Chapter One

Differential Equations

A ***Differential Equation*** is an equation that contains one or more derivatives of a differentiable function. An equation with partial derivatives is called a ***Partial Differential Equation***. While, an equation with ordinary derivatives, that is, derivatives of a function of a single variable, is called an ***Ordinary Differential Equation***.

The ***order*** of a differential equation is the order of the equation's highest order derivative. A differential equation is ***linear*** if it can be put in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)$$

The ***degree*** of a differential equation is the power (exponent) of the equation's highest order derivative.

<u>Example</u>

First order, first degree, linear $\qquad \frac{dy}{dx} = 5y, \qquad 3\frac{dy}{dx} - \sin x = 0$

Third order, second degree, nonlinear $\qquad \left(\frac{d^3 y}{dx^3}\right)^2 + \left(\frac{d^2 y}{dx^2}\right)^5 - \frac{dy}{dx} = e^x$

<u>Solution of First Order Differential Equations</u>

1) Separable Equations

A first order differential equations is separable if it can be put in the form

$$M(x)dx + N(y)dy = 0$$

Steps for Solving a Separable First Order Differential Equation

i. Write the equation in the form $M(x)dx + N(y)dy = 0$.

ii. Integrate M with respect to x and N with respect to y to obtain an equation that relates y and x.

Example

Solve the following differential equations

(a) $\dfrac{dy}{dx} = (1+y^2)e^x$, (b) $\dfrac{dy}{dx} = \dfrac{x(2\ln x+1)}{\sin y + y\cos y}$, (c) $e^{x+y}dx = \dfrac{dy}{x}$

Solution

(a) $\dfrac{dy}{dx} = (1+y^2)e^x \quad \Rightarrow \quad e^x dx - \dfrac{1}{1+y^2}dy = 0$

$$\int e^x dx - \int \frac{1}{1+y^2}dy = C \quad \Rightarrow \quad e^x - \tan^{-1} y = C$$

$$\tan^{-1} y = e^x - C \quad \Rightarrow \quad y = \tan(e^x - C)$$

(b) $\dfrac{dy}{dx} = \dfrac{x(2\ln x+1)}{\sin y + y\cos y} \quad \Rightarrow \quad (\sin y + y\cos y)dy = x(2\ln x + 1)dx$

$$\int (\sin y + y\cos y)dy - \int x(2\ln x+1)dx = C$$

$$\int \sin(y)dy + \int y\cos(y)dy - 2\int x\ln(x)dx - \int x dx = C$$

$$-\cos(y) + \left[y\sin y - \int \sin(y)dy\right] - 2\left[\ln(x)\times\frac{x^2}{2} - \int \frac{x^2}{2}\times\frac{1}{x}dx\right] - \frac{x^2}{2} = C$$

$$-\cos(y) + y\sin y + \cos y - x^2\ln x + \frac{x^2}{2} - \frac{x^2}{2} = C$$

$$y\sin y - x^2\ln x = C.$$

(c) $e^{x+y}dx = \dfrac{dy}{x} \quad \Rightarrow \quad e^x e^y dx = \dfrac{dy}{x}$

$xe^x dx = \dfrac{dy}{e^y} \quad \Rightarrow \quad \int xe^x dx - \int e^{-y} dy = C$

$xe^x - \int e^x dx + e^{-y} = C \quad \Rightarrow \quad xe^x - e^x + e^{-y} = C$

Notes

1) $f_1(x)g_1(y)dy + f_2(x)g_2(y)dx = 0$ *Separable*

2) $\dfrac{f_1(x)}{g_1(y)}dy + \dfrac{f_2(x)}{g_2(y)}dx = 0$ *Separable*

3) $[f_1(x) \pm g_1(y)]dy + [f_2(x) \pm g_2(y)]dx$ *Not Separable*

Example

$f_1(x) = x, \quad f_2(x) = \sin(x), \quad g_1(y) = y, \quad g_2(y) = \tan(y)$

1) $xydy + \sin(x)\tan(y)dx = 0 \quad \Rightarrow \quad xydy = -\sin(x)\tan(y)dx$

$\dfrac{y}{\tan(y)}dy = -\dfrac{\sin(x)}{x}dx$ *Separable*

2) $\dfrac{x}{y}dy + \dfrac{\sin(x)}{\tan(y)}dx = 0 \quad \Rightarrow \quad \dfrac{x}{y}dy = -\dfrac{\sin(x)}{\tan(y)}dx$

$\dfrac{\tan(y)}{y}dy = -\dfrac{\sin(x)}{x}dx$ *Separable*

3) $(x+y)dy + (\sin(x) + \tan(y))dx = 0$

$(x+y)dy = -(\sin(x) + \tan(y))dx$ *Not Separable*

Special Type of Separable Equations

If $\frac{dy}{dx} = f(ax+by+c)$; then let $z = ax+by+c$ and the resultant equation may be reduced to a separable equation.

