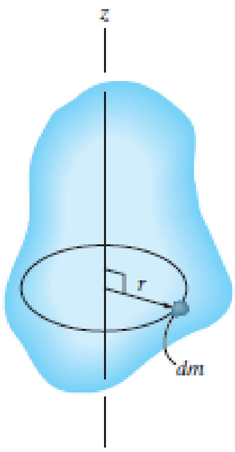
CHAPTER FOUR
PLANE MOTION OF RIGID BODIES:
FORCES AND ACCELERATIONS

4-1 Introduction

In this chapter we will be concerned with the *kinetics* of rigid bodies, i.e., relations between the forces acting on a rigid body, the shape and mass of the body, and the motion produced.

4-2 The Mass Moment of Inertia

Since a body has a definite size and shape, an applied nonconcurrent force system can cause the body to both translate and rotate. The translational aspects of the motion were studied in Chapter 13 and are governed by the equation **F** = *m***a**. It will be shown in the next section that the rotational aspects, caused by a moment **M**, are governed by an equation of the form **M** = *I*A. The symbol *I* in this equation is termed the mass moment of inertia. By comparison, the *moment of inertia* is a measure of the resistance of a body to *angular acceleration* (**M** = *I*A) in the same way that *mass* is a measure of the body's resistance to *acceleration* (**F** = *m***a**). We define the *moment of inertia* as the integral of the "second moment" about an axis of all the elements of mass *dm* which compose the body. For example, the body's moment of inertia about the *z* axis in Fig. is



$$I = \int_m r^2\, dm$$

If the body consists of material having a variable density, r = r (*x*,*y*,*z*), the elemental mass *dm* of the body can be expressed in terms of its density and volume as *dm* = r *dV*. Substituting *dm* into Eq. above, the body's moment of inertia is then computed using *volume elements* for integration; i.e.,

$$I = \int_V r^2 \rho\, dV$$

In the special case of r being a *constant*, this term may be factored out of the integral, and the integration is then purely a function of geometry,

$$I = \rho \int_V r^2\, dV$$

If the moment of inertia of the body about an axis passing through the body's mass center is known, then the moment of inertia about any other *parallel axis* can be determined by using the *parallel-axis theorem*. the moment of inertia about the *z* axis can be written as

$$I = I_G + md^2$$

where

I_G = moment of inertia about the *z*-axis passing through the mass center *G*

m = mass of the body

d = perpendicular distance between the parallel *z* and *z*- axes

Occasionally, the moment of inertia of a body about a specified axis is reported in handbooks using the *radius of gyration*, *k*. This is a geometrical property which has units of length. When it and the body's mass *m* are known, the body's moment of inertia is determined from the equation

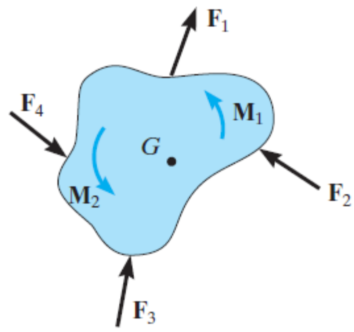
$$I = mk^2 \quad \text{or} \quad k = \sqrt{\frac{I}{m}}$$

4-3 Planar Kinetic Equations of Motion

In the following analysis we will limit our study of planar kinetics to rigid bodies which, along with their loadings, are considered to be symmetrical with respect to a fixed reference plane.

