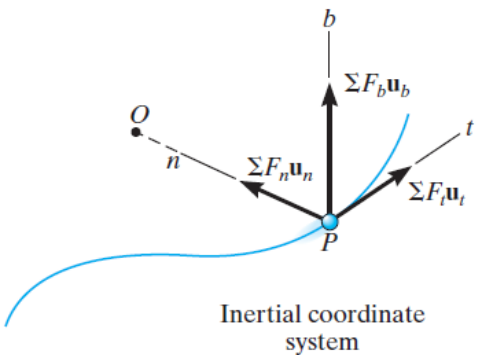
2-7 Equations of Motion: Normal and Tangential Coordinates

When a particle moves along a curved path which is known, the equation of motion for the particle may be written in the tangential, normal, and binomial directions. Note that there is no motion of the particle in the binomial direction, since the particle is constrained to move along the path. We have



Inertial coordinate system

$$\Sigma \mathbf{F} = m\mathbf{a}$$

$$\Sigma F_t \mathbf{u}_t + \Sigma F_n \mathbf{u}_n + \Sigma F_b \mathbf{u}_b = m\mathbf{a}_t$$

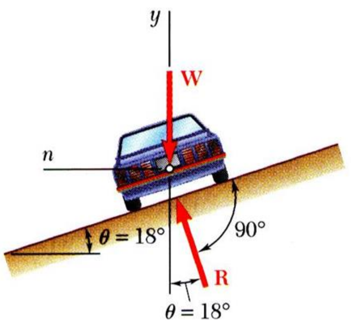
This equation is satisfied provided

$$\Sigma F_t = ma_t$$

$$\Sigma F_n = ma_n$$

Recall that $a_t (= dv/dt)$ represents the time rate of change in the magnitude of velocity. So if $\Sigma \mathbf{F}_t$ acts in the direction of motion, the particle's speed will increase, whereas if it acts in the opposite direction, the particle will slow down. Likewise, $a_n (= v^2/\rho)$ represents the time rate of change in the velocity's direction. It is caused by $\Sigma \mathbf{F}_n$, which *always* acts in the positive n direction, i.e., toward the path's center of curvature. For this reason it is often referred to as the *centripetal force*.

Example 5: Determine the rated speed of a highway curve of radius r = 400 ft banked through an angle θ = 18°. The rated speed of a banked highway curve is the speed at which a car should travel if no lateral friction force is to be exerted at its wheels.

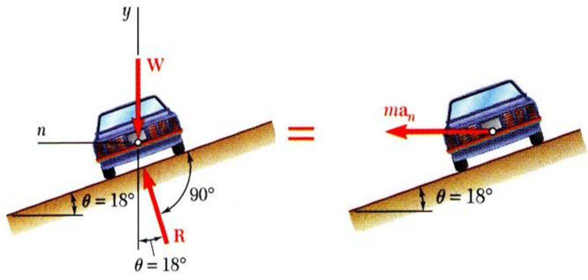


SOLUTION:

Resolve the equation of motion for the car into vertical and normal components:

$$\sum F_y = 0 : R\cos\theta - W = 0 \text{..........} R = \frac{W}{\cos\theta}$$

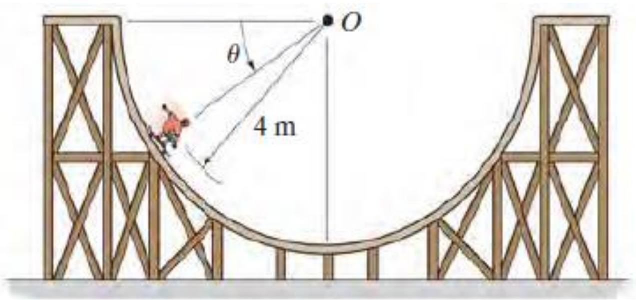
$$R\sin\theta = \frac{W}{g}a_n \text{.......} \frac{W}{\cos\theta}\sin\theta = \frac{W}{g}\frac{v^2}{\rho}$$



Solve for the vehicle speed:

$$v^2 = g\rho\tan\theta = (32.2\,\text{ft/s}^2)(400\,\text{ft})\tan 18° = 64.7\,\text{ft/s} = 44.1\,\text{mi/h}$$

Example 6: The 60-kg skateboarder in Fig. coasts down the circular track. If he starts from rest when $\theta = 0^o$, determine the magnitude of the normal reaction the track exerts on him when $\theta = 60^o$. Neglect his size for the calculation.



