Chapter Two

Second Order Differential Equations

Second Order Linear Homogeneous Equations

The linear equation

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_1(x)\frac{dy}{dx}+a_0(x)y=F(x)$$

if $F(x)=0$ then it is called ***homogeneous***; otherwise it is called ***non-homogeneous***.

Linear Differential Operator

It is convenient to introduce the symbol D to represent the operation of differentiation with respect to x. That is, we write $Df(x)$ to mean df/dx. Furthermore, we define powers of D to mean taking successive derivatives:

$$D^2 f(x)=D\{Df(x)\}=\frac{d^2 f}{dx^2},\qquad D^3 f(x)=D\{D^2 f(x)\}=\frac{d^3 f}{dx^3}$$

$$(D^2+D-2)f(x)=D^2 f(x)+Df(x)-2f(x)=\frac{d^2 f}{dx^2}+\frac{df}{dx}-2f(x)$$

The Characteristic Equation

The linear second order equation with constant real-number coefficients is

$$\frac{d^2 y}{dx^2}+2a\frac{dy}{dx}+by=0$$

or, in operator notation

$$(D^2+2aD+b)y=0$$

$$(D-r_1)(D-r_2)y=0$$

Solution of $\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+by=0$	
Roots r_1 ***&*** r_2	***Solution***
Real and unequal	$y=C_1e^{r_1x}+C_2e^{r_2x}$
Real and equal	$y=(C_1x+C_2)e^{r_2x}$
Complex conjugate, $\alpha\pm j\beta$	$y=e^{\alpha x}(C_1\cos\beta x \;+\; C_2\sin\beta x)$

Example

Solve the following differential equations:

(a) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$, (b) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$

(c) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+6y=0$, (d) $\frac{d^2y}{dx^2}+4y=0$

Solution

(a)

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$$

The characteristic equation is

$$D^2+D-2=0$$

$$(D-1)(D+2)=0 \;\Rightarrow\; r_1=1 \;\text{ and }\; r_2=-2$$

The solution is

$$y=C_1e^{x}+C_2e^{-2x}$$

(b) $$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$$

The characteristic equation is

$$D^2+4D+4=0$$

$$(D+2)^2=0 \qquad \Rightarrow \qquad r_1=r_2=-2$$

The solution is

$$y=(C_1x+C_2)e^{-2x}$$

(c) $$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+6y=0$$

The characteristic equation is

$$D^2+4D+6=0$$

$$r_{1,2}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

$$r_{1,2}=\frac{-4\pm\sqrt{(4)^2-4(1)(6)}}{2(1)}=\frac{-4\pm\sqrt{16-24}}{2}$$

$$r_{1,2}=\frac{-4\pm\sqrt{-8}}{2}=\frac{-4\pm j2\sqrt{2}}{2}$$

$$r_{1,2}=-2\pm j\sqrt{2} \quad \Rightarrow \quad r_1=-2+j\sqrt{2} \quad \text{and} \quad r_2=-2-j\sqrt{2}$$

$$\Rightarrow \quad \alpha=-2 \quad \text{and} \quad \beta=\sqrt{2}$$

The solution is

$$y=e^{-2x}(C_1\cos\sqrt{2}x \; + \; C_2\sin\sqrt{2}x)$$

(d) $$\frac{d^2y}{dx^2}+4y=0$$

The characteristic equation is

$$D^2+4=0$$

$$(D-j2)(D+j2)=0 \quad \Rightarrow \quad r_1=j2 \quad \text{and} \quad r_2=-j2$$

$$\Rightarrow \quad \alpha=0 \quad \text{and} \quad \beta=2$$

The solution is

$$y=C_1\cos 2x \quad + \quad C_2\sin 2x$$

Exercises

Find the solution of the following Differential Equations

1) $y''-4y'+3y=0$

2) $y''-16y=0$

3) $y''+16y=0$

4) $y''-y'-6y=0$

5) $y''+2y'=0$

6) $y''-2y'+2y=0$

7) $y''+\omega^2y=0,\ (\omega\neq 0)$

8) $y''+4y'+5y=0$

9) $y''-y=0,\ y(0)=6,\ y'(0)=-4$

10) $y''-9y=0,\ y(0)=2,\ y'(0)=0$

11) $y''-4y'+3y=0,\ y(0)=-1,\ y'(0)=-5$

12) $y''-3y'+2y=0,\ y(0)=-1,\ y'(0)=0$

13) $y''+2y'+2y=0$

14) $4y''+4y'+y=0$

15) $y''-9y=0$

16) $y''+6y'+12y=0$

17) $y''-4y'=0$

18) $4y''+4y'+17y=0$

Second Order Non-homogeneous Linear Equations

Now, we solve non-homogeneous equations of the form

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = F(x)$$

The procedure has three basic steps. First, we find the homogeneous solution y_h (h stands for "homogeneous") of the ***reduced equation***