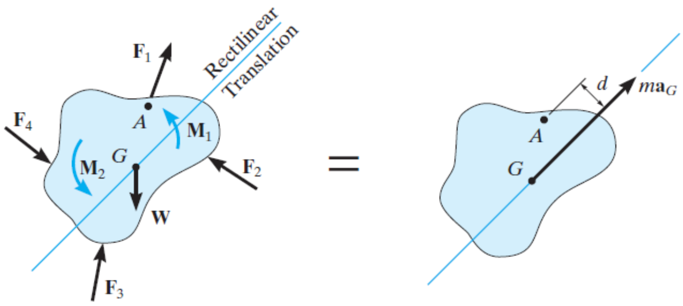
4-3-1 Equations of Motion: Translation

When the rigid body in Fig. *a* undergoes a *translation*, all the particles of the body have the *same acceleration*. Furthermore, A = **0**, in which case the rotational equation of motion applied at point *G* reduces to a simplified form, namely, $\sum M_G = 0$. Application of this and the force equations of motion will now be discussed for each of the two types of translation. When a body is subjected to *rectilinear translation*, all the particles of the body (slab) travel along parallel straightline paths. The free-body and kinetic diagrams are shown in Fig. *b*. Since I_G A = **0**, only $m\mathbf{a}_G$ is shown on the kinetic diagram. Hence, the equations of motion which apply in this case become



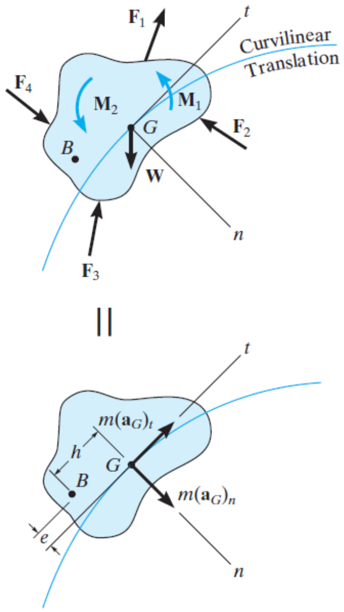
(a)

$$\Sigma F_x = m(a_G)_x$$
$$\Sigma F_y = m(a_G)_y$$
$$\Sigma M_G = 0$$



(b)

When a rigid body is subjected to *curvilinear translation*, all the particles of the body have the same accelerations as they travel along *curved*. For analysis, it is often convenient to use an inertial coordinate system having an origin which coincides with the body's mass center at the instant considered, and axes which are oriented in the normal and tangential directions to the path of motion, Fig. *c*. The three scalar equations of motion are then



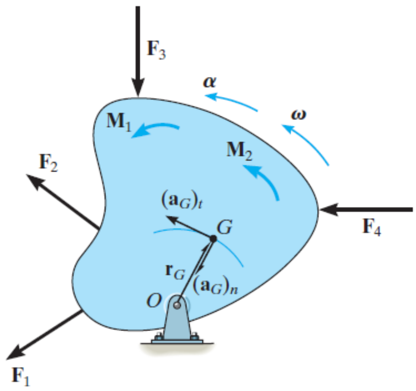
(c)

$$\Sigma F_n = m(a_G)_n$$
$$\Sigma F_t = m(a_G)_t$$
$$\Sigma M_G = 0$$

4-3-2 Equations of Motion: Rotation about a Fixed Axis

Consider the rigid body (or slab) shown in Fig. *a*, which is constrained to rotate in the vertical plane about a fixed axis perpendicular to the page and passing through the pin at *O*. The angular velocity and angular acceleration are caused by the external force and couple moment system acting on the body. Because the body's center of mass *G* moves around a *circular path*, the acceleration of this point is best represented by its tangential and normal components. The *tangential component of acceleration* has a *magnitude* of $(a_G)_t$ = a$_{rG}$ and must act in a *direction* which is *consistent* with the body's angular acceleration A. The *magnitude* of the *normal component of acceleration* is $(a_G)_n$ = v^2*rG*. This component is *always directed* from point *G* to *O*, regardless of the rotational sense of V. The free-body and kinetic diagrams for the body are shown in Fig. *b*. The two components $m(\mathbf{a}_G)_t$ and $m(\mathbf{a}_G)_n$, shown on the kinetic diagram, are associated with the tangential and normal components of acceleration of the body's mass center. The *IG* A vector acts in the same *direction* as A and has a *magnitude* of *IG*a, where *IG* is the body's moment of inertia calculated about an axis which is perpendicular to the page and passes through *G*. The equations of motion which apply to the body can be written in the form

$$\Sigma F_n = m(a_G)_n = m\omega^2 r_G$$
$$\Sigma F_t = m(a_G)_t = m\alpha r_G$$
$$\Sigma M_G = I_G\alpha$$



