CHAPTER TWO
KINETICS OF PARTICLES: IMPULSE AND MOMENTUM

2-14 Principle of Linear Impulse and Momentum

Using kinematics, the equation of motion for a particle of mass m can be written as

$$\Sigma \mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}$$

Where **a** and **v** are both measured from an inertial frame of reference. Rearranging the terms and integrating between the limits $\mathbf{v} = \mathbf{v}_1$ at $t = t_1$ and $\mathbf{v} = \mathbf{v}_2$ at $t = t_2$, we have

$$\Sigma \int_{t_1}^{t_2} \mathbf{F} dt = m \int_{\mathbf{v}_1}^{\mathbf{v}_2} d\mathbf{v} \qquad \Sigma \int_{t_1}^{t_2} \mathbf{F} dt = m\mathbf{v}_2 - m\mathbf{v}_1$$

………(1)

This equation is referred to as the *principle of linear impulse and momentum*.
Each of the two vectors of the form $\mathbf{L} = m\mathbf{v}$ in Eq. 1 is referred to as the particle's linear momentum. The integral $\mathbf{I} = \int \mathbf{F}\, dt$ in Eq. 1is referred to as the *linear impulse*.
For problem solving, Eq. 1will be rewritten in the form

$$m\mathbf{v}_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}\, dt = m\mathbf{v}_2$$

equation states that the initial linear momentum of the system plus the impulses of all the *external forces* acting on the system from t_1 to t_2 is equal to the system's final linear momentum.

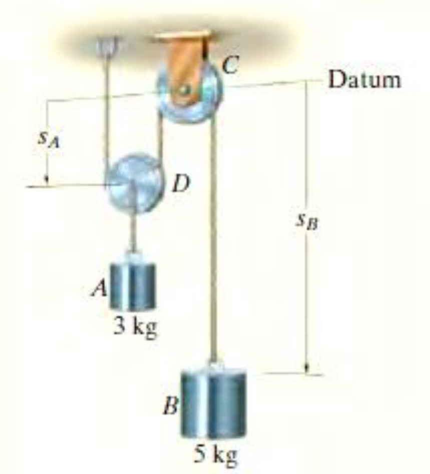
$$\Sigma m_i(\mathbf{v}_i)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}_i\, dt = \Sigma m_i(\mathbf{v}_i)_2$$

……..(2)

When the sum of the *external impulses* acting on a system of particles is *zero*, Eq. 2 reduces to a simplified form, namely,

$$\Sigma m_i(\mathbf{v}_i)_1 = \Sigma m_i(\mathbf{v}_i)_2$$

This equation is referred to as the *conservation of linear momentum*.



Example 14: Blocks *A* and *B* shown in Fig. have a mass of 3 kg and 5 kg, respectively. If the system is released from rest, determine the velocity of block *B* in 6 s. Neglect the mass of the pulleys and cord.

SOLUTION:

Since the weight of each block is constant, the cord tensions will also be constant. Furthermore, since the mass of pulley *D* is neglected, the cord tension $T_A = 2T_B$. Note that the blocks are both assumed to be moving downward in the positive coordinate directions, s_A and s_B .

Principle of Impulse and Momentum.

Block *A*:

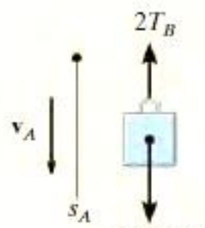
$(+\downarrow)$ $$m(v_A)_1 + \Sigma\int_{t_1}^{t_2} F_y\,dt = m(v_A)_2$$

$$0 - 2T_B(6\text{ s}) + 3(9.81)\text{ N}(6\text{ s}) = (3\text{ kg})(v_A)_2 \qquad (1)$$

Block *B*:

$(+\downarrow)$ $$m(v_B)_1 + \Sigma\int_{t_1}^{t_2} F_y\,dt = m(v_B)_2$$

$$0 + 5(9.81)\text{ N}(6\text{ s}) - T_B(6\text{ s}) = (5\text{ kg})(v_B)_2 \qquad (2)$$



$$2s_A + s_B = l$$

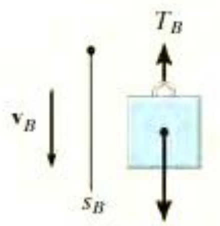
Taking the time derivative yields

$$2v_A = -v_B \qquad (3)$$

As indicated by the negative sign, when *B* moves downward *A* moves upward. Substituting this result into Eq. 1 and solving Eqs. 1 and 2 yields

