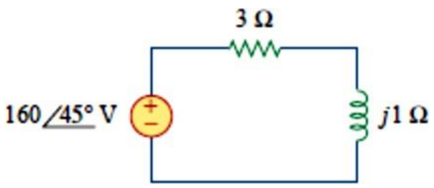
$$P = \frac{1}{2}(4.472)(1.118) = 2.5 \text{ W}$$

which is the same as the average power supplied. Zero average power is absorbed by the capacitor.

Practice Problem

In the circuit of Fig. 11.4, calculate the average power absorbed by the resistor and inductor. Find the average power supplied by the voltage source.

Answer: 3.84 kW, 0 W, 3.84 kW.



Example

Determine the average power generated by each source and the average power absorbed by each passive element in the circuit of Fig. 11.5(a).

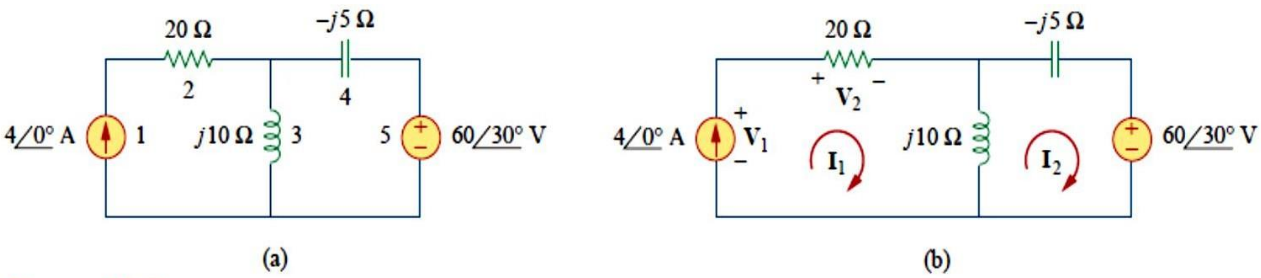


Figure 11.5

Solution:

We apply mesh analysis as shown in Fig. 11.5(b). For mesh 1,

$$\mathbf{I}_1 = 4 \text{ A}$$

For mesh 2,

$$(j10 - j5)\mathbf{I}_2 - j10\mathbf{I}_1 + 60\angle 30^\circ = 0, \qquad \mathbf{I}_1 = 4 \text{ A}$$

or

$$j5\mathbf{I}_2 = -60\angle 30^\circ + j40 \qquad \Rightarrow \qquad \mathbf{I}_2 = -12\angle -60^\circ + 8$$
$$= 10.58\angle 79.1^\circ \text{ A}$$

For the voltage source, the current flowing from it is $\mathbf{I}_2 = 10.58\angle 79.1^\circ$ A and the voltage across it is $60\angle 30^\circ$ V, so that the average power is

$$P_5 = \frac{1}{2}(60)(10.58)\cos(30^\circ - 79.1^\circ) = 207.8 \text{ W}$$

Following the passive sign convention (see Fig. 1.8), this average power is absorbed by the source, in view of the direction of $\mathbf{I}_2$ and the polarity of the voltage source. That is, the circuit is delivering average power to the voltage source.

For the current source, the current through it is $\mathbf{I}_1 = 4\angle 0^\circ$ and the voltage across it is

$$\mathbf{V}_1 = 20\mathbf{I}_1 + j10(\mathbf{I}_1 - \mathbf{I}_2) = 80 + j10(4 - 2 - j10.39)$$
$$= 183.9 + j20 = 184.984\angle 6.21^\circ \text{ V}$$

The average power supplied by the current source is

$$P_1 = -\frac{1}{2}(184.984)(4)\cos(6.21^\circ - 0) = -367.8 \text{ W}$$

It is negative according to the passive sign convention, meaning that the current source is supplying power to the circuit.

For the resistor, the current through it is $\mathbf{I}_1 = 4\angle 0^\circ$ and the voltage across it is $20\mathbf{I}_1 = 80\angle 0^\circ$, so that the power absorbed by the resistor is

$$P_2 = \frac{1}{2}(80)(4) = 160 \text{ W}$$

For the capacitor, the current through it is $\mathbf{I}_2 = 10.58\angle 79.1^\circ$ and the voltage across it is $-j5\mathbf{I}_2 = (5\angle -90^\circ)(10.58\angle 79.1^\circ) = 52.9\angle 79.1^\circ - 90^\circ$. The average power absorbed by the capacitor is

$$P_4 = \frac{1}{2}(52.9)(10.58)\cos(-90^\circ) = 0$$

For the inductor, the current through it is $\mathbf{I}_1 - \mathbf{I}_2 = 2 - j10.39 = 10.58\angle -79.1^\circ$. The voltage across it is $j10(\mathbf{I}_1 - \mathbf{I}_2) = 10.58\angle -79.1^\circ + 90^\circ$. Hence, the average power absorbed by the inductor is

$$P_3 = \frac{1}{2}(105.8)(10.58)\cos 90^\circ = 0$$

Notice that the inductor and the capacitor absorb zero average power and that the total power supplied by the current source equals the power absorbed by the resistor and the voltage source, or

$$P_1 + P_2 + P_3 + P_4 + P_5 = -367.8 + 160 + 0 + 0 + 207.8 = 0$$

indicating that power is conserved.

Practice Problem

Calculate the average power absorbed by each of the five elements in the circuit of Fig. 11.6.

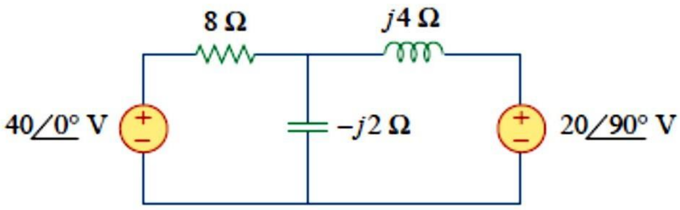


Figure 11.6

Answer: 40-V Voltage source: −60 W; j20-V Voltage source: −40 W; resistor: 100 W; others: 0 W.

11.3 Maximum Average Power Transfer

In Section 4.8 we solved the problem of maximizing the power delivered by a power-supplying resistive network to a load R_L. Representing the circuit by its Thevenin equivalent, we proved that the maximum power would be delivered to the load if the load resistance is equal to the Thevenin resistance $R_L = R_{Th}$. We now extend that result to ac circuits.

Consider the circuit in Fig. 11.7, where an ac circuit is connected to a load $\mathbf{Z}_L$ and is represented by its Thevenin equivalent. The load is usually represented by an impedance, which may model an electric

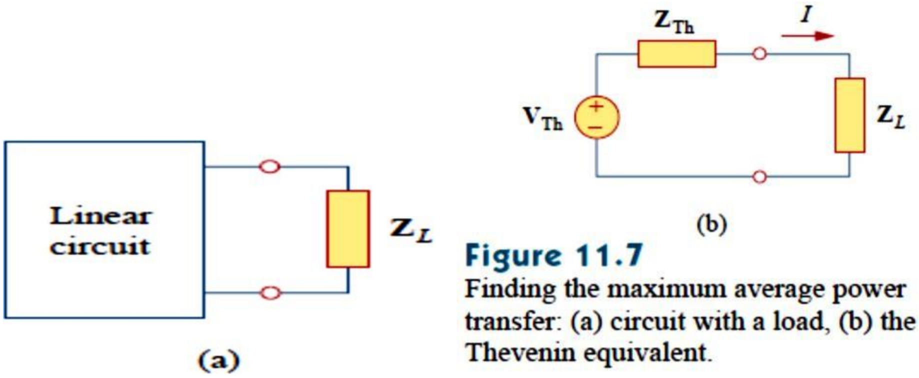


Figure 11.7
Finding the maximum average power transfer: (a) circuit with a load, (b) the Thevenin equivalent.