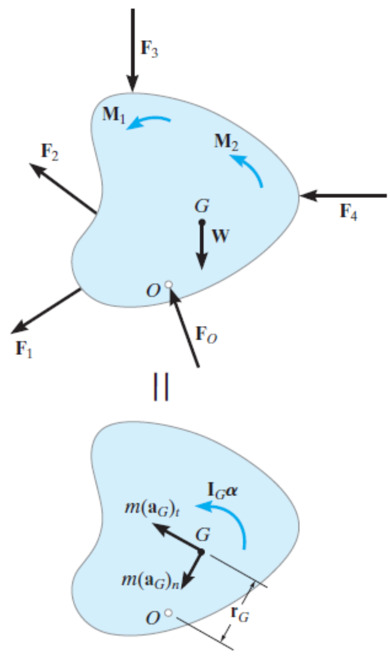
(a)

Often it is convenient to sum moments about the pin at *O* in order to eliminate the *unknown* force $\mathbf{F}_O$. From the kinetic diagram, Fig. *b*, this requires

$$\circlearrowleft + \Sigma M_O = \Sigma(\mathcal{M}_k)_O; \qquad \Sigma M_O = r_G m(a_G)_t + I_G\alpha$$

We can write the three equations of motion for the body as

$$\Sigma F_n = m(a_G)_n = m\omega^2 r_G$$
$$\Sigma F_t = m(a_G)_t = m\alpha r_G$$
$$\Sigma M_O = I_O\alpha$$



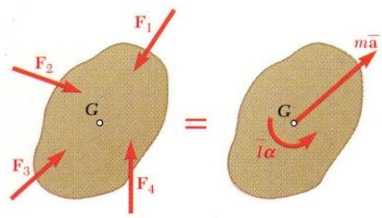
(b)

When using these equations, remember that "$I_O\alpha$" accounts for the "moment" of *both* $m(\mathbf{a}_G)_t$ *and* $I_G\,\alpha$ about point *O*, Fig. *b*. In other words, $\sum M_O = \sum(M_k)O = I_O\alpha$,

4-5 ***Plane Motion of a Rigid Body: D'Alembert's Principle***

Motion of a rigid body in plane motion is completely defined by the resultant and moment resultant about *G* of the external forces

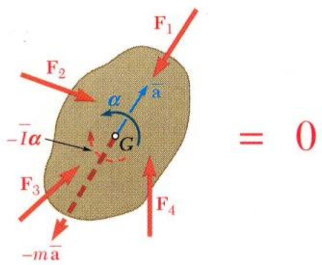
$$\sum F_x = m\bar{a}_x \quad \sum F_y = m\bar{a}_y \quad \sum M_G = \bar{I}\alpha$$



d'Alembert's Principle: The external forces acting on a rigid body are equivalent to the effective forces of the various particles forming the body. The most general motion of a rigid body that is symmetrical with respect to the reference plane can be replaced by the sum of a translation and a centroidal rotation. The fundamental relation between the forces acting on a rigid body in plane motion and the acceleration of its mass center and the angular acceleration of the body is illustrated in a free-body-diagram equation.

The techniques for solving problems of static equilibrium may be applied to solve problems of plane motion by utilizing

- d'Alembert's principle, or
- principle of dynamic equilibrium



These techniques may also be applied to problems involving plane motion of connected rigid bodies by drawing a free-body-diagram equation for each body and solving the corresponding equations of motion simultaneously.

4-6 Uniform Free Body Diagrams and Kinetic Diagrams

The free body diagram is the same as you have done in statics , we will add the kinetic diagram in our dynamic analysis.