Example

Solve the differential equation $\frac{dy}{dx} = \tan^2(x+y)$

Solution

$$z = x+y \quad \Rightarrow \quad \frac{dz}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1 \quad \Rightarrow \quad \frac{dz}{dx} - 1 = \tan^2(z)$$

$$\frac{dz}{dx} = \tan^2(z) + 1 \quad \Rightarrow \quad \frac{dz}{dx} = \sec^2(z)$$

$$\frac{dz}{\sec^2(z)} = dx \quad \Rightarrow \quad \cos^2(z)dz = dx$$

$$\int \cos^2(z)dz - \int dx = C \quad \Rightarrow \quad \int \frac{1+\cos(2z)}{2} dz - \int dx = C$$

$$\frac{1}{2}z + \frac{1}{4}\sin(2z) - x = C$$

While $z = x+y$, then the solution is

$$\frac{1}{2}(x+y) + \frac{1}{4}\sin(2(x+y)) - x = C$$

Note

For the differential equation $\frac{dy}{dx} = \tan(x+y) - \sec(x-y)$, we can not use the assumption because the difference between the arguments of $\tan(x+y)$ and $\sec(x-y)$. So, the differential equation can not be converted to a separable equation.

Exercises

Find the solution of the following Differential Equations

1) $y' = ky$

2) $y' = -xy$

3) $y' - 2y + a = 0$

4) $xy' + by = 0$

5) $(x \ln x)y' = y$

6) $(x+2)y' - xy = 0$

7) $y' = 2x^{-1}\sqrt{y-1}$

8) $yy' = 2x\exp(y^2)$

9) $2y' = y\cot(x)$

10) $y' + \csc(y) = 0$

11) $y' = (1+x)(1+y^2)$

12) $yy' = 0.5\sin^2(\omega x)$

13) $y' = y\tanh(x)$

14) $y'\sin(2x) = y\cos(2x)$

15) $y' = y\tan(2x),\ y(0) = 2$

16) $(x+1)y' = 2y,\ y(0) = 1$

17) $2xy' = 3y,\ y(1) = 4$

18) $y'x\ln(x) = y,\ y(2) = \ln(4)$

19) $y' = 2e^x y^3,\ y(0) = 0.5$

20) $(x^2+1)yy' = 1,\ y(0) = -3$

21) $dr\sin(\theta) = 2r\cos(\theta)d\theta, r(\pi/2) = 2$

22) $\upsilon\, d\upsilon/dx = C,\ \upsilon(x_0) = \upsilon_0$

2) Homogeneous Function

If $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ then $f(x, y)$ is homogeneous function and n represents the degree of the homogeneous function.

Example

For the function $f(x, y) = x^2 + y^2$ then

$$f(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^2$$
$$= \lambda^2 x^2 + \lambda^2 y^2$$
$$= \lambda^2 \left(x^2 + y^2\right) = \lambda^2 f(x, y)$$

So, the function $f(x, y)$ is homogeneous with degree 2.

Example

For the function $f(x, y) = x + y^2$ then

$$f(\lambda x, \lambda y) = \lambda x + (\lambda y)^2$$
$$= \lambda x + \lambda^2 y^2$$
$$= \lambda\left(x + \lambda y^2\right)$$

So, the function $f(x, y)$ is not homogeneous.

Example

$f(x, y) = x^2 + y^2 + 5$ (Non-homogeneous)

$f(x, y) = x^3 + xy + x$ (Non-homogeneous)

$f(x, y) = \cos(xy)$ (Non-homogeneous)

$f(x, y) = \cos(x^2 \pm y^2)$ (Non-homogeneous)

$f(x,y)=\cos\left(\frac{x}{y}\right)$ (Homogeneous)

$f(x,y)=\cos\left(\frac{x^2}{y}\right)$ (Non-homogeneous)

Homogeneous Equations

The differential equation $M(x,y)dx+N(x,y)dy$ is homogeneous if M and N are homogeneous functions of the same degree.

Example

1) $(x^2+y^2)dx+xydy=0$

This is homogeneous because M and N are both homogeneous with degree 2.

2) $(x^3+y^3)dx+xydy=0$

This is not homogeneous because M is homogeneous with degree 3 while N is homogeneous with degree 2.

3) $xdx+(x^2+y)dy=0$

This is not homogeneous because N is not homogeneous.