motor, an antenna, a TV, and so forth. In rectangular form, the Thevenin impedance $\mathbf{Z}_{Th}$ and the load impedance $\mathbf{Z}_L$ are

$$\mathbf{Z}_{Th} = R_{Th} + jX_{Th} \tag{11.13a}$$

$$\mathbf{Z}_L = R_L + jX_L \tag{11.13b}$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{Th}}{\mathbf{Z}_{Th} + \mathbf{Z}_L} = \frac{\mathbf{V}_{Th}}{(R_{Th} + jX_{Th}) + (R_L + jX_L)} \tag{11.14}$$

From Eq. (11.11), the average power delivered to the load is

$$P = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{|\mathbf{V}_{Th}|^2 R_L/2}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2} \tag{11.15}$$

the conclusion that for maximum average power transfer $\mathbf{Z}_L$ must be selected so that $X_L = -X_{Th}$ and $R_L = R_{Th}$, i.e.,

$$\mathbf{Z}_L = R_L + jX_L = R_{Th} - jX_{Th} = \mathbf{Z}_{Th}^* \tag{11.19}$$

For **maximum average power transfer,** the load impedance $\mathbf{Z}_L$ must be equal to the complex conjugate of the Thevenin impedance $\mathbf{Z}_{Th}$.

When $\mathbf{Z}_L = \mathbf{Z}_{Th}^*$, we say that the load is matched to the source.

This result is known as the *maximum average power transfer theorem* for the sinusoidal steady state. Setting $R_L = R_{Th}$ and $X_L = -X_{Th}$ in Eq. (11.15) gives us the maximum average power as

$$P_{max} = \frac{|\mathbf{V}_{Th}|^2}{8R_{Th}} \qquad \textbf{(11.20)}$$

In a situation in which the load is purely real, the condition for maximum power transfer is obtained from Eq. (11.18) by setting $X_L = 0$; that is,

$$R_L = \sqrt{R_{Th}^2 + X_{Th}^2} = |\mathbf{Z}_{Th}| \qquad \textbf{(11.21)}$$

This means that for maximum average power transfer to a purely resistive load, the load impedance (or resistance) is equal to the magnitude of the Thevenin impedance

Example

Determine the load impedance $\mathbf{Z}_L$ that maximizes the average power drawn from the circuit of Fig. 11.8. What is the maximum average power?

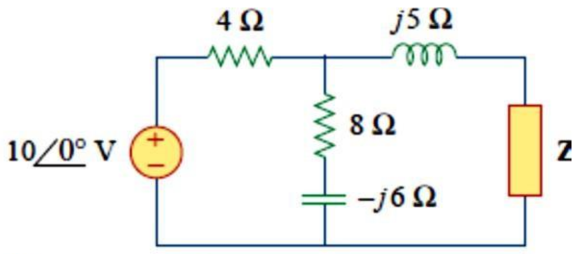


Figure 11.8

Solution:

First we obtain the Thevenin equivalent at the load terminals. To get $\mathbf{Z}_{Th}$, consider the circuit shown in Fig. 11.9(a). We find

$$\mathbf{Z}_{Th} = j5 + 4 \parallel (8 - j6) = j5 + \frac{4(8 - j6)}{4 + 8 - j6} = 2.933 + j4.467\ \Omega$$

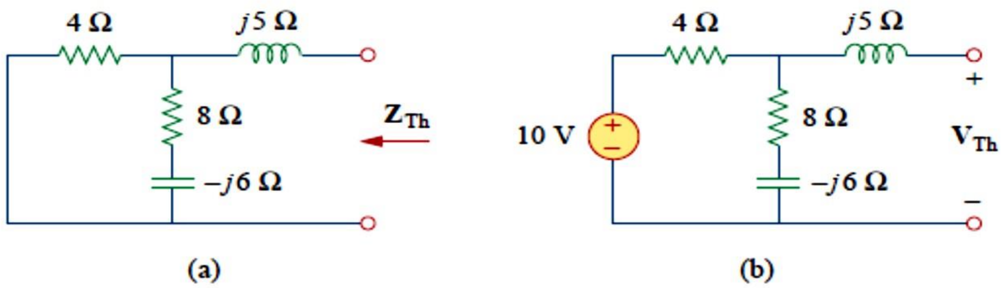


Figure 11.9
Finding the Thevenin equivalent of the circuit in Fig. 11.8.

To find $\mathbf{V}_{Th}$, consider the circuit in Fig. 11.8(b). By voltage division,

$$\mathbf{V}_{Th} = \frac{8 - j6}{4 + 8 - j6}(10) = 7.454\underline{/-10.3^\circ}\ \text{V}$$

The load impedance draws the maximum power from the circuit when

$$\mathbf{Z}_L = \mathbf{Z}_{Th}^* = 2.933 - j4.467\ \Omega$$

According to Eq. (11.20), the maximum average power is

$$P_{max} = \frac{|\mathbf{V}_{Th}|^2}{8R_{Th}} = \frac{(7.454)^2}{8(2.933)} = 2.368\ \text{W}$$

Practice Problem

For the circuit shown in Fig. 11.10, find the load impedance $\mathbf{Z}_L$ that absorbs the maximum average power. Calculate that maximum average power.

Answer: $3.415 - j0.7317\ \Omega$, 12.861 W.

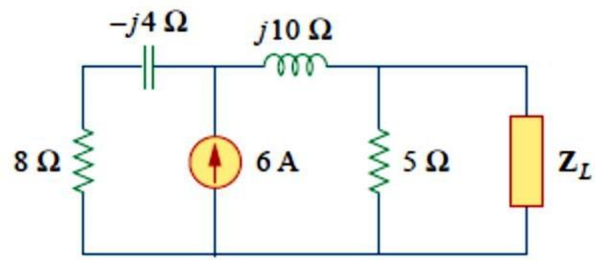


Figure 11.10

Example

In the circuit in Fig. 11.11, find the value of R_L that will absorb the maximum average power. Calculate that power.

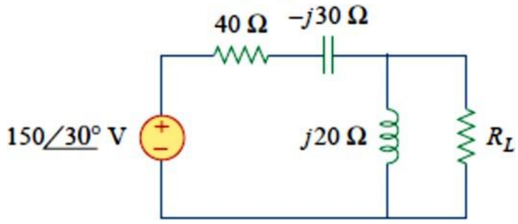


Figure 11.11

Solution:

We first find the Thevenin equivalent at the terminals of R_L.

$$\mathbf{Z}_{Th} = (40 - j30) \parallel j20 = \frac{j20(40 - j30)}{j20 + 40 - j30} = 9.412 + j22.35\ \Omega$$

By voltage division,

$$\mathbf{V}_{Th} = \frac{j20}{j20 + 40 - j30}(150\underline{/30^\circ}) = 72.76\underline{/134^\circ}\text{ V}$$

The value of R_L that will absorb the maximum average power is

$$R_L = |\mathbf{Z}_{Th}| = \sqrt{9.412^2 + 22.35^2} = 24.25\ \Omega$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{Th}}{\mathbf{Z}_{Th} + R_L} = \frac{72.76\underline{/134^\circ}}{33.66 + j22.35} = 1.8\underline{/100.42^\circ}\text{ A}$$

The maximum average power absorbed by R_L is

$$P_{max} = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{1}{2}(1.8)^2(24.25) = 39.29\text{ W}$$

Practice Problem

In Fig. 11.12, the resistor R_L is adjusted until it absorbs the maximum average power. Calculate R_L and the maximum average power absorbed by it.

Answer: 30 Ω, 6.863 W.