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = 0$$

Second, we find a particular solution y_p of the ***complete*** equation. Finally, we add y_p to y_h to form the general solution of the complete equation. So, the final solution is

$$y = y_h + y_p$$

Variation of Parameters

This method assumes we already know the homogeneous solution

$$y_h = C_1u_1(x) + C_2u_2(x)$$

The method consists of replacing the constants C_1 and C_2 by functions $v_1(x)$ and $v_2(x)$ and then requiring that the new expression

$$y_h = v_1u_1 + v_2u_2$$

and by solving the following two equations

$$v_1'u_1 + v_2'u_2 = 0$$

$$v_1'u_1' + v_2'u_2' = F(x)$$

for the unknown functions v_1' and v_2' using the following matrix notation

$$\begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ F(x) \end{bmatrix}$$

Finally v_1 and v_2 can be found by integration.

In applying the method of ***variation of parameters*** to find the particular solution, the following steps are taken:

i. Find v_1' and v_2' using the following equations

$$v_1' = \frac{\begin{vmatrix} 0 & u_2 \\ F(x) & u_2' \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} = \frac{-u_2 F(x)}{D}, \qquad v_2' = \frac{\begin{vmatrix} u_1 & 0 \\ u_1' & F(x) \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} = \frac{u_1 F(x)}{D}$$

where
$$D = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$$

ii. Integrate v_1' and v_2' to find v_1 and v_2.

iii. Write the particular solution as

$$y_p = v_1 u_1 + v_2 u_2$$

Example

Solve the equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6$

Solution

The homogeneous solution y_h can be found using the reduced equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$$

The characteristic equation is $D^2+2D-3=0$ and the roots of this equation are $r_1=-3$ and $r_2=1$, so

$$y_h=C_1e^{-3x}+C_2e^{x}$$

Then $$u_1=e^{-3x},\ u_2=e^{x}$$

$$D=\begin{vmatrix} e^{-3x} & e^{x} \\ -3e^{-3x} & e^{x} \end{vmatrix}=e^{-2x}+3e^{-2x}=4e^{-2x}$$

$$v_1'=\frac{\begin{vmatrix} 0 & e^{x} \\ 6 & e^{x} \end{vmatrix}}{4e^{-2x}}=\frac{-6e^{x}}{4e^{-2x}}=-\frac{3}{2}e^{3x}, \qquad v_2'=\frac{\begin{vmatrix} e^{-3x} & 0 \\ -3e^{-3x} & 6 \end{vmatrix}}{4e^{-2x}}=\frac{6e^{-3x}}{4e^{-2x}}=\frac{3}{2}e^{-x}$$

$$v_1=\int -\frac{3}{2}e^{3x}dx=-\frac{1}{2}e^{3x}, \qquad v_2=\int \frac{3}{2}e^{-x}dx=-\frac{3}{2}e^{-x}$$

$$y_p=v_1u_1+v_2u_2=\left(-\frac{1}{2}e^{3x}\right)e^{-3x}+\left(-\frac{3}{2}e^{-x}\right)e^{x}=-2$$

$$y=y_h+y_p=C_1e^{-3x}+C_2e^{x}-2$$

Example

Solve the equation $y''-2y'+y=e^{x}\ln(x)$

Solution

The homogeneous solution y_h can be found using the reduced equation

$$y''-2y'+y=0$$

The characteristic equation is

$$D^2-2D+1=0$$

$$(D-1)^2=0$$

The roots are $$r_1=r_2=1$$

The solution is $$y_h=(C_1x+C_2)e^x$$

$$y_h=C_1xe^x+C_2e^x$$

From that we have $u_1(x)=xe^x$, and $u_2(x)=e^x$.