Solution of Homogeneous Equations

A homogeneous first order differential equation can be put in the form

$$\frac{dy}{dx}=F\left(\frac{y}{x}\right)$$

This equation can be changed into separable equation with the substitutions

$$v=\frac{y}{x} \quad \Rightarrow \quad y=vx \quad \Rightarrow \quad \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Then becomes $$v+x\frac{dv}{dx}=F(v)$$

which can be rearranged algebraically to give

$$\frac{dx}{x}+\frac{dv}{v-F(v)}=0$$

with the variables now separated, the equation can now be solved by integrating with respect to x and v. We can then return to x and y by substituting $v=y/x$.

Example

Find the solution of the differential equation

$$\frac{dy}{dx}=-\frac{x^2+y^2}{2xy}$$

that satisfies the condition $y(1)=1$.

Solution

Dividing the numerator and denominator of the right-hand side by x^2 gives

$$\frac{dy}{dx}=-\frac{1+(y/x)^2}{2(y/x)} \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{1+v^2}{2v}=F(v)$$

$$v-F(v)=v+\frac{1+v^2}{2v}=\frac{2v^2+1+v^2}{2v}=\frac{3v^2+1}{2v}$$

$$\frac{dx}{x}+\frac{dv}{v-F(v)}=0 \quad \Rightarrow \quad \frac{dx}{x}+\frac{2vdv}{3v^2+1}=0$$

The solution of this equation can be written as

$$\int \frac{dx}{x} + \int \frac{2v\,dv}{3v^2+1} = C \qquad \Rightarrow \qquad \ln x + \frac{1}{3}\ln(1+3v^2) = C$$

$$3\ln x + \ln(1+3v^2) = 3C \quad \Rightarrow \quad \ln x^3 + \ln(1+3v^2) = 3C$$

$$e^{\ln x^3 + \ln(1+3v^2)} = e^{3C} \qquad \Rightarrow \qquad e^{\ln x^3} \times e^{\ln(1+3v^2)} = e^{3C}$$

$$x^3(1+3v^2) = C' \qquad \Rightarrow \qquad x^3\left(1+3\frac{y^2}{x^2}\right) = C'$$

$$x^3 + 3xy^2 = C'$$

The condition is that when $x=1$ then $y=1$ and the constant C' can be found

$$(1)^3 + 3(1)(1)^2 = C' \qquad \Rightarrow \qquad C' = 4$$

The final solution is $x^3 + 3xy^2 = 4$.

Reducible to Homogeneous

If the differential equation has the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

Case 1: if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$ then $z = a_1x + b_1y$

Case 2: if $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ then intersect the two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ to find the intersection point (h,k) and let

$$x = X + h \quad \Rightarrow \quad dx = dX, \text{ and } \quad y = Y + k \quad \Rightarrow \quad dy = dY$$

Example

Solve the differential equation $\dfrac{dx}{dy}=\dfrac{4x+6y+5}{3y+2x+4}$

Solution

$$\frac{dy}{dx}=\frac{2x+3y+4}{4x+6y+5}$$

$$a_1=2,\quad a_2=4 \qquad \Rightarrow \qquad \frac{a_1}{a_2}=\frac{2}{4}=\frac{1}{2}$$

$$b_1=3,\quad b_2=6 \qquad \Rightarrow \qquad \frac{b_1}{b_2}=\frac{3}{6}=\frac{1}{2}$$

So $\quad \dfrac{a_1}{a_2}=\dfrac{b_1}{b_2} \qquad \Rightarrow \qquad$ ***Case 1***

Let $\quad z=2x+3y \qquad \Rightarrow \qquad \dfrac{dz}{dx}=2+3\dfrac{dy}{dx}$

$$\frac{dy}{dx}=\frac{1}{3}\left(\frac{dz}{dx}-2\right) \quad \Rightarrow \quad \frac{1}{3}\frac{dz}{dx}-\frac{2}{3}=\frac{z+4}{2z+5} \quad \Rightarrow \quad \frac{dz}{dx}=\frac{3z+12}{2z+5}+2$$

$$\frac{dz}{dx}=\frac{3z+12+4z+10}{2z+5} \quad \Rightarrow \quad \frac{dz}{dx}=\frac{7z+22}{2z+5} \quad \Rightarrow \quad \frac{2z+5}{7z+22}dz=dx$$

$$\int\frac{2z+5}{7z+22}dz-\int dx=C \quad \Rightarrow \quad \int\left(\frac{2}{7}-\frac{9}{7}\times\frac{1}{7z+22}\right)dz-\int dx=C$$