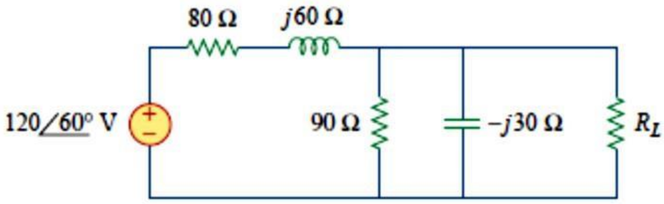


Figure 11.12

11.4 Effective or RMS Value

The idea of *effective value* arises from the need to measure the effectiveness of a voltage or current source in delivering power to a resistive load.

> The **effective value** of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current.

In Fig. 11.13, the circuit in (a) is ac while that of (b) is dc. Our objective is to find I_{eff} that will transfer the same power to resistor R as the sinusoid i. The average power absorbed by the resistor in the ac circuit is

$$P = \frac{1}{T}\int_0^T i^2 R\,dt = \frac{R}{T}\int_0^T i^2\,dt \qquad (11.22)$$

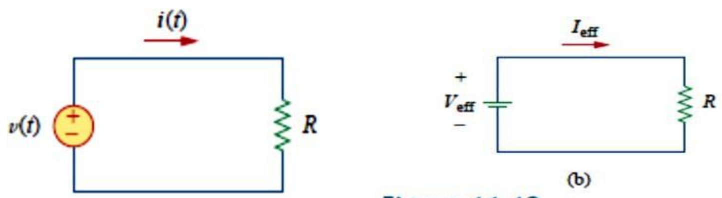


Figure 11.13
Finding the effective current: (a) ac circuit, (b) dc circuit.

while the power absorbed by the resistor in the dc circuit is

$$P = I_{eff}^2 R \qquad (11.23)$$

Equating the expressions in Eqs. (11.22) and (11.23) and solving for I_{eff}, we obtain

$$I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} \qquad (11.24)$$

The effective value of the voltage is found in the same way as current; that is,

$$V_{eff} = \sqrt{\frac{1}{T}\int_0^T v^2\,dt} \qquad (11.25)$$

This indicates that the effective value is the (square) *root* of the *mean* (or average) of the *square* of the periodic signal. Thus, the effective value is often known as the *root-mean-square* value, or *rms* value for short; and we write

$$I_{eff} = I_{rms}, \qquad V_{eff} = V_{rms} \qquad (11.26)$$

For any periodic function $x(t)$ in general, the rms value is given by

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt} \qquad (11.27)$$

The **effective value** of a periodic signal is its root mean square (rms) value.

Equation 11.27 states that to find the rms value of $x(t)$, we first find its *square* x^2 and then find the *mean* of that, or

$$\frac{1}{T}\int_0^T x^2\,dt$$

and then the square *root* ($\sqrt{\quad}$) of that mean. The rms value of a constant is the constant itself. For the sinusoid $i(t) = I_m \cos\omega t$, the effective or rms value is

$$\begin{aligned} I_{\text{rms}} &= \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2 \omega t\,dt} \\ &= \sqrt{\frac{I_m^2}{T}\int_0^T \frac{1}{2}(1 + \cos 2\omega t)\,dt} = \frac{I_m}{\sqrt{2}} \end{aligned} \tag{11.28}$$

Similarly, for $v(t) = V_m \cos\omega t$,

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}} \tag{11.29}$$

Keep in mind that Eqs. (11.28) and (11.29) are only valid for sinusoidal signals.

The average power in Eq. (11.8) can be written in terms of the rms values.

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v - \theta_i)$$
$$= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) \tag{11.30}$$

Similarly, the average power absorbed by a resistor R in Eq. (11.11) can be written as

$$P = I_{\text{rms}}^2 R = \frac{V_{\text{rms}}^2}{R} \tag{11.31}$$

When a sinusoidal voltage or current is specified, it is often in terms of its maximum (or peak) value or its rms value, since its average value is zero. The power industries specify phasor magnitudes in terms of their rms values rather than peak values. For instance, the 110 V available at every household is the rms value of the voltage from the power company. It is convenient in power analysis to express voltage and current in their rms values. Also, analog voltmeters and ammeters are designed to read directly the rms value of voltage and current, respectively.

Example

Determine the rms value of the current waveform in Fig. 11.14. If the current is passed through a 2-Ω resistor, find the average power absorbed by the resistor.

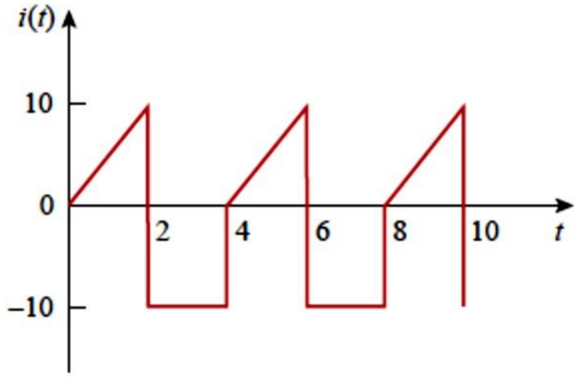


Figure 11.14

Solution:
The period of the waveform is $T = 4$. Over a period, we can write the current waveform as

$$i(t) = \begin{cases} 5t, & 0 < t < 2 \\ -10, & 2 < t < 4 \end{cases}$$

The rms value is

$$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} = \sqrt{\frac{1}{4}\left[\int_0^2 (5t)^2\,dt + \int_2^4 (-10)^2\,dt\right]}$$

$$= \sqrt{\frac{1}{4}\left[25\frac{t^3}{3}\Big|_0^2 + 100t\Big|_2^4\right]} = \sqrt{\frac{1}{4}\left(\frac{200}{3} + 200\right)} = 8.165\text{ A}$$

The power absorbed by a 2-Ω resistor is

$$P = I_{\text{rms}}^2 R = (8.165)^2(2) = 133.3\text{ W}$$

Practice Problem

Find the rms value of the current waveform of Fig. 11.15. If the current flows through a 9-Ω resistor, calculate the average power absorbed by the resistor.

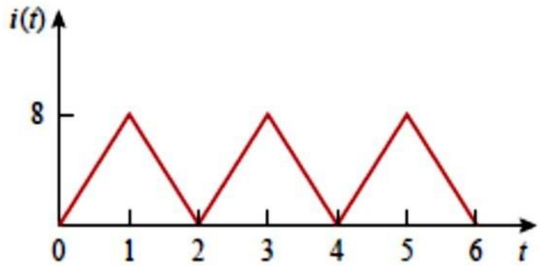


Figure 11.15
For Practice Prob. 11.7.

Answer: 4.318 A, 192 W.

Example

The waveform shown in Fig. 11.16 is a half-wave rectified sine wave. Find the rms value and the amount of average power dissipated in a 10-Ω resistor.

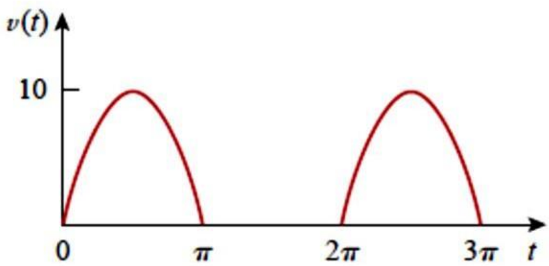


Figure 11.16

Solution:
The period of the voltage waveform is $T = 2\pi$, and

$$v(t) = \begin{cases} 10 \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

The rms value is obtained as

$$V_{rms}^2 = \frac{1}{T}\int_0^T v^2(t)\,dt = \frac{1}{2\pi}\left[\int_0^{\pi}(10\sin t)^2\,dt + \int_{\pi}^{2\pi} 0^2\,dt\right]$$

But $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$. Hence,

$$V_{rms}^2 = \frac{1}{2\pi}\int_0^{\pi}\frac{100}{2}(1 - \cos 2t)\,dt = \frac{50}{2\pi}\left(t - \frac{\sin 2t}{2}\right)\Big|_0^{\pi}$$

$$= \frac{50}{2\pi}\left(\pi - \frac{1}{2}\sin 2\pi - 0\right) = 25, \qquad V_{rms} = 5\text{ V}$$

The average power absorbed is

$$P = \frac{V_{rms}^2}{R} = \frac{5^2}{10} = 2.5\text{ W}$$

Practice Problem

Find the rms value of the full-wave rectified sine wave in Fig. 11.17. Calculate the average power dissipated in a 6-Ω resistor.