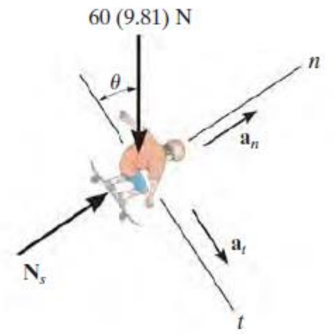
SOLUTION:

Equations of Motion.

$$+\nearrow \Sigma F_n = ma_n; \quad N_s - [60(9.81)\text{N}]\sin\theta = (60\text{ kg})\left(\frac{v^2}{4\text{ m}}\right)$$

$$+\searrow \Sigma F_t = ma_t; \quad [60(9.81)\text{N}]\cos\theta = (60\text{ kg})\,a_t$$

$$a_t = 9.81\cos\theta$$



Since a_t is expressed in terms of θ, the equation $v\,dv = a_t\,ds$ must be used to determine the speed of the skateboarder when $\theta = 60^o$. Using the geometric relation $s = \theta r$, where $ds = r\,d\theta = (4\text{ m})\,d\theta$, and the initial condition $v = 0$ at $\theta = 0^o$, we have,

$$v\,dv = a_t\,ds$$

$$\int_0^v v\,dv = \int_0^{60°} 9.81\cos\theta(4\,d\theta)$$

$$\left.\frac{v^2}{2}\right|_0^v = 39.24\sin\theta\Big|_0^{60°}$$

$$\frac{v^2}{2} - 0 = 39.24(\sin 60^\circ - 0)$$

$$v^2 = 67.97\ \text{m}^2/\text{s}^2$$

Substituting this result and $\theta = 60^o$ into Eq. of N_s, yields

$$N_s = 1529.23\ \text{N} = 1.53\ \text{kN}$$

2-8 Equations of Motion: Cylindrical Coordinates

When all the forces acting on a particle are resolved into cylindrical components, i.e., along the unit-vector directions $\boldsymbol{u}_r$, $\boldsymbol{u}_\theta$, $\boldsymbol{u}_z$, the equation of motion can be expressed as

$$\Sigma\mathbf{F} = m\mathbf{a}$$

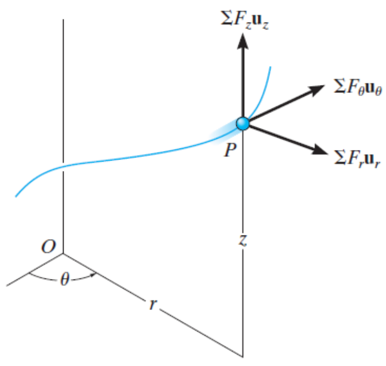
$$\Sigma F_r\mathbf{u}_r + \Sigma F_\theta\mathbf{u}_\theta + \Sigma F_z\mathbf{u}_z = ma_r\mathbf{u}_r + ma_\theta\mathbf{u}_\theta + ma_z\mathbf{u}_z$$

To satisfy this equation, we require

$$\Sigma F_r = ma_r$$

$$\Sigma F_\theta = ma_\theta$$

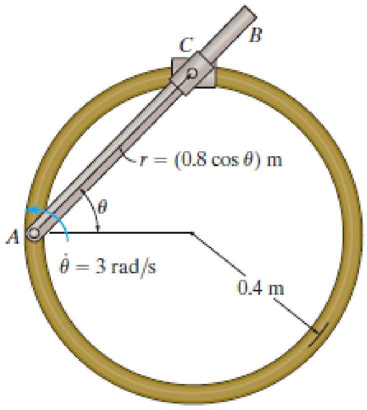
$$\Sigma F_z = ma_z$$



Inertial coordinate system

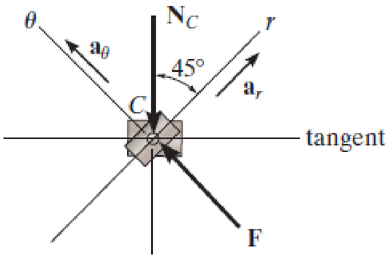
Example 7: The smooth 0.5-kg double-collar in Fig. can freely slide on arm *AB* and the circular guide rod.

If the arm rotates with a constant angular velocity of $\dot{\theta}$ = 3 rad/s, determine the force the arm exerts on the ollar at the instant θ= 45^o. Motion is in the horizontal plane.