$$\int\frac{2}{7}dz-\int\frac{9}{7\times7}\times\frac{7}{7z+22}dz-\int dx=C$$

$$\frac{2}{7}z-\frac{9}{49}\times\ln(7z+22)-x=C$$

$$\frac{2}{7}(2x+3y)-\frac{9}{49}\times\ln\big(7(2x+3y)+22\big)-x=C$$

Example

Solve the differential equation $(2x+y-3)dy=(x+2y-3)dx$

Solution

$$\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$$

$$a_1=1,\quad a_2=2 \qquad \Rightarrow \qquad \frac{a_1}{a_2}=\frac{1}{2}$$

$$b_1=2,\quad b_2=1 \qquad \Rightarrow \qquad \frac{b_1}{b_2}=2$$

So $\quad \dfrac{a_1}{a_2}\neq\dfrac{b_1}{b_2} \qquad \Rightarrow \qquad$ ***Case 2***

$$\begin{array}{rrrrl} x & +2y & -3 & =0 & \ldots\ldots (1) \\ 2x & +y & -3 & =0 & \ldots\ldots (2) \\ \hline +2x & +4y & -6 & =0 & \\ \mp 2x & \mp y & \pm 3 & =0 & \\ \hline & 3y & -3 & =0 & \end{array}$$

$$\Rightarrow \quad y=1$$

Substituting into (2), we get

$$2x+1-3=0 \quad \Rightarrow \quad x=1$$

The intersection point $(h,k)=(1,1)$.

Let $\quad x=X+1 \quad \Rightarrow \quad dx=dX$

$$y=Y+1 \quad \Rightarrow \quad dy=dY$$

$$\frac{dY}{dX}=\frac{(X+1)+2(Y+1)-3}{2(X+1)+(Y+1)-3}$$

$$=\frac{X+1+2Y+2-3}{2X+2+Y+1-3} \qquad \Rightarrow \qquad \frac{dY}{dX}=\frac{X+2Y}{2X+Y}$$

$$\frac{dY}{dX}=\frac{1+2\frac{Y}{X}}{2+\frac{Y}{X}}$$

Let $\quad v=\frac{Y}{X} \quad \Rightarrow \quad \frac{dY}{dX}=\frac{1+2v}{2+v}=F(v)$

$$\frac{dX}{X}+\frac{dv}{v-F(v)}=0$$

$$v-F(v)=v-\frac{1+2v}{2+v}$$

$$=\frac{2v+v^2-1-2v}{2+v}=\frac{v^2-1}{2+v}$$

$$\frac{dX}{X}+\frac{2+v}{v^2-1}dv=0 \quad \Rightarrow \quad \ln X+\int\frac{2+v}{v^2-1}dv=C$$

$$\frac{2+v}{v^2-1}=\frac{A}{v+1}+\frac{B}{v-1}=\frac{A(v-1)+B(v+1)}{v^2-1}$$

$$2+v=A(v-1)+B(v+1)$$

At $\quad v=1 \quad \Rightarrow \quad B=\frac{3}{2}$

At $\quad v=-1 \quad \Rightarrow \quad A=\frac{-1}{2}$

$$\ln X+\int\frac{-1/2}{v+1}dv+\int\frac{3/2}{v-1}dv=C$$

$$\ln X-\frac{1}{2}\ln(v+1)+\frac{3}{2}\ln(v-1)=C$$

$$\ln X - \frac{1}{2}\ln(\frac{Y}{X}+1) + \frac{3}{2}\ln(\frac{Y}{X}-1) = C$$

$$\ln(x-1) - \frac{1}{2}\ln(\frac{(y-1)}{(x-1)}+1) + \frac{3}{2}\ln(\frac{(y-1)}{(x-1)}-1) = C$$

Exercises

Find the solution of the following Differential Equations

1) $xy' = x + y$

2) $xy' = 2x + 2y$

3) $x^2 y' = y^2 + xy + x^2$

4) $x^2 y' = y^2 + 5xy + 4x^2$

5) $y' = \dfrac{y-x}{y+x}$

6) $y' = \dfrac{y+x}{y-x}$

7) $y' = (y-x)^2$

8) $y' = \tan(x+y) - 1$

9) $y' = \dfrac{y-x+1}{y-x+5}$

10) $y' = \dfrac{1-2y-4x}{1+y+2x}$

3) Linear First Order Equations

A differential equation that can be written in the form

$$\frac{dy}{dx}+P(x)y=Q(x)$$

where P and Q are functions of x, is called a ***Linear First Order Equation***. The solution is

$$y=\frac{1}{\rho(x)}\int \rho(x)Q(x)dx$$

where $$\rho(x)=e^{\int P(x)dx}$$

Steps for Solving a Linear First Order Equation

i. Put it in standard form and identify the functions P and Q.

