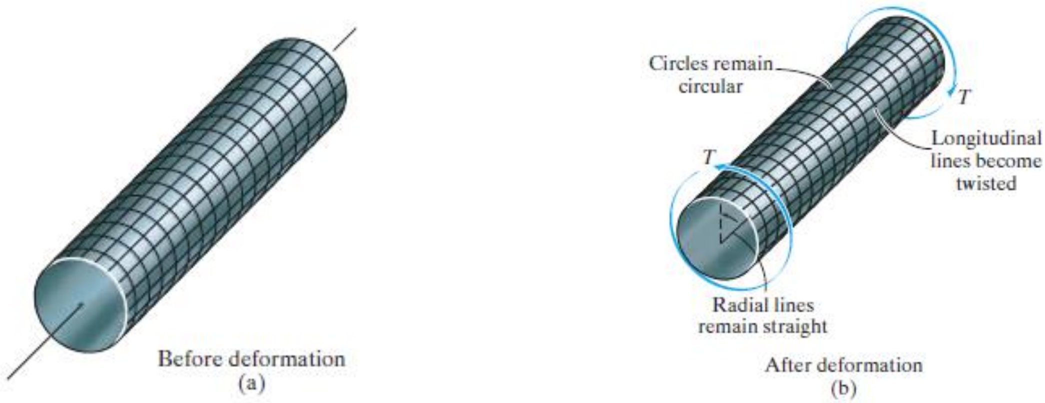
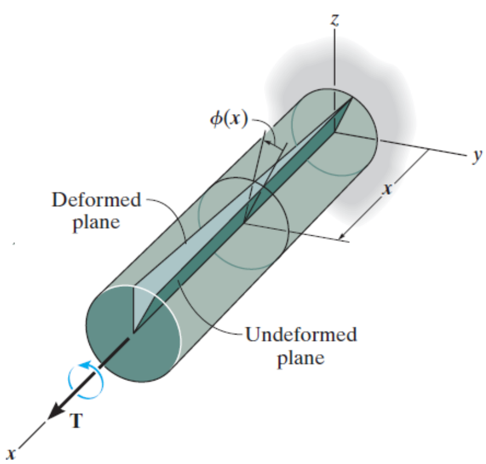
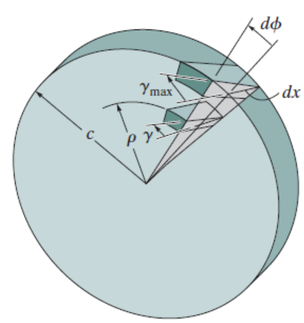
Chapter 6

Torsion







$$\gamma dx = d\phi\rho \rightarrow \gamma = \rho\frac{d\phi}{dx}$$

Since $dx\ and\ d\phi$ is the same for all elements located at points on the cross section at x, then $\frac{dx}{d\phi}$ is constant over the cross section, γ increases linearly with distance from the center of the axis of the shaft,

$$\gamma = \frac{\rho}{c}\gamma_{max}$$

If the material is linear-elastic, then Hooke's law applies, $\tau = G\gamma$, and consequently a ***linear variation in shear strain***, as noted in the previous section, leads to a corresponding ***linear variation in shear stress*** along any radial line on the cross section.

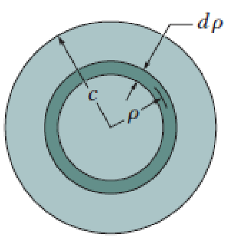
$$\tau = \left(\frac{\rho}{c}\right)\tau_{max}$$

$$T = \int_A \rho(\tau\, dA) = \int_A \rho\left(\frac{\rho}{c}\right)\tau_{max}\, dA$$

$$T = \frac{\tau_{max}}{c}\int_A \rho^2\, dA$$

$$\tau_{max} = \frac{Tc}{J}$$

Combing above equations, $$\tau = \frac{T\rho}{J}$$



Solid Shaft. If the shaft has a solid circular cross section, the polar moment of inertia J can be determined using an area element in the form of a *differential ring* or annulus having a thickness $d\rho$ and circumference $2\pi\rho$, Fig. 5–6. For this ring, $dA = 2\pi\rho\, d\rho$, and so