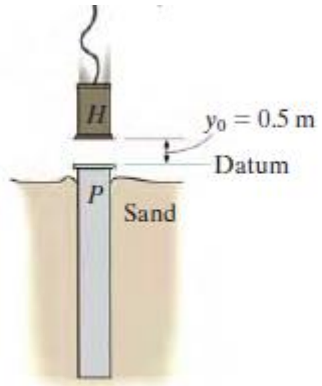
$$(v_B)_2 = 35.8\text{ m/s}\downarrow \qquad \textit{Ans.}$$

$$T_B = 19.2\text{ N}$$



Example 15: An 800-kg rigid pile is driven into the ground using a 300-kg hammer. The hammer falls from rest at a height y_0 = 0.5 m and strikes the top of the pile. Determine the impulse which the pile exerts on the hammer if the pile is surrounded entirely by loose sand so that after striking, the hammer does *not* rebound off the pile.

SOLUTION



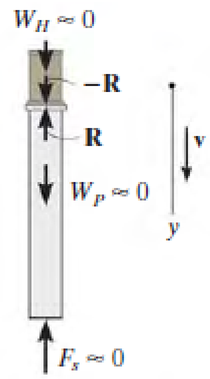
Conservation of Energy. The velocity at which the hammer strikes the pile can be determined using the conservation of energy
equation applied to the hammer. With the datum at the top of the pile, Fig. 15–10*a*, we have

$$T_0 + V_0 = T_1 + V_1$$

$$\frac{1}{2}m_H(v_H)_0^2 + W_H y_0 = \frac{1}{2}m_H(v_H)_1^2 + W_H y_1$$

$$0 + 300(9.81)\text{ N}(0.5\text{ m}) = \frac{1}{2}(300\text{ kg})(v_H)_1^2 + 0$$

$$(v_H)_1 = 3.132\text{ m/s}$$



Conservation of Momentum. Since the hammer does not rebound off the pile just after collision, then $(v_H)_2 = (v_P)_2 = v_2$.

$$(+\downarrow) \qquad m_H(v_H)_1 + m_P(v_P)_1 = m_H v_2 + m_P v_2$$

$$(300\text{ kg})(3.132\text{ m/s}) + 0 = (300\text{ kg})v_2 + (800\text{ kg})v_2$$

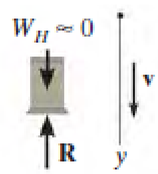
$$v_2 = 0.8542\text{ m/s}$$

Principle of Impulse and Momentum. The impulse which the pile imparts to the hammer can now be determined since v_2 is known. From the free-body diagram for the hammer, Fig. *c*, we have

$$(+\downarrow) \qquad m_H(v_H)_1 + \Sigma\int_{t_1}^{t_2} F_y\,dt = m_H v_2$$

$$(300\text{ kg})(3.132\text{ m/s}) - \int R\,dt = (300\text{ kg})(0.8542\text{ m/s})$$

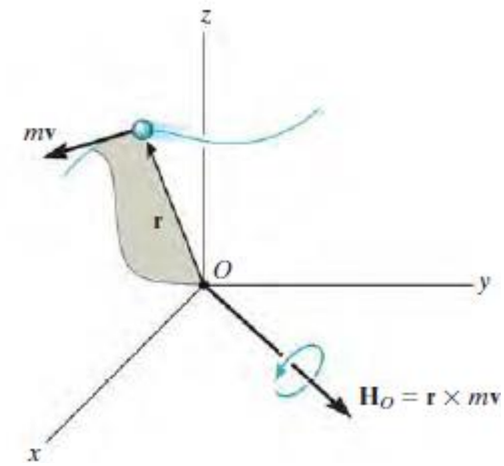
$$\int R\,dt = 683\text{ N}\cdot\text{s} \qquad \textit{Ans.}$$



NOTE: The equal but opposite impulse acts on the pile. Try finding this impulse by applying the principle of impulse and momentum to the pile.