ii. Find an anti-derivative of $P(x)$.

iii. Find the integrating factor $\rho(x)=e^{\int P(x)dx}$.

iv. Find y using the following equation

$$y=\frac{1}{\rho(x)}\int \rho(x)Q(x)dx$$

Example

Solve the equation $x\frac{dy}{dx}-3y=x^2$

Solution

Step 1: ***Put the equation in standard form and identify the functions*** P ***and*** Q. To do so, we divide both sides of the equation by the coefficient of dy/dx, in this case x, obtaining

$$\frac{dy}{dx}-\frac{3}{x}y=x \qquad \Rightarrow \qquad P(x)=-\frac{3}{x}, \qquad Q(x)=x.$$

Step 2: ***Find an anti-derivative of*** $P(x)$.

$$\int P(x)dx=\int -\frac{3}{x}dx=-3\int \frac{1}{x}dx=-3\ln(x)$$

Step 3: ***Find the integrating factor*** $\rho(x)$.

$$\rho(x)=e^{\int P(x)dx}=e^{-3\ln x}=e^{\ln x^{-3}}=e^{\ln\frac{1}{x^3}}=\frac{1}{x^3}$$

Step 4: ***Find the solution***.

$$y=\frac{1}{\rho(x)}\int \rho(x)Q(x)dx=\frac{1}{(1/x^3)}\int\left(\frac{1}{x^3}\right)(x)dx$$

$$= x^3 \int \frac{1}{x^2}\,dx = x^3\left(-\frac{1}{x} + C\right) = Cx^3 - x^2$$

The solution is the function $y = Cx^3 - x^2$.

Example

Solve the equation $(1+x^2)dy + (y - \tan^{-1}(x))dx = 0$.

Solution

Dividing the two sides by $(1+x^2)dx$

$$\frac{dy}{dx} + \frac{y}{1+x^2} - \frac{\tan^{-1}(x)}{1+x^2} = 0$$

$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{\tan^{-1}(x)}{1+x^2} \quad \Rightarrow \quad P(x) = \frac{1}{1+x^2}, \quad Q = \frac{\tan^{-1}(x)}{1+x^2}$$

$$\int P(x)dx = \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)$$

$$\rho(x) = e^{\tan^{-1}(x)}$$

$$e^{\tan^{-1}(x)} y = \int e^{\tan^{-1}(x)} \frac{\tan^{-1}(x)}{1+x^2}\,dx + C$$

$$z = \tan^{-1}(x) \quad \Rightarrow \quad dz = \frac{1}{1+x^2}\,dx$$

$$e^{\tan^{-1}(x)} y = \int e^z \times z\,dz + C$$

$$= ze^z - \int e^z dz + C$$

$$= ze^z - e^z + C$$

$$e^{\tan^{-1}(x)} y = \tan^{-1}(x) e^{\tan^{-1}(x)} - e^{\tan^{-1}(x)} + C$$

Steps for Solving other Form of Linear First Order Equation

There is another form of differential equation that can be written in the form

$$\frac{dx}{dy} + P(y)x = Q(y)$$

where P and Q are functions of y. The solution is found as follows:

i. Put it in standard form and identify the functions P and Q.

ii. Find an anti-derivative of $P(y)$.

iii. Find the integrating factor $\rho(y) = e^{\int P(y)dy}$.

iv. Find x using the following equation

$$x = \frac{1}{\rho(y)} \int \rho(y) Q(y) dy$$

Example

Solve the equation $e^{2y}dx + 2(xe^{2y} - y)dy = 0$.

Solution

Dividing the differential equation by $e^{2y}dy$ to get

$$\frac{dx}{dy} + 2x - 2ye^{-2y} = 0$$

$$\frac{dx}{dy} + 2x = 2ye^{-2y} \qquad \Rightarrow \qquad P(y) = 2, \quad Q(y) = 2ye^{-2y}$$

$$\int P(y)dy = \int 2dy = 2y, \qquad \rho(y) = e^{\int P(y)dy} = e^{2y}$$

$$x = \frac{1}{e^{2y}} \int (e^{2y})(2ye^{-2y})dy + C \quad \Rightarrow \quad e^{2y}x = 2\int ydy + C$$

$$e^{2y}x = 2\frac{y^2}{2} + C \qquad \Rightarrow \qquad e^{2y}x = y^2 + C$$

Reducible to Linear

- The general form

$$\frac{dy}{dx}+P(x)y=Q(x)f(y)$$

where the function f is y to any power n.