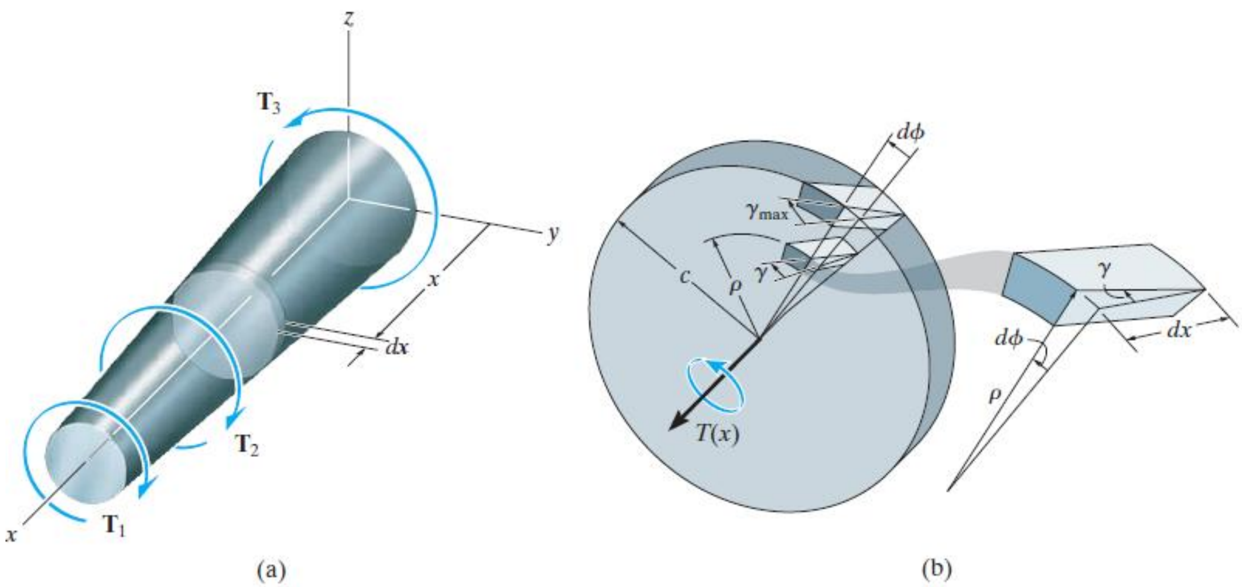
$$J = \int_A \rho^2\, dA = \int_0^c \rho^2(2\pi\rho\, d\rho) = 2\pi\int_0^c \rho^3\, d\rho = 2\pi\left(\frac{1}{4}\right)\rho^4\Big|_0^c$$

$$J = \frac{\pi}{2}c^4$$

Tabular shafts

$$J = \frac{\pi}{2}(c_o^4 - c_i^4)$$

Angle of twist



(a) (b)

$$d\phi = \gamma \frac{dx}{\rho}$$

Since Hooke's law, $\gamma = \tau/G$, applies and the shear stress can be expressed in terms of the applied torque using the torsion formula $\tau = T(x)\rho/J(x)$, then $\gamma = T(x)\rho/J(x)G$. Substituting

$$d\phi = \frac{T(x)}{J(x)G} dx$$

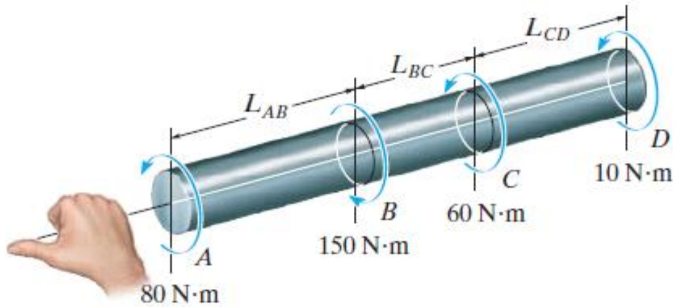
Integrating over the entire length L of the shaft, we obtain the angle of twist for the entire shaft, namely,

$$\phi = \int_0^L \frac{T(x)\, dx}{J(x)G}$$

Multiple torques

$$\phi = \sum \frac{TL}{JG}$$

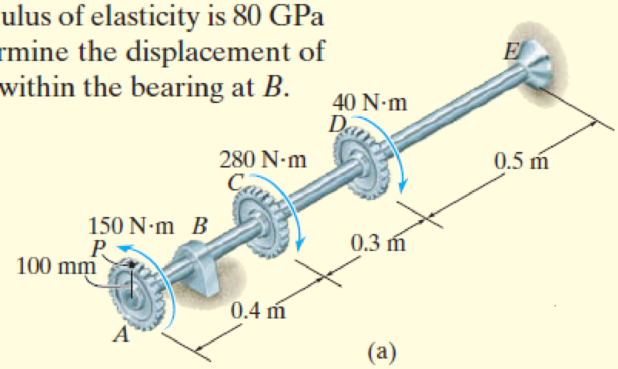
Example



$$\phi_{A/D} = \frac{(+80\ \text{N}\cdot\text{m})\ L_{AB}}{JG} + \frac{(-70\ \text{N}\cdot\text{m})\ L_{BC}}{JG} + \frac{(-10\ \text{N}\cdot\text{m})\ L_{CD}}{JG}$$

Example

The gears attached to the fixed-end steel shaft are subjected to the torques shown in Fig. 5–19*a*. If the shear modulus of elasticity is 80 GPa and the shaft has a diameter of 14 mm, determine the displacement of the tooth *P* on gear *A*. The shaft turns freely within the bearing at *B*.