2-15 Angular Momentum

The *angular momentum* of a particle about point *O* is defined as the "moment" of the particle's linear momentum about *O*. Since this concept is analogous to finding the moment of a force about a point, the angular momentum, $\mathbf{H}_O$, is sometimes referred to as the *moment of momentum*.



$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \quad \text{……..(1)}$$

$$\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O \quad \text{……..(2)}$$

The equation 2 states that *the resultant moment about point O of all the forces acting on the particle is equal to the time rate of change of the particle's angular momentum about point O.*

If Eq. 2 is rewritten in the form $\sum \mathbf{M}_O\, dt = d\,\mathbf{H}_O$ and integrated, assuming that at time $t = t_1$, $\mathbf{H}_O = (\mathbf{H}_O)_1$ and at time $t = t_2$, $\mathbf{H}_O = (\mathbf{H}_O)_2$, we have

$$\Sigma \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1 \qquad (\mathbf{H}_O)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2 \quad \text{…….(3)}$$

$$\text{angular impulse} = \int_{t_1}^{t_2} \mathbf{M}_O\, dt = \int_{t_1}^{t_2} (\mathbf{r} \times \mathbf{F})\, dt$$

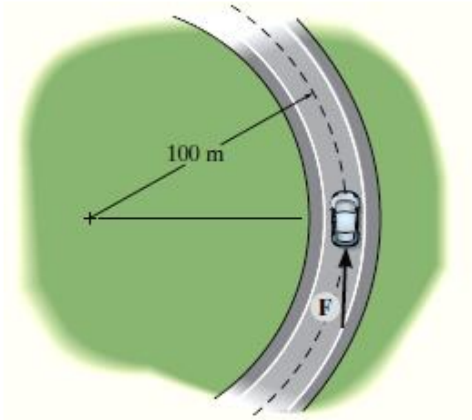
When the angular impulses acting on a particle are all zero during the time t_1 to t_2, Eq. 3 reduces to the following simplified form:

$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2$$

This equation is known as the *conservation of angular momentum*. we can also write the conservation of angular momentum for a system of particles as

$$\Sigma(\mathbf{H}_O)_1 = \Sigma(\mathbf{H}_O)_2$$

Example 16: The 1.5-Mg car travels along the circular road as shown in Fig. If the traction force of the wheels on the road is $F = (150t^2)$ N, where t is in seconds, determine the speed of the car when $t = 5$ s. The car initially travels with a speed of 5 m/s. Neglect the size of the car.



SOLUTION:

Principle of Angular Impulse and Momentum.

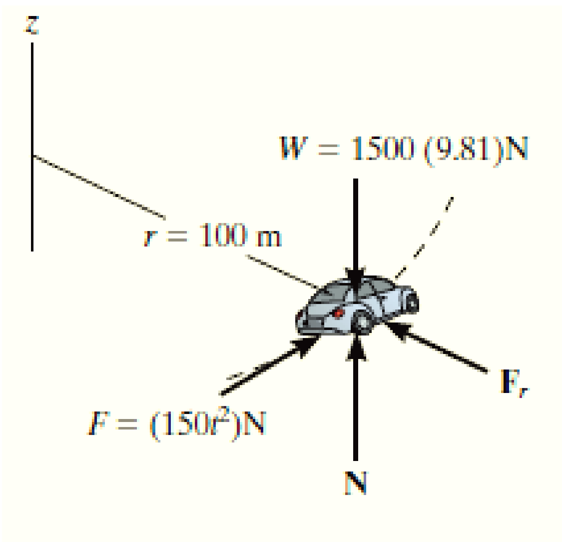
$$(H_z)_1 + \Sigma \int_{t_1}^{t_2} M_z\, dt = (H_z)_2$$