SOLUTION:

Free-Body Diagram. The normal reaction $\boldsymbol{N_C}$ of the circular guide rod and the force $\boldsymbol{F}$ of arm AB act on the collar in the plane of motion. Note that $\boldsymbol{F}$ acts perpendicular to the axis of arm AB, that is, in the direction of the u axis, while $\boldsymbol{N_C}$ acts perpendicular to the tangent of the circular path at $\theta = 45^{o}$. The four unknowns are N_C, F, a_r, a_θ.



Equations of Motion.

$$+\nearrow \Sigma F_r = ma_r: \qquad -N_C \cos 45° = (0.5 \text{ kg})\, a_r \qquad (1)$$

$$+\nwarrow \Sigma F_\theta = ma_\theta: \qquad F - N_C \sin 45° = (0.5 \text{ kg})\, a_\theta \qquad (2)$$

Kinematics. Using the chain rule, the first and second time derivatives of r when $\theta = 45^{o}$, $\dot{\theta}$ = 3 rad/s, θ = 0, are

$$r = 0.8 \cos \theta = 0.8 \cos 45° = 0.5657 \text{ m}$$

$$\dot{r} = -0.8 \sin \theta\, \dot{\theta} = -0.8 \sin 45°(3) = -1.6971 \text{ m/s}$$

$$\ddot{r} = -0.8\left[\sin \theta\, \ddot{\theta} + \cos \theta\, \dot{\theta}^2\right]$$

$$= -0.8[\sin 45°(0) + \cos 45°(3^2)] = -5.091 \text{ m/s}^2$$

We have

$$a_r = \ddot{r} - r\dot{\theta}^2 = -5.091 \text{ m/s}^2 - (0.5657 \text{ m})(3 \text{ rad/s})^2 = -10.18 \text{ m/s}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (0.5657 \text{ m})(0) + 2(-1.6971 \text{ m/s})(3 \text{ rad/s})$$

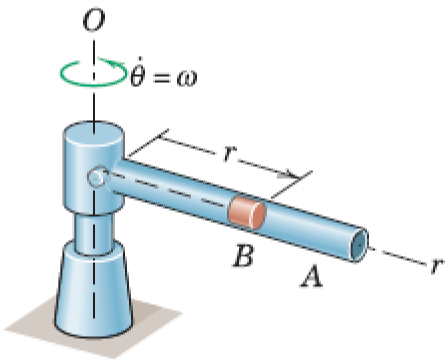
$$= -10.18 \text{ m/s}^2$$

Substituting these results into Eqs. (1) and (2) and solving, we get

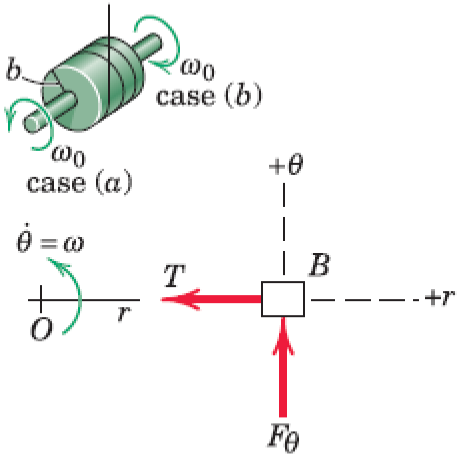
$$N_C = 7.20 \text{ N}$$

$$F = 0 \qquad \textit{Ans.}$$

Example 8: Tube *A* rotates about the vertical *O*-axis with a constant angular rate $\dot{\theta} = \omega$ and contains a small cylindrical plug *B* of mass *m* whose radial position is controlled by the cord which passes freely through the tube and shaft and is wound around the drum of radius *b*. Determine the tension *T* in the cord and the horizontal component F_θ of force exerted by the tube on the plug if the constant angular rate of rotation of the drum is ω_0 first in the direction for case (*a*) and second in the direction for case (*b*). Neglect friction.



SOLUTION:



$[\Sigma F_r = ma_r]$ $$-T = m(\ddot{r} - r\dot{\theta}^2)$$

$[\Sigma F_\theta = ma_\theta]$ $$F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

Case (a). With $\dot{r} = +b\omega_0$, $\ddot{r} = 0$, and $\ddot{\theta} = 0$, the forces become

$$T = mr\omega^2 \qquad F_\theta = 2mb\omega_0\omega$$

Case (b). With $\dot{r} = -b\omega_0$, $\ddot{r} = 0$, and $\ddot{\theta} = 0$, the forces become

$$T = mr\omega^2 \qquad F_\theta = -2mb\omega_0\omega$$