Answer: 7.071 V, 8.333 W.

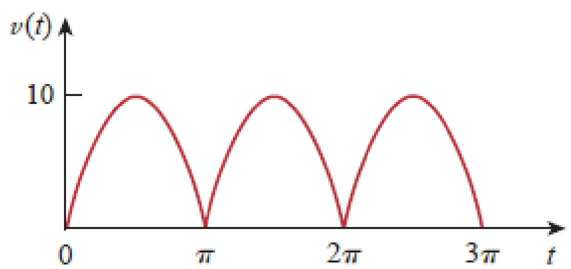


Figure 11.17

11.5 Apparent Power and Power Factor

In Section 11.2 we saw that if the voltage and current at the terminals of a circuit are

$$v(t) = V_m \cos(\omega t + \theta_v) \qquad \text{and} \qquad i(t) = I_m \cos(\omega t + \theta_i) \tag{11.32}$$

or, in phasor form, $\mathbf{V} = V_m \underline{/\theta_v}$ and $\mathbf{I} = I_m \underline{/\theta_i}$, the average power is

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \tag{11.33}$$

In Section 11.4, we saw that

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) = S \cos(\theta_v - \theta_i) \tag{11.34}$$

We have added a new term to the equation:

$$S = V_{\text{rms}} I_{\text{rms}} \tag{11.35}$$

The average power is a product of two terms. The product $V_{\text{rms}} I_{\text{rms}}$ is known as the *apparent power* S. The factor $\cos(\theta_v - \theta_i)$ is called the *power factor* (pf).

The **apparent power** (in VA) is the product of the rms values of voltage and current.

The apparent power is so called because it seems apparent that the power should be the voltage-current product, by analogy with dc resistive circuits. It is measured in volt-amperes or VA to distinguish it from the average or real power, which is measured in watts. The power factor is dimensionless, since it is the ratio of the average power to the apparent power,

$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i) \tag{11.36}$$

The angle $\theta_v - \theta_i$ is called the *power factor angle*, since it is the angle whose cosine is the power factor. The power factor angle is equal to the angle of the load impedance if **V** is the voltage across the load and **I** is the current through it. This is evident from the fact that

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m\angle\theta_v}{I_m\angle\theta_i} = \frac{V_m}{I_m}\angle\theta_v - \theta_i \tag{11.37}$$

Alternatively, since

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}}\angle\theta_v \tag{11.38a}$$

and

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}}\angle\theta_i \tag{11.38b}$$

the impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{V_{\text{rms}}}{I_{\text{rms}}}\angle\theta_v - \theta_i \tag{11.39}$$

The **power factor** is the cosine of the phase difference between voltage and current. It is also the cosine of the angle of the load impedance.

From Eq. (11.36), the power factor may be seen as that factor by which the apparent power must be multiplied to obtain the real or average power. The value of pf ranges between zero and unity. For a purely resistive load, the voltage and current are in phase, so that $\theta_v - \theta_i = 0$ and pf = 1. This implies that the apparent power is equal to the average power. For a purely reactive load, $\theta_v - \theta_i = \pm 90°$ and pf = 0. In this case the average power is zero. In between these two extreme cases, pf is said to be *leading* or *lagging*. Leading power factor means that current leads voltage, which implies a capacitive load. Lagging power factor means that current lags voltage, implying an inductive load.

Example

A series-connected load draws a current $i(t) = 4\cos(100\pi t + 10^\circ)$ A when the applied voltage is $v(t) = 120\cos(100\pi t - 20^\circ)$ V. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

Solution:

The apparent power is

$$S = V_{\text{rms}} I_{\text{rms}} = \frac{120}{\sqrt{2}} \frac{4}{\sqrt{2}} = 240 \text{ VA}$$

The power factor is

$$\text{pf} = \cos(\theta_v - \theta_i) = \cos(-20^\circ - 10^\circ) = 0.866 \qquad \text{(leading)}$$

The pf is leading because the current leads the voltage. The pf may also be obtained from the load impedance.

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{120\angle -20^\circ}{4\angle 10^\circ} = 30\angle -30^\circ = 25.98 - j15\ \Omega$$

$$\text{pf} = \cos(-30^\circ) = 0.866 \qquad \text{(leading)}$$

The load impedance $\mathbf{Z}$ can be modeled by a 25.98-Ω resistor in series with a capacitor with

$$X_C = -15 = -\frac{1}{\omega C}$$

or

$$C = \frac{1}{15\omega} = \frac{1}{15 \times 100\pi} = 212.2\ \mu\text{F}$$

Practice Problem Example

Obtain the power factor and the apparent power of a load whose impedance is $\mathbf{Z} = 60 + j40\ \Omega$ when the applied voltage is $v(t) = 160\cos(377t + 10^\circ)$ V.

Answer: 0.8321 lagging, $177.5\angle 33.69^\circ$ VA.

Example

Determine the power factor of the entire circuit of Fig. 11.18 as seen by the source. Calculate the average power delivered by the source.

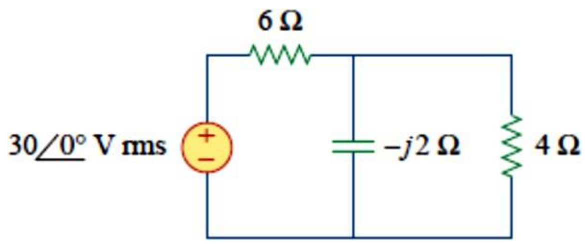


Figure 11.18

Solution:

The total impedance is

$$\mathbf{Z} = 6 + 4 \parallel (-j2) = 6 + \frac{-j2 \times 4}{4 - j2} = 6.8 - j1.6 = 7\underline{/-13.24^\circ}\ \Omega$$

The power factor is

$$\text{pf} = \cos(-13.24) = 0.9734 \qquad \text{(leading)}$$

since the impedance is capacitive. The rms value of the current is

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{Z}} = \frac{30\underline{/0^\circ}}{7\underline{/-13.24^\circ}} = 4.286\underline{/13.24^\circ}\text{ A}$$

The average power supplied by the source is

$$P = V_{\text{rms}} I_{\text{rms}}\,\text{pf} = (30)(4.286)0.9734 = 125\text{ W}$$

or

$$P = I_{\text{rms}}^2 R = (4.286)^2(6.8) = 125\text{ W}$$

where R is the resistive part of $\mathbf{Z}$.

Practice Problem

Calculate the power factor of the entire circuit of Fig. 11.19 as seen by the source. What is the average power supplied by the source?

Answer: 0.936 lagging, 1.062 kW.