$$r\, m_c (v_c)_1 + \int_{t_1}^{t_2} r\, F\, dt = r\, m_c (v_c)_2$$

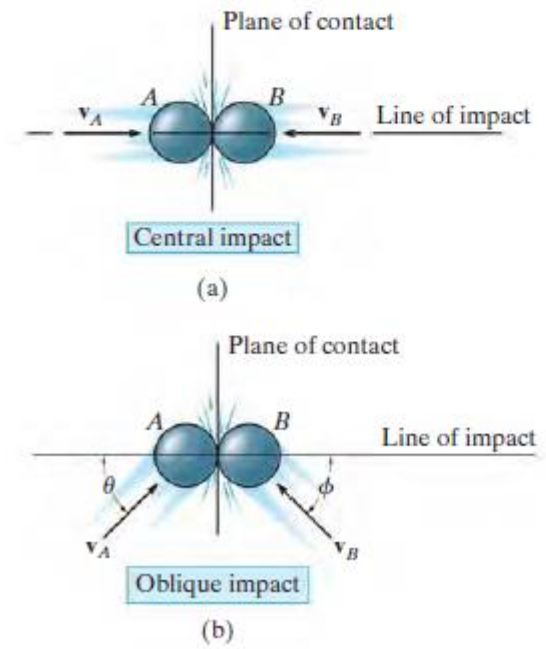
$$(100\text{ m})(1500\text{ kg})(5\text{ m/s}) + \int_0^{5\text{ s}} (100\text{ m})[(150t^2)\text{ N}]\, dt = (100\text{ m})(1500\text{ kg})(v_c)_2$$

$$750(10^3) + 5000t^3\Big|_0^{5\text{ s}} = 150(10^3)(v_c)_2$$

$$(v_c)_2 = 9.17\text{ m/s} \qquad \textit{Ans.}$$



2-15 Impact



Impact occurs when two bodies collide with each other during a very *short* period of time, causing relatively large (impulsive) forces to be exerted between the bodies. The striking of a hammer on a nail, or a golf club on a ball, are common examples of impact loadings.

In general, there are two types of impact. *Central impact* occurs when the direction of motion of he mass centers of the two colliding particles is along a line passing through the mass centers of the particles. This line is called the *line of impact*, which is perpendicular to the plane of contact, Fig. *a*. When the motion of one or both of the particles make an angle with the line of impact, Fig. *b*, the impact is said to be *oblique impact*.

2-15-1 Central Impact

In most cases the *final velocities* of two smooth particles are to be determined *just after* they are subjected to direct central impact. Provided the coefficient of restitution, the mass of each particle, and each particle's initial velocity *just before* impact are known, the solution to this problem can be obtained using the following two equations:

- The conservation of momentum applies to the system of particles,

$$\sum mv_1 = \sum mv_2$$

- The coefficient of restitution *e*,

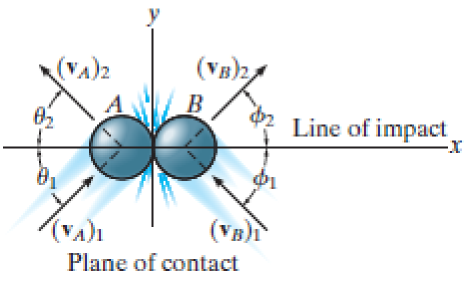
$$e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$$

e is equal to the ratio of the relative velocity of the particles' separation *just after impact*, $(v_B)_2 - (v_A)_2$, to the relative velocity of the particles' approach *just before impact*, $(v_B)_1 - (v_A)_1$. By measuring these relative velocities experimentally, it has been found that *e* varies appreciably with impact velocity as well as with the size and shape of the colliding bodies. For these reasons the coefficient of restitution is reliable only when used with data which closely approximate the conditions which were known to exist when measurements of it were made. In general *e* has a value between zero and one, and one should be aware of the physical meaning of these two limits. If the solution yields a negative magnitude, the velocity acts in the opposite sense.

- If the collision between the two particles is *perfectly elastic*, the deformation impulse (∫**P** *dt*) is equal and opposite to the restitution impulse (∫**R** *dt*). Although in reality this can never be achieved, $e = 1$ for an elastic collision.
- The impact is said to be *inelastic or plastic* when $e = 0$. In this case there is no restitution impulse (∫**R** *dt*= **0**), so that after collision both particles couple or stick *together* and move with a common velocity.

