Chapter Three
Kinematics of Rigid Bodies

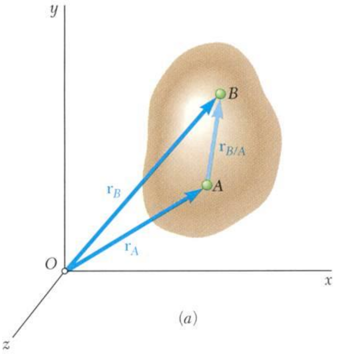
3-1 Introduction to Dynamics

Kinematics of rigid bodies: relations between time and the positions, velocities, and accelerations of the particles forming a rigid body.

Classification of rigid body motions:

- translation:
  - rectilinear translation
  - curvilinear translation
- rotation about a fixed axis
- general plane motion
- motion about a fixed point
- general motion

3-2 Translation



(a)

- Consider rigid body in translation:
  - direction of any straight line inside the body is constant,
  - all particles forming the body move in parallel lines.

- For any two particles in the body:

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

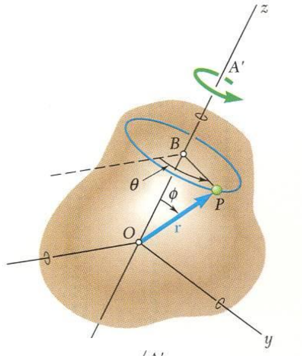
- Differentiating with respect to time: $\dot{\vec{r}}_B = \dot{\vec{r}}_A + \dot{\vec{r}}_{B/A} = \dot{\vec{r}}_A \ldots \vec{v}_B = \vec{v}_A$

  Then all particles have the same velocity.
- Differentiating with respect to time again:

  $\ddot{\vec{r}}_B = \ddot{\vec{r}}_A + \ddot{\vec{r}}_{B/A} = \ddot{\vec{r}}_A \ldots\ldots\ldots \vec{a}_B = \vec{a}_A$

And all particles have the same acceleration.

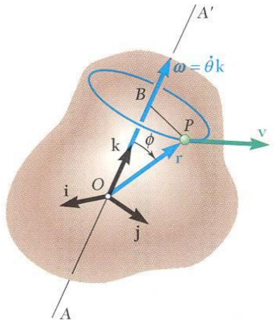
3-3 Rotation About a Fixed Axis: Velocity and Acceleration



Consider rotation of rigid body about a fixed axis *AA'*. Velocity vector of the particle *P* is tangent to the path with magnitude:

$$\Delta s = (BP)\Delta\theta = (r\sin\phi)\Delta\theta$$

$$v = \frac{ds}{dt} = \lim_{\Delta t \to 0}(r\sin\phi)\frac{\Delta\theta}{\Delta t} = r\dot{\theta}\sin\phi$$



The same result is obtained from:

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r}$$

$$\vec{\omega} = \omega\vec{k} = \dot{\theta}\vec{k} = angular\ velocity$$

Differentiating to determine the acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega}\times\vec{r}) = \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt} = \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\vec{v}$$

$$\frac{d\vec{\omega}}{dt} = \vec{\alpha} = angular\ acceleration = \alpha\vec{k} = \dot{\omega}\vec{k} = \ddot{\theta}\vec{k}$$

Acceleration of *P* is combination of two vectors: $\vec{a} = \vec{\alpha}\times\vec{r} + \vec{\omega}\times\vec{\omega}\times\vec{r}$

$\vec{\alpha}\times\vec{r} = \text{tangential acceleration component}$

$\vec{\omega}\times\vec{\omega}\times\vec{r} = \text{radial acceleration component}$

Motion of a rigid body rotating around a fixed axis is often specified by the type of angular acceleration.

$$\omega = \frac{d\theta}{dt} \qquad \text{or} \qquad dt = \frac{d\theta}{\omega} ..... \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega\frac{d\omega}{d\theta}$$

- *Uniform Rotation, a* = 0: $\theta = \theta_0 + \omega t$
- *Uniformly Accelerated Rotation, a = constant:*

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$

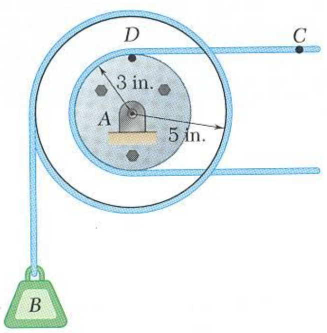
$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

3-4 Comparison Between Rotational and Linear Equations

The kinematics equations for rotational and translation motion:

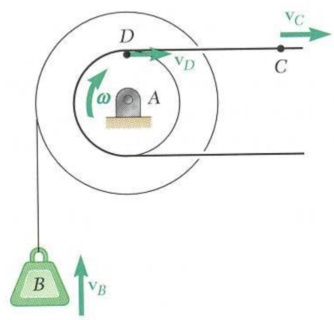
Rigid Body Under Constant Angular Acceleration	Particle Under Constant Acceleration
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

Example1: Cable C has a constant acceleration of 9 in/s2 and an initial velocity of 12 in/s, both directed to the right. Determine (a) the number of revolutions of the pulley in 2 s, (b) the velocity and change in position of the load B after 2 s, and (c) the acceleration of the point D on the rim of the inner pulley at t = 0.