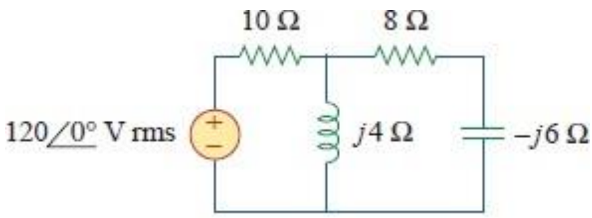


Figure 11.19

11.6 Complex Power

Power engineers have coined the term *complex power*, which they use to find the total effect of parallel loads. Complex power is important in power analysis because it contains *all* the information pertaining to the power absorbed by a given load

Consider the ac load in Fig. 11.20. Given the phasor form $\mathbf{V} = V_m\underline{/\theta_v}$ and $\mathbf{I} = I_m\underline{/\theta_i}$ of voltage $v(t)$ and current $i(t)$, the *complex power* $\mathbf{S}$ absorbed by the ac load is the product of the voltage and the complex conjugate of the current, or

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* \tag{11.40}$$

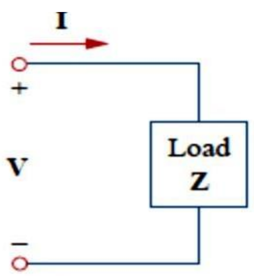


Figure 11.20
The voltage and current phasors associated with a load.

assuming the passive sign convention (see Fig. 11.20). In terms of the rms values,

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}} \tag{11.41}$$

where

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}}\underline{/\theta_v} \tag{11.42}$$

and

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}}\underline{/\theta_i} \tag{11.43}$$

Thus we may write Eq. (11.41) as

$$\begin{aligned}\mathbf{S} &= V_{\text{rms}}I_{\text{rms}}\underline{/\theta_v - \theta_i} \\ &= V_{\text{rms}}I_{\text{rms}}\cos(\theta_v - \theta_i) + jV_{\text{rms}}I_{\text{rms}}\sin(\theta_v - \theta_i)\end{aligned} \tag{11.44}$$

This equation can also be obtained from Eq. (11.9). We notice from Eq. (11.44) that the magnitude of the complex power is the apparent power; hence, the complex power is measured in volt-amperes (VA). Also, we notice that the angle of the complex power is the power factor angle. The complex power may be expressed in terms of the load impedance **Z**. From Eq. (11.37), the load impedance **Z** may be written as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \underline{/\theta_v - \theta_i} \qquad \textbf{(11.45)}$$

Thus, $\mathbf{V}_{\text{rms}} = \mathbf{Z}\mathbf{I}_{\text{rms}}$. Substituting this into Eq. (11.41) gives

$$\mathbf{S} = I_{\text{rms}}^2 \mathbf{Z} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^* \qquad \textbf{(11.46)}$$

Since $\mathbf{Z} = R + jX$, Eq. (11.46) becomes

$$\mathbf{S} = I_{\text{rms}}^2 (R + jX) = P + jQ \qquad \textbf{(11.47)}$$

where P and Q are the real and imaginary parts of the complex power; that is,

$$P = \text{Re}(\mathbf{S}) = I_{\text{rms}}^2 R \qquad \textbf{(11.48)}$$

$$Q = \text{Im}(\mathbf{S}) = I_{\text{rms}}^2 X \qquad \textbf{(11.49)}$$

P is the average or real power and it depends on the load's resistance R. Q depends on the load's reactance X and is called the *reactive* (or quadrature) power.

Comparing Eq. (11.44) with Eq. (11.47), we notice that

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i), \qquad Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i) \qquad \textbf{(11.50)}$$

The real power P is the average power in watts delivered to a load; it is the only useful power. It is the actual power dissipated by the load. The reactive power Q is a measure of the energy exchange between the source and the reactive part of the load. The unit of Q is the *volt-ampere reactive* (VAR) to distinguish it from the real power, whose unit is the watt. We know from Chapter 6 that energy storage elements neither dissipate nor supply power, but exchange power back and forth with the rest of the network. In the same way, the reactive power is being transferred back and forth between the load and the source. It represents a lossless interchange between the load and the source.

Notice that:

1. $Q = 0$ for resistive loads (unity pf).
2. $Q < 0$ for capacitive loads (leading pf).
3. $Q > 0$ for inductive loads (lagging pf).

Thus,

Complex power (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. As a complex quantity, its real part is real power P and its imaginary part is reactive power Q.

Introducing the complex power enables us to obtain the real and reactive powers directly from voltage and current phasors.

$$\begin{aligned}
\text{Complex Power} &= \mathbf{S} = P + jQ = \mathbf{V}_{\text{rms}}(\mathbf{I}_{\text{rms}})^* \\
&= \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}\angle\theta_v - \theta_i \\
\text{Apparent Power} &= S = |\mathbf{S}| = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}} = \sqrt{P^2 + Q^2} \\
\text{Real Power} &= P = \text{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i) \\
\text{Reactive Power} &= Q = \text{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i) \\
\text{Power Factor} &= \frac{P}{S} = \cos(\theta_v - \theta_i)
\end{aligned} \tag{11.51}$$

S contains *all* power information of a load. The real part of **S** is the real power P; its imaginary part is the reactive power Q; its magnitude is the apparent power S; and the cosine of its phase angle is the power factor pf.

This shows how the complex power contains *all* the relevant power information in a given load
It is a standard practice to represent **S**, *P*, and *Q* in the form of a triangle, known as the *power triangle*, shown in Fig. 11.21(a). This issimilar to the impedance triangle showing the relationship between **Z**, *R*, and *X*, illustrated in Fig. 11.21(b). The power triangle has four items—the apparent/complex power, real power, reactive power, and

the power factor angle. Given two of these items, the other two can easily be obtained from the triangle. As shown in Fig. 11.22, when **S** lies in the first quadrant, we have an inductive load and a lagging pf. When **S** lies in the fourth quadrant, the load is capacitive and the pf is leading. It is also possible for the complex power to lie in the second or third quadrant. This requires that the load impedance have a negative resistance, which is possible with active circuits.

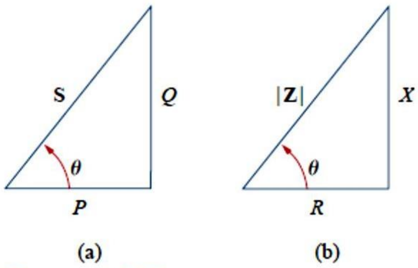


Figure 11.21
(a) Power triangle, (b) impedance triangle.

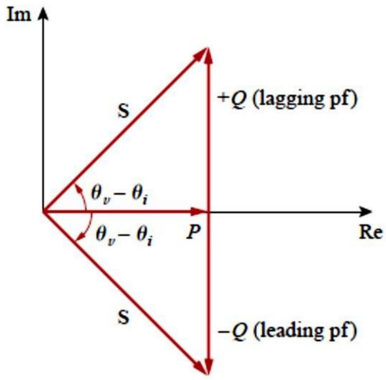


Figure 11.22
Power triangle.

Example

The voltage across a load is $v(t) = 60\cos(\omega t - 10^\circ)$ V and the current through the element in the direction of the voltage drop is $i(t) = 1.5\cos(\omega t + 50^\circ)$ A. Find: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Solution:

(a) For the rms values of the voltage and current, we write

$$\mathbf{V}_{rms} = \frac{60}{\sqrt{2}}\angle -10^\circ, \qquad \mathbf{I}_{rms} = \frac{1.5}{\sqrt{2}}\angle +50^\circ$$

The complex power is

$$\mathbf{S} = \mathbf{V}_{rms}\mathbf{I}^*_{rms} = \left(\frac{60}{\sqrt{2}}\angle -10^\circ\right)\left(\frac{1.5}{\sqrt{2}}\angle -50^\circ\right) = 45\angle -60^\circ \text{ VA}$$

The apparent power is

$$S = |\mathbf{S}| = 45 \text{ VA}$$

(b) We can express the complex power in rectangular form as

$$\mathbf{S} = 45\angle -60^\circ = 45[\cos(-60^\circ) + j\sin(-60^\circ)] = 22.5 - j38.97$$

Since $\mathbf{S} = P + jQ$, the real power is

$$P = 22.5 \text{ W}$$

while the reactive power is

$$Q = -38.97 \text{ VAR}$$

(c) The power factor is

$$\text{pf} = \cos(-60^\circ) = 0.5 \text{ (leading)}$$

It is leading, because the reactive power is negative. The load impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{60\angle -10^\circ}{1.5\angle +50^\circ} = 40\angle -60^\circ\ \Omega$$

which is a capacitive impedance.