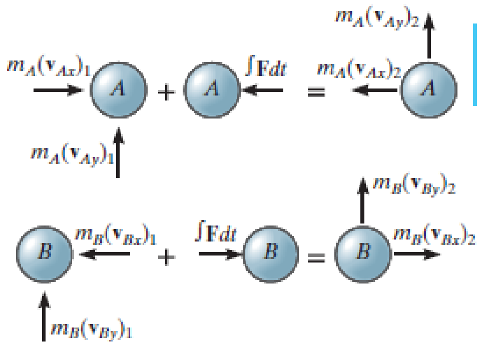
2-15-2 Oblique Impact

When oblique impact occurs between two smooth particles, the particles move away from each other with velocities having unknown directions as well as unknown magnitudes. Provided the initial velocities are known, then four unknowns are present in the problem. As shown in Fig. *a*, these unknowns may be represented either as $(v_A)_2$, $(v_B)_2$, θ_2, and ϕ_2 , or as the *x* and *y* components of the final velocities.



(a)

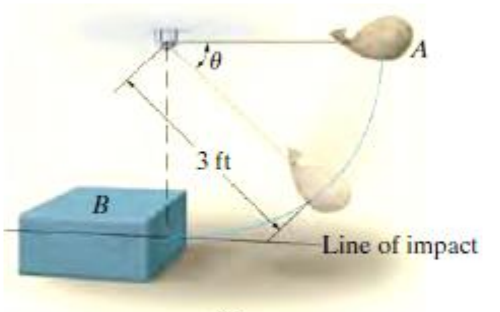
If the *y* axis is established within the plane of contact and the *x* axis along the line of impact, the impulsive forces of deformation and restitution act *only in the x direction*, Fig. *b*. By resolving the velocity or momentum vectors into components along the *x* and *y* axes, Fig. *b*, it is then possible to write four independent scalar equations in order to determine $(v_{Ax})_2$, $(v_{Ay})_2$, $(v_{Bx})_2$, and $(v_{By})_2$.



(b)

- Momentum of the system is conserved *along the line of impact*, *x* axis, so that $\sum mv_1 = \sum mv_2$.
- The coefficient of restitution, $e = [(v_{Bx})_2 - (v_{Ax})_2]/[(v_{Ax})_1 - (v_{Bx})_1]$, relates the relative-velocity *components* of the particles *along the line of impact* (*x* axis).
- If these two equations are solved simultaneously, we obtain $(v_{Ax})_2$ and $(v_{Bx})_2$.
- Momentum of particle *A* is conserved along the *y* axis, perpendicular to the line of impact, since no impulse acts on particle *A* in this direction. As a result $m_A(v_{Ay})_1 = m_A(v_{Ay})_2$ or $(v_{Ay})_1 = (v_{Ay})_2$
- Momentum of particle *B* is conserved along the *y* axis, perpendicular to the line of impact, since no impulse acts on particle *B* in this direction. Consequently $(v_{By})_1 = (v_{By})_2$.

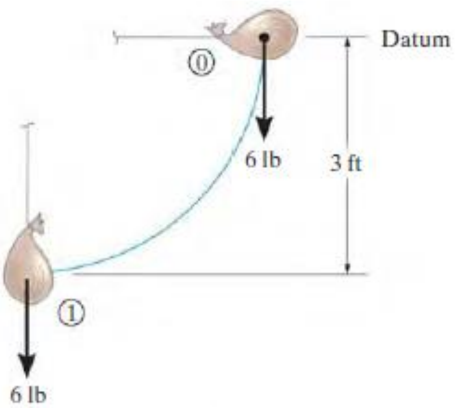
Example 17: The bag *A*, having a weight of 6 lb, is released from rest at the position $\theta = 0^o$, as shown in Fig. *a*. After falling to $\theta = 90^o$, it strikes an 18-lb box *B*. If the oefficient of restitution between the bag and box is *e* = 0.5, determine the velocities of the bag and box just after impact. What is the loss of energy during collision?



(a)

SOLUTION

This problem involves central impact. Why? Before analyzing the mechanics of the impact, however, it is first necessary to obtain the
velocity of the bag *just before* it strikes the box.