SOLUTION:

The tangential velocity and acceleration of *D* are equal to the velocity and acceleration of *C*.

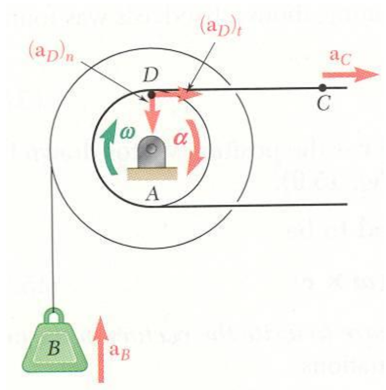


$$(\vec{v}_D)_0 = (\vec{v}_C)_0 = 12\,\text{in./s} \rightarrow \qquad (\vec{a}_D)_t = \vec{a}_C = 9\,\text{in./s} \rightarrow$$

$$(v_D)_0 = r\omega_0 \qquad (a_D)_t = r\alpha$$

$$\omega_0 = \frac{(v_D)_0}{r} = \frac{12}{3} = 4\,\text{rad/s} \qquad \alpha = \frac{(a_D)_t}{r} = \frac{9}{3} = 3\,\text{rad/s}^2$$

Apply the relations for uniformly accelerated rotation to determine velocity and angular position of pulley after 2 s.



$$\omega = \omega_0 + \alpha t = 4\,\text{rad/s} + (3\,\text{rad/s}^2)(2\,\text{s}) = 10\,\text{rad/s}$$

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 = (4\,\text{rad/s})(2\,\text{s}) + \tfrac{1}{2}(3\,\text{rad/s}^2)(2\,\text{s})^2 = 14\,\text{rad}$$

$$N = (14\,\text{rad})\left(\frac{1\,\text{rev}}{2\pi\,\text{rad}}\right) = \text{number of revs} \qquad N = 2.23\,\text{rev}$$

$$v_B = r\omega = (5\,\text{in.})(10\,\text{rad/s}) \ldots\ldots\ldots \vec{v}_B = 50\,\text{in./s} \uparrow$$

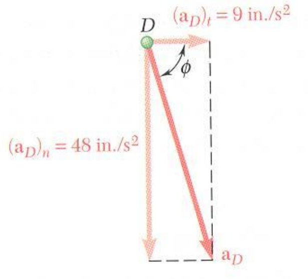
$$\Delta y_B = r\theta = (5\,\text{in.})(14\,\text{rad}) \ldots\ldots\ldots\ldots \Delta y_B = 70\,\text{in.}$$

Evaluate the initial tangential and normal acceleration components of D.

$$(\vec{a}_D)_t = \vec{a}_C = 9\,\text{in./s} \rightarrow$$

$$(a_D)_n = r_D\omega_0^2 = (3\,\text{in.})(4\,\text{rad/s})^2 = 48\,\text{in/s}^2 \downarrow$$

Magnitude and direction of the total acceleration:

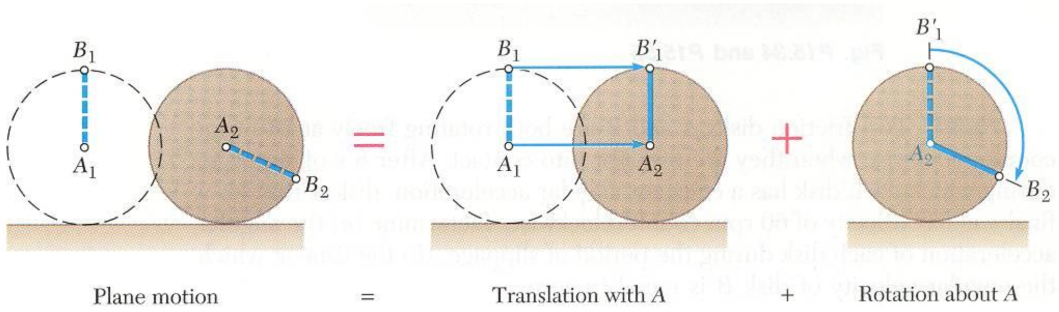


$$a_D = \sqrt{(a_D)_t^2 + (a_D)_n^2} = \sqrt{9^2 + 48^2} = 48.8\,\text{in./s}^2$$

$$\tan\phi = \frac{(a_D)_n}{(a_D)_t} = \frac{48}{9} \quad \ldots\ldots\ldots\ \phi = 79.4°$$

3-5 General Plane Motion

General plane motion is neither a translation nor a rotation. General plane motion can be considered as the *sum* of a translation and rotation.