- Also, it may be in the following form

$$\frac{dy}{dx}+P(x)g(y)=Q(x)h(y)$$

where the function g and h are functions of y.

Example

Solve the equation $\dfrac{dy}{dx}+\dfrac{y}{x}=\ln(x)y^2$

Solution

Dividing the two sides of the equation by y^2

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{xy}=\ln(x)$$

Let $z=\dfrac{1}{y} \quad\Rightarrow\quad \dfrac{dz}{dx}=-\dfrac{1}{y^2}\dfrac{dy}{dx} \quad\Rightarrow\quad \dfrac{dy}{dx}=-y^2\dfrac{dz}{dx}$

$$-\frac{dz}{dx}+\frac{1}{x}z=\ln(x)$$

$$\frac{dz}{dx}-\frac{1}{x}z=-\ln(x) \qquad\Rightarrow\qquad P=\frac{-1}{x},\quad Q=-\ln(x)$$

$$\int P(x)dx=\int\frac{-1}{x}dx=-\ln(x)$$

$$\rho(x) = e^{\int P(x)dx} = e^{-\ln(x)} = e^{\ln(x)^{-1}} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$$

$$\rho(x)z = \int \rho(x)Q(x)dx + C$$

$$\frac{1}{x}z = -\int \frac{1}{x}\ln(x)dx + C$$

$$\frac{1}{x} \times \frac{1}{y} = -\frac{(\ln(x))^2}{2} + C \quad \Rightarrow \quad \frac{1}{xy} = -\frac{(\ln(x))^2}{2} + C$$

Example

Solve the equation $\dfrac{dy}{dx} + x\sin(2y) = x\cos^2(y)$

Solution

Dividing the two sides of the equation by $\cos^2(y)$

$$\frac{1}{\cos^2(y)}\frac{dy}{dx} + x\frac{\sin(2y)}{\cos^2(y)} = x \;\Rightarrow\; \sec^2(y)\frac{dy}{dx} + x\frac{2\sin(y)\cos(y)}{\cos^2(y)} = x$$

$$\sec^2(y)\frac{dy}{dx} + x\frac{2\sin(y)}{\cos(y)} = x \;\Rightarrow\; \sec^2(y)\frac{dy}{dx} + 2x\tan(y) = x$$

Let $z = \tan(y) \quad \Rightarrow \quad \dfrac{dz}{dx} = \sec^2(y)\dfrac{dy}{dx} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\sec^2(y)}\dfrac{dz}{dx}$

$$\frac{dz}{dx} + 2xz = x \;\Rightarrow\; P = 2x, \quad Q = x$$

$$\int P(x)dx = \int 2xdx = x^2 \quad \Rightarrow \quad \rho(x) = e^{\int P(x)dx} = e^{x^2}$$

$$\rho(x)z = \int \rho(x)Q(x)dx + C$$

$$e^{x^2}z = \int e^{x^2}(x)dx + C \quad \Rightarrow \quad e^{x^2}\tan(y) = \frac{e^{x^2}}{2} + C$$

Another Form of Reducible to Linear

- The general form may be as follows

$$\frac{dx}{dy}+P(y)x=Q(y)f(x)$$

where the function f is x to any power n.

- Also, it may be in the following form

$$\frac{dx}{dy}+P(y)g(x)=Q(y)h(x)$$

where the function g and h are functions of x.

Example

Solve the equation $\cos(y)dx = x(\sin(y)-x)dy$

Solution

Dividing the two sides of the equation by $\cos(y)dy$

$$\frac{dx}{dy}=\frac{\sin(y)}{\cos(y)}x-\frac{x^2}{\cos(y)} \quad \Rightarrow \quad \frac{dx}{dy}-x\tan(y)=-x^2\sec(y)$$

Dividing by x^2, we get

$$\frac{1}{x^2}\frac{dx}{dy}-\frac{1}{x}\tan(y)=-\sec(y)$$

Let $z=\dfrac{1}{x} \quad \Rightarrow \quad \dfrac{dz}{dy}=\dfrac{-1}{x^2}\dfrac{dx}{dy} \quad \Rightarrow \quad \dfrac{dx}{dy}=-x^2\dfrac{dz}{dy}$

$$-\frac{dz}{dy}-z\tan(y)=-\sec(y)$$

$$\frac{dz}{dy}+z\tan(y)=\sec(y) \quad \Rightarrow \quad P=\tan(y), \quad Q=\sec(y)$$

$$\int P(y)dy = \int \tan(y)dy = \int \frac{\sin(y)}{\cos(y)}dy = -\ln(\cos(y))$$

$$\rho(y) = e^{\int P(y)dy} = e^{-\ln(\cos(y))} = e^{\ln(\cos(y))^{-1}} = e^{\ln\left(\frac{1}{\cos(y)}\right)} = \sec(y)$$

$$\rho(y)z = \int \rho(y)Q(y)dy + C$$

$$\sec(y) \times \frac{1}{x} = \int \sec(y)\sec(y)dy + C$$

$$\frac{\sec(y)}{x} = \int \sec^2(y)dy + C \quad \Rightarrow \quad \frac{\sec(y)}{x} = \tan(y) + C$$