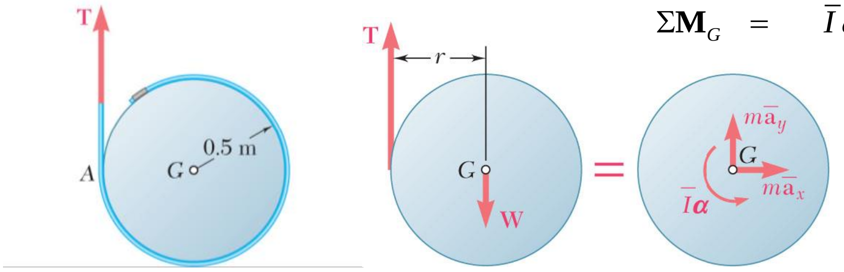
1. Isolate the body of interest (free body)
2. Draw your axis system (Cartesian, polar, path)
3. Add in applied forces (e.g., weight)
4. Replace supports with forces (e.g., tension force)
5. Draw appropriate dimensions (angles and distances)

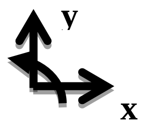
Put the inertial terms for the body of interest on the kinetic diagram.

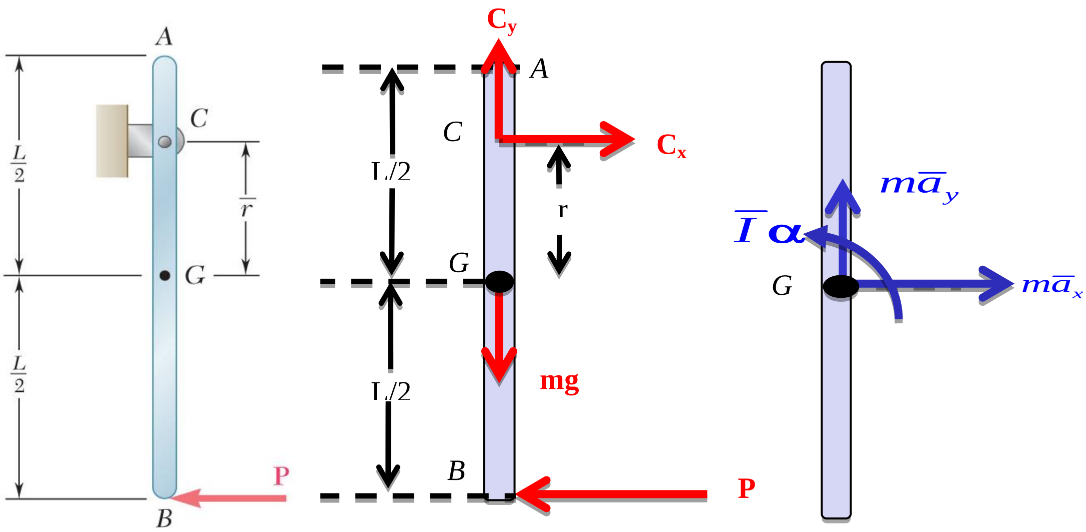
1. Isolate the body of interest (free body)
2. Draw in the mass times acceleration of the particle; if unknown, do this in the positive direction according to your chosen axes. For rigid bodies, also include the rotational term, $I_G\alpha$.