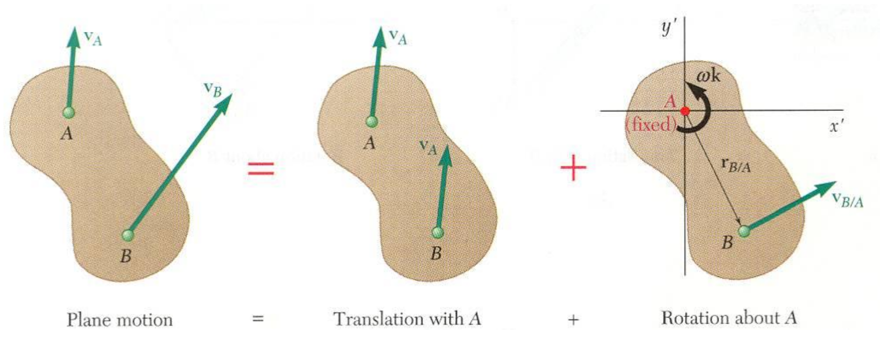


Displacement of particles *A* and *B* to A_2 and B_2 can be divided into two parts:

- translation to A_2 and
- rotation of about A_2 to B_2

3-6 Absolute and Relative Velocity in Plane Motion

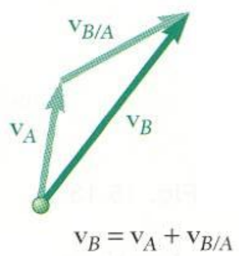
Any plane motion can be replaced by a translation of an arbitrary reference point *A* and a simultaneous rotation about *A*.



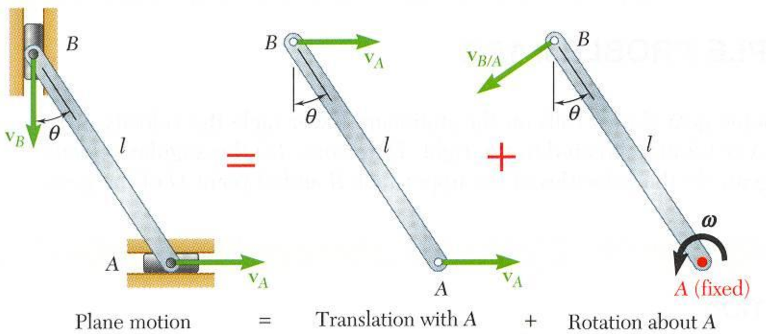
$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

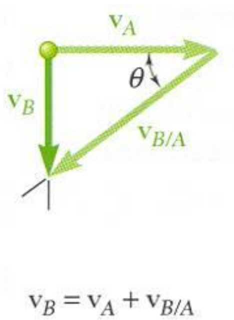
$$\vec{v}_{B/A} = \omega\vec{k} \times \vec{r}_{B/A} \qquad v_{B/A} = r\omega$$

$$\vec{v}_B = \vec{v}_A + \omega\vec{k} \times \vec{r}_{B/A}$$



Assuming that the velocity v_A of end A is known, wish to determine the velocity v_B of end B and the angular velocity ω in terms of v_A, l, and θ.



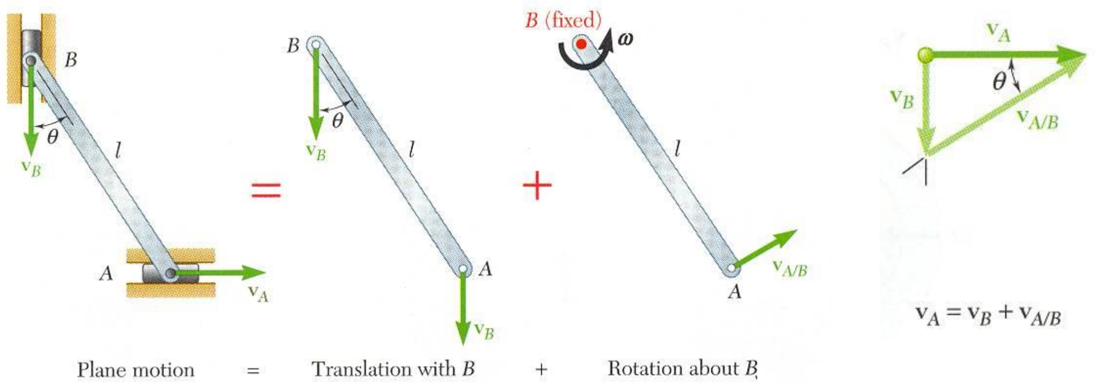


The direction of v_B and $v_{B/A}$ are known. Complete the velocity diagram.