Exercises

Find the solution of the following Differential Equations

1) $y' - y = 3$

2) $y' + 2xy = 0$

3) $y' + 2y = 6e^x$

4) $y' - 4y = 2x - 4x^2$

5) $y' + y = \sin(x)$

6) $y' + 2y = \cos(x)$

7) $y' + ky = e^{-kx}$

8) $y' = (y-1)\cot(x)$

9) $xy' - 2y = x^3e^x$

10) $x^2y' + 2xy = \sinh(3x)$

11) $y' - y = e^x,\ y(1) = 0$

12) $y' + y = (x+1)^2,\ y(0) = 0$

13) $y' + xy = xy^{-1}$

14) $y' + x^{-1}y = x^{-1}y^2$

15) $2xy' = 10x^3y^5 + y$

16) $y' + y = xy^{-1}$

4) Exact Differential Equations

Example

If $f(x,y)=C$ and $f(x,y)=\sin(xy)$ then

$$\frac{df}{dx}=y\cos(xy)+x\cos(xy)\frac{dy}{dx}=0\text{, or}$$

$$df=y\cos(xy)dx+x\cos(xy)dy=0$$

i.e., $y\cos(xy)dx+x\cos(xy)dy=0$

From the above equation, we see that $M(x,y)=y\cos(xy)=\dfrac{\partial f}{\partial x}$, and $N(x,y)=x\cos(xy)=\dfrac{\partial f}{\partial y}$. The solution of this differential equation is $f(x,y)=C$.

Exact Differential Equation Test

A differential equation $M(x,y)dx+N(x,y)dy=0$ is said to be ***exact*** if for some function $f(x,y)$

$$M(x,y)dx+N(x,y)dy=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=df$$

is exact if and only if $$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Example

- The equation $(x^2+y^2)dx+(2xy+\cos(y))dy=0$ is exact because the partial derivatives

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2)=2y,\qquad \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(2xy+\cos(y))=2y$$

are equal.

- The equation $(x+3y)dx+(x^2+\cos(y))dy=0$ is not exact because the partial derivatives

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(x+3y)=3, \qquad \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(x^2+\cos(y))=2x$$

are not equal.

Steps for Solving an Exact Differential Equation

i. Match the equation to the form $M(x,y)dx+N(x,y)dy=0$ to identify M and N.

ii. Integrate M (or N) with respect to x (or y), writing the constant of integration as $g(y)$ (or $g(x)$).

iii. Differentiate with respect to y (or x) and set the result equal to N (or M) to find $g'(y)$ (or $g'(x)$).

iv. Integrate to find $g(y)$ (or $g(x)$).

v. Write the solution of the exact equation as $f(x,y)=C$.

Example

Solve the differential equation

$$(x^2+y^2)dx+(2xy+\cos(y))dy=0.$$

Solution

Step 1: ***Match the equation to the form*** $M(x,y)dx+N(x,y)dy=0$ ***to identify*** M.

$$M(x,y)=x^2+y^2$$

Step 2: ***Integrate*** M ***with respect to*** x, ***writing the constant of integration as*** $g(y)$.

$$f(x,y)=\int M(x,y)dx=\int (x^2+y^2)dx=\frac{x^3}{3}+xy^2+g(y)$$

Step 3: ***Differentiate with respect to*** y ***and set the result equal to*** N ***to find*** $g'(y)$.

$$\frac{\partial}{\partial y}\left(\frac{x^3}{3}+xy^2+g(y)\right)=2xy+g'(y)$$

$$2xy+g'(y)=2xy+\cos(y) \quad \Rightarrow \quad g'(y)=\cos(y)$$

Step 4: ***Integrate to find*** $g(y)$.

$$\int g'(y)dy=\int \cos(y)dy=\sin(y)$$

Step 5: ***Write the solution of the exact equation as*** $f(x,y)=C$.