PracticeProblem

For a load, $\mathbf{V}_{rms} = 110\angle 85^\circ$ V, $\mathbf{I}_{rms} = 0.4\angle 15^\circ$ A. Determine: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Answer: (a) $44\angle 70^\circ$ VA, 44 VA, (b) 15.05 W, 41.35 VAR, (c) 0.342 lagging, $94.06 + j258.4\ \Omega$.

Example

A load **Z** draws 12 kVA at a power factor of 0.856 lagging from a 120-V rms sinusoidal source. Calculate: (a) the average and reactive powers delivered to the load, (b) the peak current, and (c) the load impedance.

Solution:

(a) Given that pf = $\cos\theta = 0.856$, we obtain the power angle as $\theta = \cos^{-1} 0.856 = 31.13°$. If the apparent power is $S = 12{,}000$ VA, then the average or real power is

$$P = S\cos\theta = 12{,}000 \times 0.856 = 10.272 \text{ kW}$$

while the reactive power is

$$Q = S\sin\theta = 12{,}000 \times 0.517 = 6.204 \text{ kVA}$$

(b) Since the pf is lagging, the complex power is

$$\mathbf{S} = P + jQ = 10.272 + j6.204 \text{ kVA}$$

From $\mathbf{S} = \mathbf{V}_{rms}\mathbf{I}^*_{rms}$, we obtain

$$\mathbf{I}^*_{rms} = \frac{\mathbf{S}}{\mathbf{V}_{rms}} = \frac{10{,}272 + j6204}{120\underline{/0°}} = 85.6 + j51.7 \text{ A} = 100\underline{/31.13°} \text{ A}$$

Thus $\mathbf{I}_{rms} = 100\underline{/-31.13°}$ and the peak current is

$$I_m = \sqrt{2}I_{rms} = \sqrt{2}(100) = 141.4 \text{ A}$$

(c) The load impedance

$$\mathbf{Z} = \frac{\mathbf{V}_{rms}}{\mathbf{I}_{rms}} = \frac{120\underline{/0°}}{100\underline{/-31.13°}} = 1.2\underline{/31.13°}\ \Omega$$

which is an inductive impedance.

Practice Problem

A sinusoidal source supplies 20 kVAR reactive power to load $\mathbf{Z} = 250\underline{/-75°}\ \Omega$. Determine: (a) the power factor, (b) the apparent power delivered to the load, and (c) the rms voltage.

Answer: (a) 0.2588 leading, (b) 20.71 kVA, (c) 2.275 kV.

11.7 †Conservation of AC Power

The principle of conservation of power applies to ac circuits as well as to dc circuits (see Section 1.5).

To see this, consider the circuit in Fig. 11.23(a), where two load impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are connected in parallel across an ac source $\mathbf{V}$. KCL gives

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 \tag{11.52}$$

The complex power supplied by the source is (from now on, unless otherwise specified, all values of voltages and currents will be assumed to be rms values)

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* = \mathbf{V}(\mathbf{I}_1^* + \mathbf{I}_2^*) = \mathbf{V}\mathbf{I}_1^* + \mathbf{V}\mathbf{I}_2^* = \mathbf{S}_1 + \mathbf{S}_2 \tag{11.53}$$

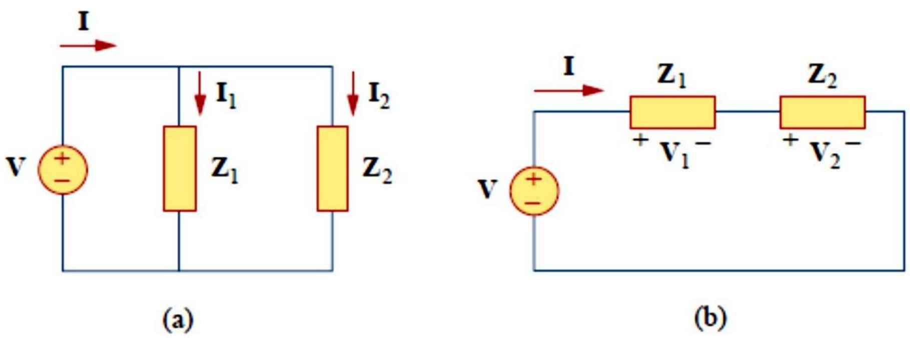


Figure 11.23
An ac voltage source supplied loads connected in: (a) parallel, (b) series.

where $\mathbf{S}_1$ and $\mathbf{S}_2$ denote the complex powers delivered to loads $\mathbf{Z}_1$ and $\mathbf{Z}_2$, respectively.

If the loads are connected in series with the voltage source, as shown in Fig. 11.23(b), KVL yields

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 \tag{11.54}$$

The complex power supplied by the source is

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* = (\mathbf{V}_1 + \mathbf{V}_2)\mathbf{I}^* = \mathbf{V}_1\mathbf{I}^* + \mathbf{V}_2\mathbf{I}^* = \mathbf{S}_1 + \mathbf{S}_2 \tag{11.55}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ denote the complex powers delivered to loads $\mathbf{Z}_1$ and $\mathbf{Z}_2$, respectively.

We conclude from Eqs. (11.53) and (11.55) that whether the loads are connected in series or in parallel (or in general), the total power *supplied* by the source equals the total power *delivered* to the load. Thus, in general, for a source connected to N loads,

$$\boxed{\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \cdots + \mathbf{S}_N} \tag{11.56}$$

This means that the total complex power in a network is the sum of the complex powers of the individual components. (This is also true of real power and reactive power, but not true of apparent power.) This expresses the principle of conservation of ac power

The complex, real, and reactive powers of the sources equal the respective sums of the complex, real, and reactive powers of the individual loads.

From this we imply that the real (or reactive) power flow from sources in a network equals the real (or reactive) power flow into the other elements in the networ

Example

Figure 11.24 shows a load being fed by a voltage source through a transmission line. The impedance of the line is represented by the $(4 + j2)\ \Omega$ impedance and a return path. Find the real power and reactive power absorbed by: (a) the source, (b) the line, and (c) the load.

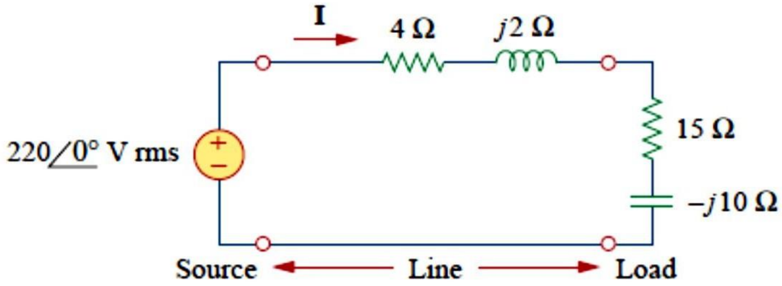


Figure 11.24

Solution:

The total impedance is

$$\mathbf{Z} = (4 + j2) + (15 - j10) = 19 - j8 = 20.62\angle -22.83^\circ\ \Omega$$

The current through the circuit is

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{220\angle 0^\circ}{20.62\angle -22.83^\circ} = 10.67\angle 22.83^\circ \text{ A rms}$$

(a) For the source, the complex power is

$$\mathbf{S}_s = \mathbf{V}_s\mathbf{I}^* = (220\angle 0^\circ)(10.67\angle -22.83^\circ)$$
$$= 2347.4\angle -22.83^\circ = (2163.5 - j910.8) \text{ VA}$$

From this, we obtain the real power as 2163.5 W and the reactive power as 910.8 VAR (leading).