$$\frac{v_B}{v_A} = \tan\theta \qquad \frac{v_A}{v_{B/A}} = \frac{v_A}{l\omega} = \cos\theta$$

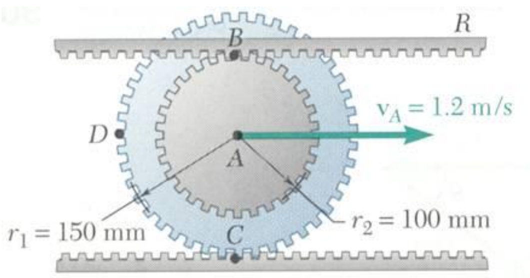
$$v_B = v_A \tan\theta \qquad \omega = \frac{v_A}{l\cos\theta}$$

Selecting point B as the reference point and solving for the velocity v_A of end A and the angular velocity ω leads to an equivalent velocity triangle.



$v_{A/B}$ has the same magnitude but opposite sense of $v_{B/A}$. The sense of the relative velocity is dependent on the choice of reference point. Angular velocity ω of the rod in its rotation about B is the same as its rotation about A. Angular velocity is not dependent on the choice of reference point.

Example 2: The double gear rolls on the stationary lower rack: the velocity of its center is 1.2 m/s. Determine *(a)* the angular velocity of the gear, and *(b)* the velocities of the upper rack R and point D of the gear.



SOLUTION:

The displacement of the gear center in one revolution is equal to the outer circumference. For $x_A > 0$ (moves to right), $\theta < 0$ (rotates clockwise).

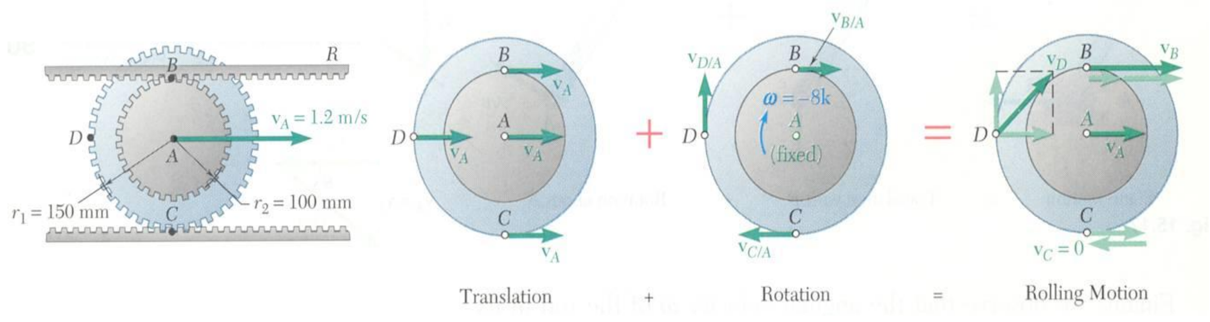
$$\frac{x_A}{2\pi r} = -\frac{\theta}{2\pi} \qquad x_A = -r_1\theta$$

Differentiate to relate the translational and angular velocities.

$$v_A = -r_1\omega$$

$$\omega = -\frac{v_A}{r_1} = -\frac{1.2\,\text{m/s}}{0.150\,\text{m}} \qquad \vec{\omega} = \omega\vec{k} = -(8\,\text{rad/s})\vec{k}$$

For any point *P* on the gear:



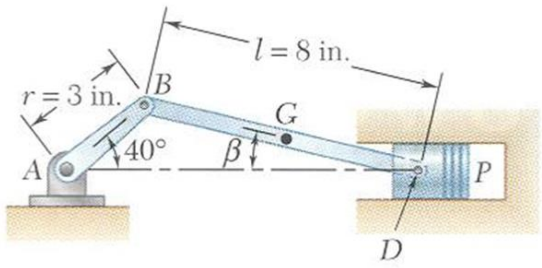
Velocity of the upper rack is equal to velocity of point B:

$$\vec{v}_R = \vec{v}_B = \vec{v}_A + \omega\vec{k} \times \vec{r}_{B/A}$$
$$= (1.2\,\mathrm{m/s})\vec{i} + (8\,\mathrm{rad/s})\vec{k} \times (0.10\,\mathrm{m})\vec{j}$$
$$= (1.2\,\mathrm{m/s})\vec{i} + (0.8\,\mathrm{m/s})\vec{i} = (2\,\mathrm{m/s})\vec{i}$$