$$\Sigma \mathbf{F} = m\mathbf{a}$$

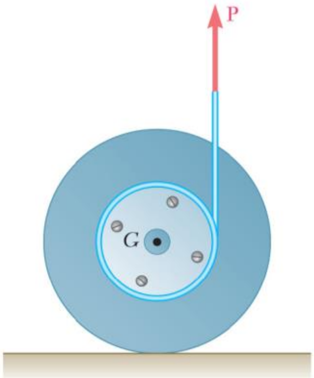
$$\Sigma \mathbf{M}_G = \bar{I}\alpha$$



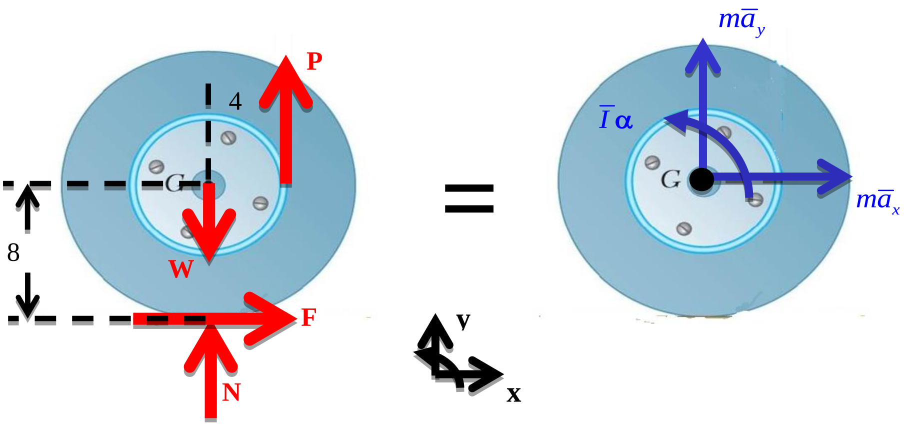




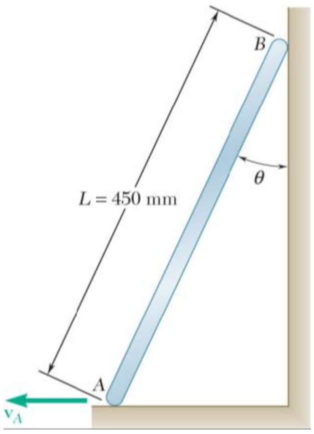
Example1: A drum of 4 inch radius is attached to a disk of 8 inch radius. The combined drum and disk had a combined mass of 10 lbs. A cord is attached as shown, and a force of magnitude P=5 lbs is applied. The coefficients of static and kinetic friction between the wheel and ground are μ_s= 0.25 and μ_k= 0.20, respectively. Draw the FBD and KD for the wheel.

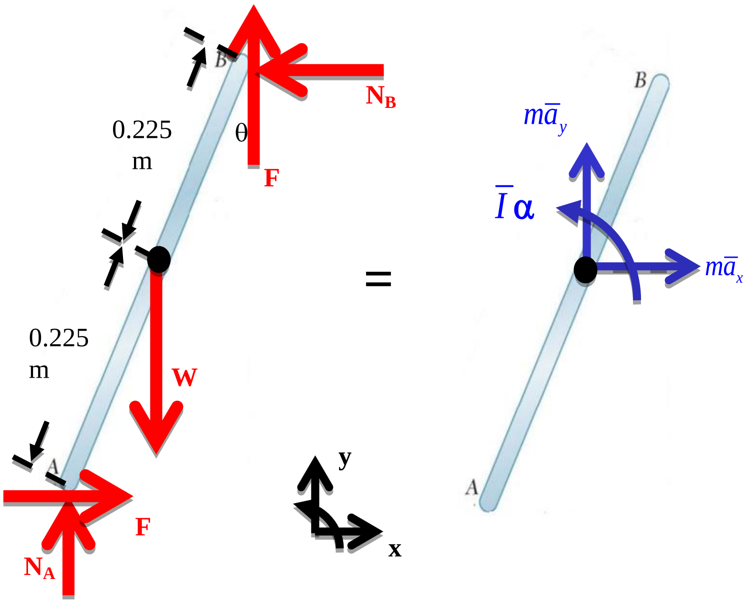


SOLUTION:

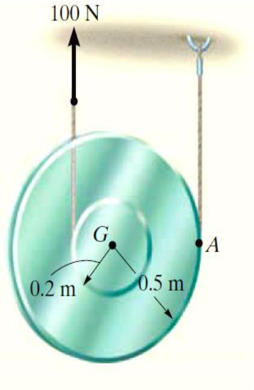


Example 2: The ladder AB slides down the wall as shown. The wall and floor are both rough. Draw the FBD and KD for the ladder.