$$\frac{x^3}{3}+xy^2+\sin(y)=C$$

Another Solution

Step 1: ***Match the equation to the form*** $M(x,y)dx+N(x,y)dy=0$ ***to identify*** N.

$$N(x,y)=2xy+\cos(y)$$

Step 2: ***Integrate*** N ***with respect to*** y, ***writing the constant of integration as*** $g(x)$.

$$f(x,y)=\int N(x,y)dy=\int (2xy+\cos(y))dy=xy^2+\sin(y)+g(x)$$

Step 3: ***Differentiate with respect to*** x ***and set the result equal to*** M ***to find*** $g'(x)$.

$$\frac{\partial}{\partial x}\left(xy^2+\sin(y)+g(x)\right)=y^2+g'(x)$$

$$y^2+g'(x)=x^2+y^2 \quad \Rightarrow \quad g'(x)=x^2$$

Step 4: ***Integrate to find*** $g(x)$.

$$\int g'(x)dx = \int x^2 dx = \frac{x^3}{3}$$

Step 5: ***Write the solution of the exact equation as*** $f(x,y) = C$.

$$\frac{x^3}{3} + xy^2 + \sin(y) = C$$

Reducible to Exact

A differential equation $M(x,y)dx + N(x,y)dy = 0$ which is not exact can be made exact by multiplying both sides by a suitable integrating factor ρ. In other words, the equation

$$\rho M(x,y)dx + \rho N(x,y)dy = 0$$

is an exact equation for an appropriate choice of ρ.

Method to Find the Integrating Factor

- If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ or ***Constant*** then $\rho(x) = e^{\int f(x)dx}$.
- If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = f(y)$ or ***Constant*** then $\rho(y) = e^{\int f(y)dy}$.

Example

Solve the equation $2ydx + xdy = 0$

Solution

$$M(x,y) = 2y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2$$

$$N(x,y) = x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1$$

This equation is not exact

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2-1}{x} = \frac{1}{x} = f(x)$$

$$\int f(x)dx = \int \frac{1}{x} dx = \ln(x)$$

$$\rho(x) = e^{\int f(x)dx} = e^{\ln(x)} = x$$

Multiplying both sides of the equation by the integrating factor $\rho(x) = x$, we get

$$x(2ydx + xdy) = 0 \quad \Rightarrow \quad 2xydx + x^2 dy = 0$$

which is exact because $\frac{\partial M}{\partial y} = 2x$ and $\frac{\partial N}{\partial x} = 2x$, and the solution is

$$f(x,y) = \int 2xydx = x^2 y + g(y)$$

$$\frac{\partial}{\partial y}\left(x^2 y + g(y)\right) = x^2 + g'(y)$$

$$x^2 + g'(y) = x^2 \quad \Rightarrow \quad g'(y) = 0$$

$$g(y) = \int g'(y)dy = C \quad \Rightarrow \quad x^2 y = C$$

Exercises

Find the solution of the following Differential Equations

1) $ydx + xdy = 0$

2) $(xdy - ydx)/x^2 = 0$

3) $(2x + e^y)dx + xe^y dy = 0$

4) $2x\ln(y)dx + y^{-1}x^2 dy = 0$

5) $\sinh(x)\cos(y)dx = \cosh(x)\sin(y)dy$

6) $3re^{3\theta}d\theta + e^{3\theta}dr = 0$

7) $(1 + x^2)dy + 2xydx = 0$

8) $xdy - 4ydx = 0$

9) $ydx + x(1 + y)dy = 0$

10) $(2ydx + dy)e^{2x} = 0$

11) $\left(3y\cos(3x)dx - \sin(3x)dy\right)/y^2 = 0$

12) $\sin(\beta y)dx = -\beta\cos(\beta y)dy$

13) $xdy - ydx = 0$

14) $2\cos(\pi y)dx = \pi\sin(\pi y)dy$

15) $y\cos(x)dx + 3\sin(x)dy = 0$

16) $3ydx + 2xdy = 0$

17) $dx + (y/x)^2 dy = 0$

18) $2dx - e^{y-x}dy = 0$

19) $y\cos(x)dx + 2\sin(x)dy = 0$

20) $(y + 1)dx - (x + 1)dy = 0$

21) $2ydx + xdy = 0$

22) $\sin(y)dx + \cos(y)dy = 0$

23) $2dx + \sec(x)\cos(y) = 0,\ y(0) = 0$

24) $2x^2dx - 3xy^2dy = 0, y(1) = 0$

25) $2\sin(y)dx + \cos(y)dy = 0,$
$y(0) = \pi/2$

26) $(2y + xy)dx + 2xdy = 0,$
$y(3) = \sqrt{2}$