Velocity of the point *D*:

$$\vec{v}_D = \vec{v}_A + \omega\vec{k} \times \vec{r}_{D/A}$$
$$= (1.2\,\mathrm{m/s})\vec{i} + (8\,\mathrm{rad/s})\vec{k} \times (-0.150\,\mathrm{m})\vec{i}$$
$$= (1.2\,\mathrm{m/s})\vec{i} + (1.2\,\mathrm{m/s})\vec{j} \quad v_D = 1.697\,\mathrm{m/s}$$

Example 3: The crank *AB* has a constant clockwise angular velocity of 2000 rpm. For the crank position indicated, determine *(a)* the angular velocity of the connecting rod *BD*, and *(b)* the velocity of the piston *P*.

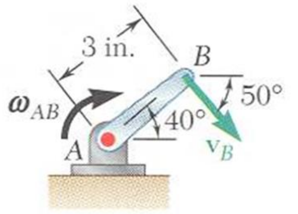


SOLUTION:

The velocity $\vec{v}_B$ is obtained from the crank rotation data:

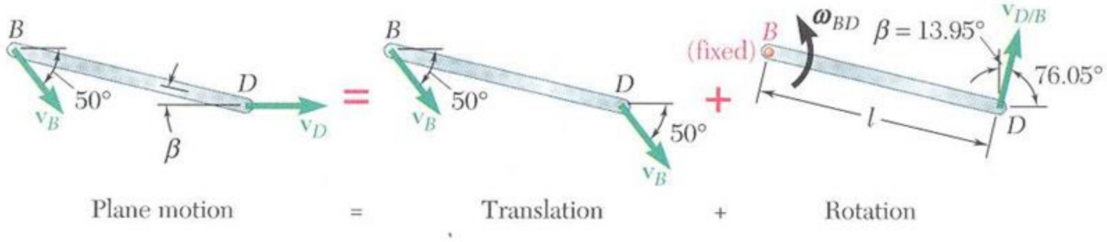
$$\omega_{AB} = \left(2000\frac{\mathrm{rev}}{\mathrm{min}}\right)\left(\frac{\mathrm{min}}{60\,\mathrm{s}}\right)\left(\frac{2\pi\,\mathrm{rad}}{\mathrm{rev}}\right) = 209.4\,\mathrm{rad/s}$$

$$v_B = (AB)\omega_{AB} = (3\,\mathrm{in.})(209.4\,\mathrm{rad/s})$$

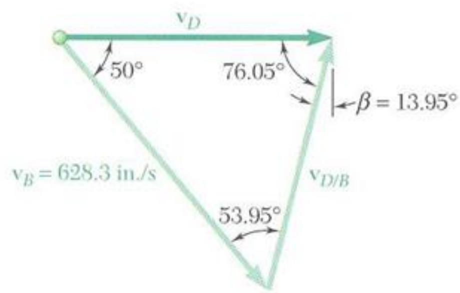


The direction of the absolute velocity $\vec{v}_D$ is horizontal. The direction of the relative velocity $\vec{v}_{D/B}$ is perpendicular to *BD*. Compute the angle between the horizontal and the connecting rod from the law of sines.

$$\frac{\sin 40^\circ}{8\,\mathrm{in.}} = \frac{\sin\beta}{3\,\mathrm{in.}} \qquad \beta = 13.95^\circ$$



Determine the velocity magnitudes from the vector triangle:



$$\frac{v_D}{\sin 53.95°} = \frac{v_{D/B}}{\sin 50°} = \frac{628.3\,\text{in./s}}{\sin 76.05°}$$

$$v_D = 523.4\,\text{in./s} = 43.6\,\text{ft/s} \qquad \ldots\ldots\ldots v_P = v_D = 43.6\,\text{ft/s}$$

$$v_{D/B} = 495.9\,\text{in./s}$$

$$v_{D/B} = l\omega_{BD} \ldots\ldots\ldots \omega_{BD} = \frac{v_{D/B}}{l} = \frac{495.9\,\text{in./s}}{8\,\text{in.}} = 62.0\,\text{rad/s}$$

$$\vec{\omega}_{BD} = (62.0\,\text{rad/s})\vec{k}$$