(b) For the line, the voltage is

$$\mathbf{V}_{\text{line}} = (4 + j2)\mathbf{I} = (4.472\angle 26.57^\circ)(10.67\angle 22.83^\circ)$$
$$= 47.72\angle 49.4^\circ \text{ V rms}$$

The complex power absorbed by the line is

$$\mathbf{S}_{\text{line}} = \mathbf{V}_{\text{line}}\mathbf{I}^* = (47.72\angle 49.4^\circ)(10.67\angle -22.83^\circ)$$
$$= 509.2\angle 26.57^\circ = 455.4 + j227.7 \text{ VA}$$

or

$$\mathbf{S}_{\text{line}} = |\mathbf{I}|^2\mathbf{Z}_{\text{line}} = (10.67)^2(4 + j2) = 455.4 + j227.7 \text{ VA}$$

That is, the real power is 455.4 W and the reactive power is 227.76 VAR (lagging).

(c) For the load, the voltage is

$$\mathbf{V}_L = (15 - j10)\mathbf{I} = (18.03\underline{/-33.7^\circ})(10.67\underline{/22.83^\circ}) = 192.38\underline{/-10.87^\circ} \text{ V rms}$$

The complex power absorbed by the load is

$$\mathbf{S}_L = \mathbf{V}_L\mathbf{I}^* = (192.38\underline{/-10.87^\circ})(10.67\underline{/-22.83^\circ}) = 2053\underline{/-33.7^\circ} = (1708 - j1139) \text{ VA}$$

The real power is 1708 W and the reactive power is 1139 VAR (leading). Note that $\mathbf{S}_s = \mathbf{S}_{\text{line}} + \mathbf{S}_L$, as expected. We have used the rms values of voltages and currents.

Practice Problem

In the circuit in Fig. 11.25, the 60-Ω resistor absorbs an average power of 240 W. Find **V** and the complex power of each branch of the circuit. What is the overall complex power of the circuit? (Assume the current through the 60-Ω resistor has no phase shift.)

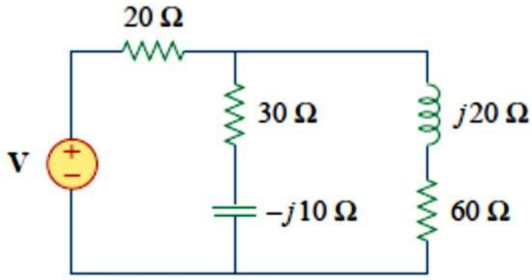


Figure 11.25

Answer: $240.67\underline{/21.45^\circ}$ V (rms); the 20-Ω resistor: 656 VA; the $(30 - j10)$ Ω impedance: $480 - j160$ VA; the $(60 + j20)$ Ω impedance: $240 + j80$ VA; overall: $1376 - j80$ VA.

Example

In the circuit of Fig. 11.26, $\mathbf{Z}_1 = 60\angle -30^\circ\ \Omega$ and $\mathbf{Z}_2 = 40\angle 45^\circ\ \Omega$. Calculate the total: (a) apparent power, (b) real power, (c) reactive power, and (d) pf, supplied by the source and seen by the source.

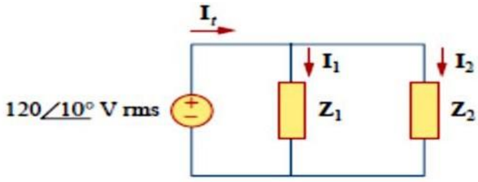


Figure 11.26

Solution:

The current through $\mathbf{Z}_1$ is

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = \frac{120\angle 10^\circ}{60\angle -30^\circ} = 2\angle 40^\circ \text{ A rms}$$

while the current through $\mathbf{Z}_2$ is

$$\mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{120\angle 10^\circ}{40\angle 45^\circ} = 3\angle -35^\circ \text{ A rms}$$

The complex powers absorbed by the impedances are

$$\mathbf{S}_1 = \frac{V_{\text{rms}}^2}{\mathbf{Z}_1^*} = \frac{(120)^2}{60\angle 30^\circ} = 240\angle -30^\circ = 207.85 - j120 \text{ VA}$$

$$\mathbf{S}_2 = \frac{V_{\text{rms}}^2}{\mathbf{Z}_2^*} = \frac{(120)^2}{40\angle -45^\circ} = 360\angle 45^\circ = 254.6 + j254.6 \text{ VA}$$

The total complex power is

$$\mathbf{S}_t = \mathbf{S}_1 + \mathbf{S}_2 = 462.4 + j134.6 \text{ VA}$$

(a) The total apparent power is

$$|\mathbf{S}_t| = \sqrt{462.4^2 + 134.6^2} = 481.6 \text{ VA.}$$

(b) The total real power is

$$P_t = \text{Re}(\mathbf{S}_t) = 462.4 \text{ W or } P_t = P_1 + P_2.$$

(c) The total reactive power is

$$Q_t = \text{Im}(\mathbf{S}_t) = 134.6 \text{ VAR or } Q_t = Q_1 + Q_2.$$

(d) The pf $= P_t/|\mathbf{S}_t| = 462.4/481.6 = 0.96$ (lagging).

We may cross check the result by finding the complex power $\mathbf{S}_s$ supplied by the source.

$$\begin{aligned}\mathbf{I}_t = \mathbf{I}_1 + \mathbf{I}_2 &= (1.532 + j1.286) + (2.457 - j1.721)\\ &= 4 - j0.435 = 4.024\angle -6.21^\circ \text{ A rms}\\ \mathbf{S}_s = \mathbf{V}\mathbf{I}_t^* &= (120\angle 10^\circ)(4.024\angle 6.21^\circ)\\ &= 482.88\angle 16.21^\circ = 463 + j135 \text{ VA}\end{aligned}$$

which is the same as before.

Practice Problem

Two loads connected in parallel are respectively 2 kW at a pf of 0.75 leading and 4 kW at a pf of 0.95 lagging. Calculate the pf of the two loads. Find the complex power supplied by the source.

Answer: 0.9972 (leading), $6 - j0.4495$ kVA.

11.8 Power Factor Correction

The process of increasing the power factor without altering the voltage or current to the original load is known as **power factor correction.**

Since most loads are inductive, as shown in Fig. 11.27(a), a load's power factor is improved or corrected by deliberately installing a capacitor in parallel with the load, as shown in Fig. 11.27(b). The effect of adding the capacitor can be illustrated using either the power triangle or the phasor diagram of the currents involved. Figure 11.28 shows the latter, where it is assumed that the circuit in Fig. 11.27(a) has a power factor of $\cos\theta_1$, while the one in Fig. 11.27(b) has a power factor of $\cos\theta_2$. It is evident from Fig. 11.28 that adding the capacitor has caused the phase angle between the supplied voltage and current to reduce from θ_1 to θ_2, thereby increasing the power factor. We also notice from the magnitudes of the vectors in Fig. 11.28 that with the same supplied voltage, the circuit in Fig. 11.27(a) draws larger current I_L than the current I drawn by the circuit in Fig. 11.27(b). Power companies charge more for larger currents, because they result in increased power losses (by a squared factor, since $P = I_L^2 R$). Therefore, it is beneficial to both the power company and the consumer that every effort is made to minimize current level or keep the power factor as close to unity as possible. By choosing a suitable size for the capacitor, the current can be made to be completely in phase with the voltage, implying unity power factor.

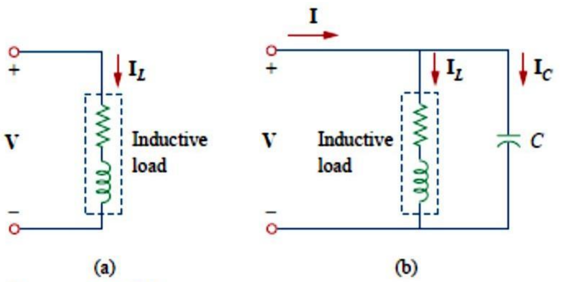


Figure 11.27
Power factor correction: (a) original inductive load, (b) inductive load with improved power factor.