(a)

SOLUTION

$$\phi_A = \sum \frac{TL}{JG} = \frac{(+150\ \text{N}\cdot\text{m})(0.4\ \text{m})}{3.771(10^{-9})\ \text{m}^4\ [80(10^9)\ \text{N/m}^2]}$$

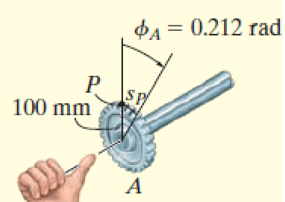
$$+ \frac{(-130\ \text{N}\cdot\text{m})(0.3\ \text{m})}{3.771(10^{-9})\ \text{m}^4\ [80(10^9)\ \text{N/m}^2)]}$$

$$+ \frac{(-170\ \text{N}\cdot\text{m})(0.5\ \text{m})}{3.771(10^{-9})\ \text{m}^4\ [80(10^9)\ \text{N/m}^2)]} = -0.2121\ \text{rad}$$

Since the answer is negative, by the right-hand rule the thumb is directed *toward* the end *E* of the shaft, and therefore gear *A* will rotate as shown in Fig. 5–19*d*.

The displacement of tooth *P* on gear *A* is

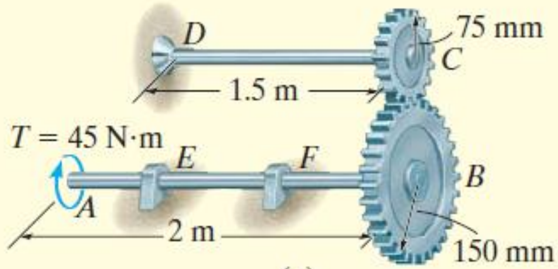
$$s_P = \phi_A r = (0.2121\ \text{rad})(100\ \text{mm}) = 21.2\ \text{mm} \qquad \textit{Ans.}$$



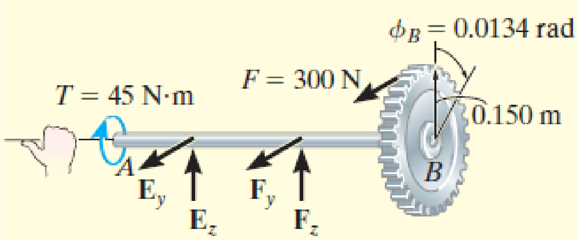
(d)

Example

The two solid steel shafts shown in Fig. 5–20*a* are coupled together using the meshed gears. Determine the angle of twist of end *A* of shaft *AB* when the torque $T = 45\ \text{N}\cdot\text{m}$ is applied. Take $G = 80$ GPa. Shaft *AB* is free to rotate within bearings *E* and *F*, whereas shaft *DC* is fixed at *D*. Each shaft has a diameter of 20 mm.



(a)



(b)

SOLUTION

Internal Torque. Free-body diagrams for each shaft are shown in Fig. 5–20*b* and 5–20*c*. Summing moments along the *x* axis of shaft *AB* yields the tangential reaction between the gears of $F = 45\ \text{N}\cdot\text{m}/0.15\ \text{m} = 300\ \text{N}$. Summing moments about the *x* axis of shaft *DC*, this force then creates a torque of $(T_D)_x = 300\ \text{N}\,(0.075\ \text{m}) = 22.5\ \text{N}\cdot\text{m}$ on shaft *DC*.

Angle of Twist. To solve the problem, we will first calculate the rotation of gear *C* due to the torque of 22.5 N · m in shaft *DC*, Fig. 5–20*c*. This angle of twist is

$$\phi_C = \frac{TL_{DC}}{JG} = \frac{(+22.5\ \text{N}\cdot\text{m})(1.5\ \text{m})}{(\pi/2)(0.010\ \text{m})^4[80(10^9)\ \text{N/m}^2]} = +0.0269\ \text{rad}$$

Since the gears at the end of the shaft are in mesh, the rotation ϕ_C of gear *C* causes gear *B* to rotate ϕ_B, Fig. 5–20*b*, where

$$\phi_B(0.15\ \text{m}) = (0.0269\ \text{rad})(0.075\ \text{m})$$

We will now determine the angle of twist of end *A* with respect to end *B* of shaft *AB* caused by the 45 N · m torque, Fig. 5–20*b*. We have

$$\phi_{A/B} = \frac{T_{AB}L_{AB}}{JG} = \frac{(+45\ \text{N}\cdot\text{m})(2\ \text{m})}{(\pi/2)(0.010\ \text{m})^4[80(10^9)\ \text{N/m}^2]} = +0.0716\ \text{rad}$$

The rotation of end *A* is therefore determined by adding ϕ_B and $\phi_{A/B}$, since both angles are in the *same direction*, Fig. 5–20*b*. We have

$$\phi_A = \phi_B + \phi_{A/B} = 0.0134\ \text{rad} + 0.0716\ \text{rad} = +0.0850\ \text{rad} \qquad \textit{Ans.}$$