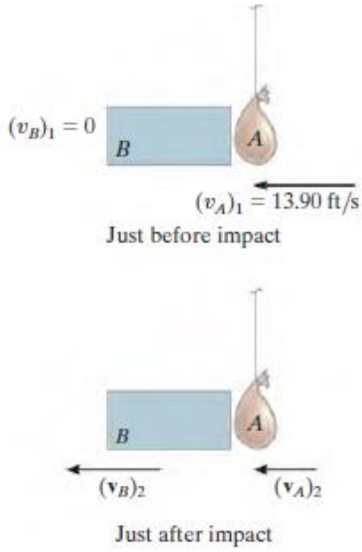
(b)

Conservation of Energy. With the datum at $\theta = 0^o$, Fig. b, we have

$$T_0 + V_0 = T_1 + V_1$$

$$0 + 0 = \frac{1}{2}\left(\frac{6\text{ lb}}{32.2\text{ ft/s}^2}\right)(v_A)_1^2 - 6\text{ lb}(3\text{ ft});(v_A)_1 = 13.90\text{ ft/s}$$

Conservation of Momentum. After impact we will assume *A* and *B* travel to the left. Applying the conservation of momentum to the system, Fig. *c*, we have



(c)

$$(\overset{+}{\leftarrow}) \qquad m_B(v_B)_1 + m_A(v_A)_1 = m_B(v_B)_2 + m_A(v_A)_2$$

$$0 + \left(\frac{6\text{ lb}}{32.2\text{ ft/s}^2}\right)(13.90\text{ ft/s}) = \left(\frac{18\text{ lb}}{32.2\text{ ft/s}^2}\right)(v_B)_2 + \left(\frac{6\text{ lb}}{32.2\text{ ft/s}^2}\right)(v_A)_2$$

$$(v_A)_2 = 13.90 - 3(v_B)_2 \qquad (1)$$

Coefficient of Restitution. Realizing that for separation to occur after collision $(v_B)_2 > (v_A)_2$, Fig. *c*, we have

$$(\overset{+}{\leftarrow}) \qquad e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}; \qquad 0.5 = \frac{(v_B)_2 - (v_A)_2}{13.90\text{ ft/s} - 0}$$

$$(v_A)_2 = (v_B)_2 - 6.950 \qquad (2)$$

Solving Eqs. 1 and 2 simultaneously yields

$(v_A)_2 = -1.74\text{ ft/s} = 1.74\text{ ft/s} \rightarrow$ and $(v_B)_2 = 5.21\text{ ft/s} \leftarrow$ *Ans.*

Loss of Energy. Applying the principle of work and energy to the bag and box just before and just after collision, we have

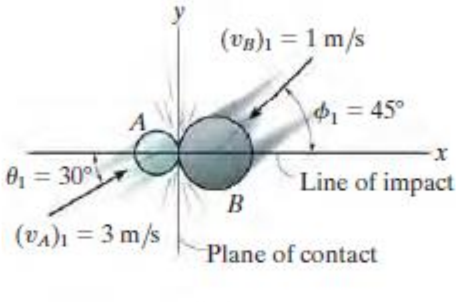
$$\Sigma U_{1-2} = T_2 - T_1;$$

$$\Sigma U_{1-2} = \left[\frac{1}{2}\left(\frac{18\text{ lb}}{32.2\text{ ft/s}^2}\right)(5.21\text{ ft/s})^2 + \frac{1}{2}\left(\frac{6\text{ lb}}{32.2\text{ ft/s}^2}\right)(1.74\text{ ft/s})^2\right] - \left[\frac{1}{2}\left(\frac{6\text{ lb}}{32.2\text{ ft/s}^2}\right)(13.9\text{ ft/s})^2\right]$$

$$\Sigma U_{1-2} = -10.1\text{ ft}\cdot\text{lb} \qquad \textit{Ans.}$$

NOTE: The energy loss occurs due to inelastic deformation during the collision.

Example 18: Two smooth disks *A* and *B*, having a mass of 1 kg and 2 kg, respectively, collide with the velocities shown in Fig. *a*. If the coefficient of restitution for the disks is *e* = 0.75, determine the *x* and *y* components of the final velocity of each disk just after collision.