Example 3: Determine the angular acceleration of the spool in Fig. *a*. The spool has a mass of 8 kg and a radius of gyration of $k_G = 0.35$ m. The cords of negligible mass are wrapped around its inner hub and outer rim.



(a)

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SOLUTION:

$$I_G = mk_G^2 = 8\text{ kg}(0.35\text{ m})^2 = 0.980\text{ kg}\cdot\text{m}^2$$

Equations of Motion.

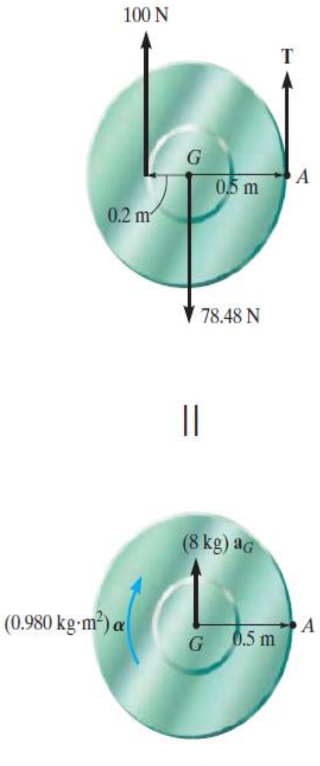
$+\uparrow\Sigma F_y = m(a_G)_y; \quad T + 100\text{ N} - 78.48\text{ N} = (8\text{ kg})a_G \qquad (1)$

$\circlearrowleft +\Sigma M_G = I_G\alpha; \quad 100\text{ N}(0.2\text{ m}) - T(0.5\text{ m}) = (0.980\text{ kg}\cdot\text{m}^2)\alpha \qquad (2)$

$(\circlearrowleft +)\ a_G = \alpha r; \qquad a_G = \alpha\,(0.5\text{ m})$

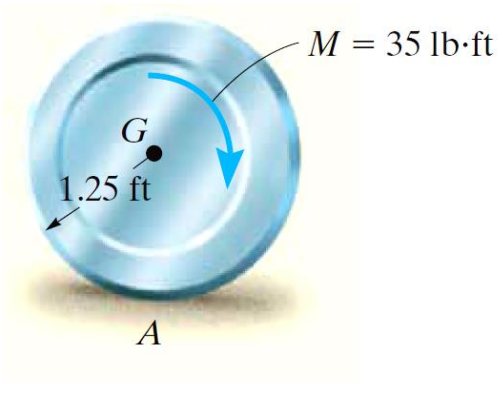
Solving Eqs. 1 to 3, we have

$$\alpha = 10.3\text{ rad/s}^2$$
$$a_G = 5.16\text{ m/s}^2$$
$$T = 19.8\text{ N}$$



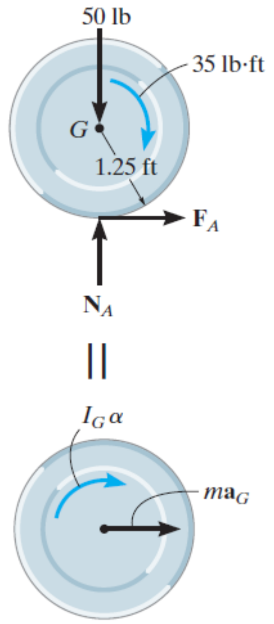
(b)

Example 4: The 50-lb wheel shown in Fig. has a radius of gyration $k_G = 0.70$ ft. If a 35 lb.ft couple moment is applied to the wheel, determine the acceleration of its mass center *G*. The coefficients of static and kinetic friction between the wheel and the plane at *A* are $\mu_s = 0.3$ and $\mu_k = 0.25$, respectively.



(a)

SOLUTION:



(b)

$$I_G = mk_G^2 = \frac{50 \text{ lb}}{32.2 \text{ ft/s}^2}(0.70 \text{ ft})^2 = 0.7609 \text{ slug} \cdot \text{ft}^2$$

The unknowns are N_A, F_A, a_G, and α.