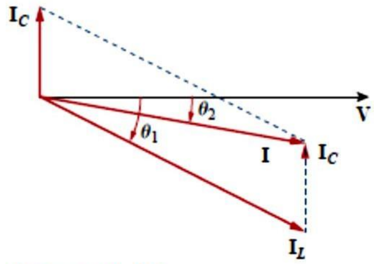


Figure 11.28
Phasor diagram showing the effect of adding a capacitor in parallel with the inductive load.

We can look at the power factor correction from another perspective. Consider the power triangle in Fig. 11.29. If the original inductive load has apparent power S_1, then

$$P = S_1 \cos\theta_1, \qquad Q_1 = S_1 \sin\theta_1 = P \tan\theta_1 \qquad (11.57)$$

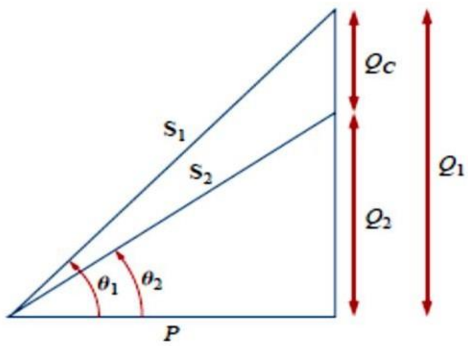


Figure 11.29
Power triangle illustrating power factor correction.

If we desire to increase the power factor from $\cos\theta_1$ to $\cos\theta_2$ without altering the real power (i.e., $P = S_2 \cos\theta_2$), then the new reactive power is

$$Q_2 = P \tan\theta_2 \qquad (11.58)$$

The reduction in the reactive power is caused by the shunt capacitor; that is,

$$Q_C = Q_1 - Q_2 = P(\tan\theta_1 - \tan\theta_2) \qquad (11.59)$$

But from Eq. (11.46), $Q_C = V_{\text{rms}}^2/X_C = \omega C V_{\text{rms}}^2$. The value of the required shunt capacitance C is determined as

$$C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{P(\tan\theta_1 - \tan\theta_2)}{\omega V_{\text{rms}}^2} \qquad (11.60)$$

Note that the real power P dissipated by the load is not affected by the power factor correction because the average power due to the capacitance is zero.

Although the most common situation in practice is that of an inductive load, it is also possible that the load is capacitive; that is, the load is operating at a leading power factor. In this case, an inductor should be connected across the load for power factor correction. The required shunt inductance L can be calculated from

$$Q_L = \frac{V_{\text{rms}}^2}{X_L} = \frac{V_{\text{rms}}^2}{\omega L} \quad \Rightarrow \quad L = \frac{V_{\text{rms}}^2}{\omega Q_L} \qquad (11.61)$$

where $Q_L = Q_1 - Q_2$, the difference between the new and old reactive powers.

.

Example

When connected to a 120-V (rms), 60-Hz power line, a load absorbs 4 kW at a lagging power factor of 0.8. Find the value of capacitance necessary to raise the pf to 0.95.

Solution:
If the pf = 0.8, then

$$\cos\theta_1 = 0.8 \quad \Rightarrow \quad \theta_1 = 36.87^\circ$$

where θ_1 is the phase difference between voltage and current. We obtain the apparent power from the real power and the pf as

$$S_1 = \frac{P}{\cos\theta_1} = \frac{4000}{0.8} = 5000 \text{ VA}$$

The reactive power is

$$Q_1 = S_1 \sin\theta = 5000 \sin 36.87 = 3000 \text{ VAR}$$

When the pf is raised to 0.95,

$$\cos\theta_2 = 0.95 \quad \Rightarrow \quad \theta_2 = 18.19^\circ$$

The real power P has not changed. But the apparent power has changed; its new value is

$$S_2 = \frac{P}{\cos\theta_2} = \frac{4000}{0.95} = 4210.5 \text{ VA}$$

The new reactive power is

$$Q_2 = S_2 \sin\theta_2 = 1314.4 \text{ VAR}$$

The difference between the new and old reactive powers is due to the parallel addition of the capacitor to the load. The reactive power due to the capacitor is

$$Q_C = Q_1 - Q_2 = 3000 - 1314.4 = 1685.6 \text{ VAR}$$

and

$$C = \frac{Q_C}{\omega V_{\text{rms}}^2} = \frac{1685.6}{2\pi \times 60 \times 120^2} = 310.5\ \mu\text{F}$$

Note: Capacitors are normally purchased for voltages they expect to see. In this case, the maximum voltage this capacitor will see is about 170 V peak. We would suggest purchasing a capacitor with a voltage rating equal to, say, 200 V.

Practice Problem

Find the value of parallel capacitance needed to correct a load of 140 kVAR at 0.85 lagging pf to unity pf. Assume that the load is supplied by a 110-V (rms), 60-Hz line.

Answer: 30.69 mF.

11.10 Summary

1. The instantaneous power absorbed by an element is the product of the element's terminal voltage and the current through the element:

$$p = vi.$$

2. Average or real power P (in watts) is the average of instantaneous power p:

$$P = \frac{1}{T}\int_0^T p\,dt$$

If $v(t) = V_m\cos(\omega t + \theta_v)$ and $i(t) = I_m\cos(\omega t + \theta_i)$, then $V_{rms} = V_m/\sqrt{2}$, $I_{rms} = I_m/\sqrt{2}$, and

$$P = \frac{1}{2}V_m I_m\cos(\theta_v - \theta_i) = V_{rms}I_{rms}\cos(\theta_v - \theta_i)$$

Inductors and capacitors absorb no average power, while the average power absorbed by a resistor is $(1/2)I_m^2 R = I_{rms}^2 R$.

3. Maximum average power is transferred to a load when the load impedance is the complex conjugate of the Thevenin impedance as seen from the load terminals, $\mathbf{Z}_L = \mathbf{Z}_{Th}^*$.
4. The effective value of a periodic signal $x(t)$ is its root-mean-square (rms) value.

$$X_{eff} = X_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt}$$

For a sinusoid, the effective or rms value is its amplitude divided by $\sqrt{2}$.

5. The power factor is the cosine of the phase difference between voltage and current:

$$\text{pf} = \cos(\theta_v - \theta_i)$$

It is also the cosine of the angle of the load impedance or the ratio of real power to apparent power. The pf is lagging if the current lags voltage (inductive load) and is leading when the current leads voltage (capacitive load).

6. Apparent power S (in VA) is the product of the rms values of voltage and current:

$$S = V_{rms}I_{rms}$$

It is also given by $S = |\mathbf{S}| = \sqrt{P^2 + Q^2}$, where P is the real power and Q is reactive power.

7. Reactive power (in VAR) is:

$$Q = \frac{1}{2}V_m I_m\sin(\theta_v - \theta_i) = V_{rms}I_{rms}\sin(\theta_v - \theta_i)$$

8. Complex power **S** (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. It is also the complex sum of real power *P* and reactive power *Q*.

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}} = V_{\text{rms}}I_{\text{rms}}\underline{/\theta_v - \theta_i} = P + jQ$$

Also,

$$\mathbf{S} = I^2_{\text{rms}}\mathbf{Z} = \frac{V^2_{\text{rms}}}{\mathbf{Z}^*}$$

9. The total complex power in a network is the sum of the complex powers of the individual components. Total real power and reactive power are also, respectively, the sums of the individual real powers and the reactive powers, but the total apparent power is not calculated by the process.
10. Power factor correction is necessary for economic reasons; it is the process of improving the power factor of a load by reducing the overall reactive power.
11. The wattmeter is the instrument for measuring the average power. Energy consumed is measured with a kilowatt-hour meter.