(a)

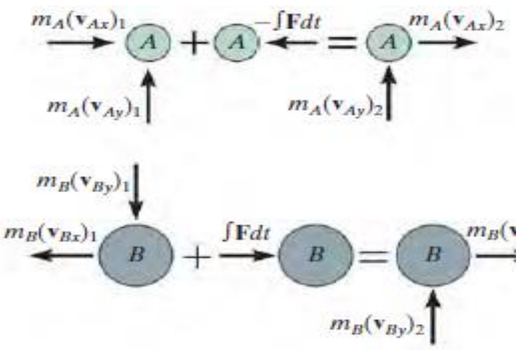
SOLUTION

This problem involves *oblique impact*. Why? In order to solve it, we have established the *x* and *y* axes along the line of impact and the plane of contact, respectively, Fig. *a*. Resolving each of the initial velocities into *x* and *y* components, we have

$$(v_{Ax})_1 = 3\cos 30° = 2.598\text{ m/s} \qquad (v_{Ay})_1 = 3\sin 30° = 1.50\text{ m/s}$$

$$(v_{Bx})_1 = -1\cos 45° = -0.7071\text{ m/s} \qquad (v_{By})_1 = -1\sin 45° = -0.7071\text{ m/s}$$

The four unknown velocity components after collision are *assumed to act in the positive directions*, Fig. *b*. Since the impact occurs in the *x* direction (line of impact), the conservation of momentum for *both* disks can be applied in this direction.



(b)

Conservation of "x" Momentum. In reference to the momentum diagrams, we have

$$(\overset{+}{\rightarrow}) \qquad m_A(v_{Ax})_1 + m_B(v_{Bx})_1 = m_A(v_{Ax})_2 + m_B(v_{Bx})_2$$

$$1\text{ kg}(2.598\text{ m/s}) + 2\text{ kg}(-0.707\text{ m/s}) = 1\text{ kg}(v_{Ax})_2 + 2\text{ kg}(v_{Bx})_2$$

$$(v_{Ax})_2 + 2(v_{Bx})_2 = 1.184 \qquad (1)$$

Coefficient of Restitution (x).

$$(\overset{+}{\rightarrow}) \qquad e = \frac{(v_{Bx})_2 - (v_{Ax})_2}{(v_{Ax})_1 - (v_{Bx})_1}; \; 0.75 = \frac{(v_{Bx})_2 - (v_{Ax})_2}{2.598 \text{ m/s} - (-0.7071 \text{ m/s})}$$

$$(v_{Bx})_2 - (v_{Ax})_2 = 2.482 \qquad (2)$$

Solving Eqs. 1 and 2 for $(v_{Ax})_2$ and $(v_{Bx})_2$ yields

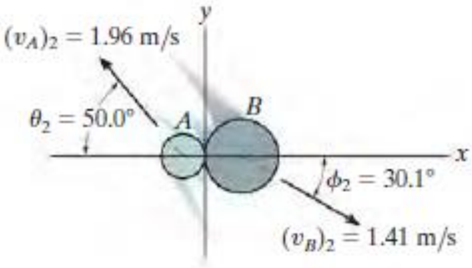
$(v_{Ax})_2 = -1.26 \text{ m/s} = 1.26 \text{ m/s} \leftarrow \quad (v_{Bx})_2 = 1.22 \text{ m/s} \rightarrow$ *Ans.*

Conservation of "*y*" Momentum. The momentum of *each disk* is *conserved* in the *y* direction (plane of contact), since the disks are smooth and therefore *no* external impulse acts in this direction. From Fig. *b*,

$(+\uparrow)\; m_A(v_{Ay})_1 = m_A(v_{Ay})_2; \quad (v_{Ay})_2 = 1.50 \text{ m/s} \uparrow$ *Ans.*

$(+\uparrow)\; m_B(v_{By})_1 = m_B(v_{By})_2; \quad (v_{By})_2 = -0.707 \text{ m/s} = 0.707 \text{ m/s} \downarrow$ *Ans.*

NOTE: Show that when the velocity components are summed vectorially, one obtains the results shown in Fig. *c*.



(c)