Equations of Motion.

$$\xrightarrow{+} \Sigma F_x = m(a_G)_x; \qquad F_A = \left(\frac{50 \text{ lb}}{32.2 \text{ ft/s}^2}\right)a_G \qquad (1)$$

$$+\uparrow \Sigma F_y = m(a_G)_y; \qquad N_A - 50 \text{ lb} = 0 \qquad (2)$$

$$\circlearrowright + \Sigma M_G = I_G\alpha; \qquad 35 \text{ lb} \cdot \text{ft} - 1.25 \text{ ft}(F_A) = (0.7609 \text{ slug} \cdot \text{ft}^2)\alpha \qquad (3)$$

A fourth equation is needed for a complete solution.

Kinematics (No Slipping). If this assumption is made, then

$$(\circlearrowright +) \qquad a_G = (1.25 \text{ ft})\alpha \qquad (4)$$

Solving Eqs. 1 to 4,

$$N_A = 50.0 \text{ lb} \qquad F_A = 21.3 \text{ lb}$$

$$\alpha = 11.0 \text{ rad/s}^2 \quad a_G = 13.7 \text{ ft/s}^2$$

This solution requires that no slipping occurs, i.e., $F_A \leq \mu_s N_A$. However, since $21.3 \text{ lb} > 0.3(50 \text{ lb}) = 15 \text{ lb}$, the wheel slips as it rolls.

(Slipping). Equation 4 is not valid, and so $F_A = \mu_k N_A$, or

$$F_A = 0.25 N_A \qquad (5)$$

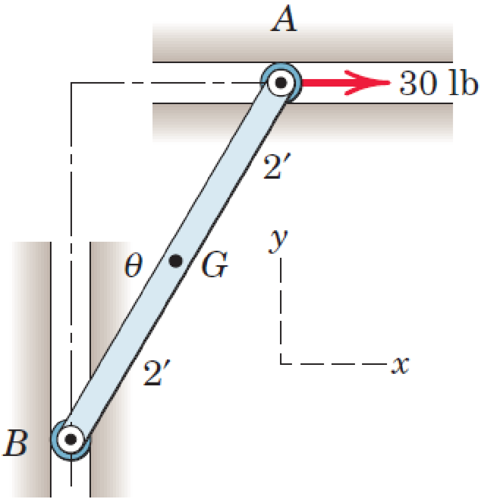
Solving Eqs. 1 to 3 and 5 yields

$$N_A = 50.0 \text{ lb} \qquad F_A = 12.5 \text{ lb}$$

$$\alpha = 25.5 \text{ rad/s}^2$$

$$a_G = 8.05 \text{ ft/s}^2 \rightarrow \qquad \textit{Ans.}$$

Example 5: The slender bar *AB* weighs 60 lb and moves in the vertical plane, with its ends constrained to follow the smooth horizontal and vertical guides. If the 30-lb force is applied at *A* with the bar initially at rest in the position for which θ =30°, calculate the resulting angular acceleration of the bar and the forces on the small end rollers at *A* and *B*.



SOLUTION:

$$\bar{a}_x = \bar{a}\cos 30° = 2\alpha\cos 30° = 1.732\alpha \text{ ft/sec}^2$$

$$\bar{a}_y = \bar{a}\sin 30° = 2\alpha\sin 30° = 1.0\alpha \text{ ft/sec}^2$$

$[\Sigma M_G = \bar{I}\alpha]$

$$30(2\cos 30°) - A(2\sin 30°) + B(2\cos 30°) = \frac{1}{12}\frac{60}{32.2}(4^2)\alpha$$

$[\Sigma F_x = m\bar{a}_x]$ $\quad 30 - B = \frac{60}{32.2}(1.732\alpha)$

$[\Sigma F_y = m\bar{a}_y]$ $\quad A - 60 = \frac{60}{32.2}(1.0\alpha)$

Solving the three equations simultaneously gives us the results

$$A = 68.2 \text{ lb} \qquad B = 15.74 \text{ lb} \qquad \alpha = 4.42 \text{ rad/sec}^2$$