$$D=\begin{vmatrix} xe^x & e^x \\ xe^x+e^x & e^x \end{vmatrix}=xe^{2x}-(xe^{2x}+e^{2x})=-e^{2x}$$

$$v_1'=\frac{\begin{vmatrix} 0 & e^x \\ e^x\ln(x) & e^x \end{vmatrix}}{-e^{2x}}=\frac{-\ln(x)e^{2x}}{-e^{2x}}=\ln(x)$$

$$v_2'=\frac{\begin{vmatrix} xe^x & 0 \\ xe^x+e^x & e^x\ln(x) \end{vmatrix}}{-e^{2x}}=\frac{x\ln(x)e^{2x}}{-e^{2x}}=-x\ln(x)$$

$$v_1=\int\ln(x)dx=x\ln(x)-x$$

$$v_2=-\int x\ln(x)dx$$

$$u=\ln(x)\quad\Rightarrow\quad du=\frac{dx}{x},\qquad dv=xdx\quad\Rightarrow\quad v=\frac{x^2}{2}$$

$$v_2=-\left(\frac{x^2}{2}\ln(x)-\int\frac{x^2}{2}\times\frac{1}{x}dx\right)=-\left(\frac{x^2}{2}\ln(x)-\int\frac{x}{2}dx\right)$$

$$=-\left(\frac{x^2}{2}\ln(x)-\frac{x^2}{4}\right)=\frac{x^2}{4}-\frac{x^2}{2}\ln(x)$$

The particular solution is

$$y_p = v_1u_1 + v_2u_2 = \left(x\ln(x) - x\right)xe^x + \left(\frac{x^2}{4} - \frac{x^2}{2}\ln(x)\right)e^x$$

$$= x^2e^x\ln(x) - x^2e^x + \frac{x^2}{4}e^x - \frac{x^2}{2}e^x\ln(x)$$

$$= \frac{x^2}{2}e^x\ln(x) - \frac{3x^2}{4}e^x$$

The complete solution is

$$y = y_h + y_p = C_1xe^x + C_2e^x + \frac{x^2}{2}e^x\ln(x) - \frac{3x^2}{4}e^x$$

Undetermined Coefficients

This method gives us the particular solution for selected equations.

The Method of Undetermined Coefficients for Selected Equations of the Form $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = F(x)$	
If $F(x)$ ***has a term of***	***The expression for*** y_p
A ***(Constant)***	C ***(Another Constant)***
e^{rx}	Ae^{rx}
$\sin(kx)$, $\cos(kx)$	$B\cos(kx) + C\sin(kx)$
$ax^2 + bx + c$	$Dx^2 + Ex + F$

Example

Solve the equation $y'' + 3y = e^x$

Solution

The homogeneous solution y_h can be found using the reduced equation

$$y'' + 3y = 0$$

The characteristic equation is

$$D^2 + 3 = 0$$

The roots are $r_1 = j\sqrt{3}$, and $r_2 = -j\sqrt{3} \Rightarrow \alpha = 0$ and $\beta = \sqrt{3}$

So, $$y_h = C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x)$$

Since $F(x) = e^x$ then let $y_p = Ae^x \Rightarrow y'_p = Ae^x \Rightarrow y''_p = Ae^x$

Substituting into the differential equation $y'' + 3y = e^x$ we get

$$Ae^x + 3Ae^x = e^x \Rightarrow A + 3A = 1 \Rightarrow A = \frac{1}{4}$$

So, $$y_p = \frac{1}{4}e^x$$

And the complete solution is

$$y = C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x) + \frac{1}{4}e^x$$

Important Note

The expression used for y_p should not have any term similar to the terms of the homogeneous solution. Otherwise, multiply the term that is similar to the homogeneous solution repeatedly by x until it becomes different.

Example

Solve the equation $y'' - 3y' + 2y = 5e^x$

Solution

The homogeneous solution y_h can be found using the reduced equation

$$y'' - 3y' + 2y = 0$$

The characteristic equation is

$$D^2 - 3D + 2 = 0$$

$$(D-1)(D-2) = 0$$

The roots are $r_1 = 1$, and $r_2 = 2$

$$y_h = C_1 e^x + C_2 e^{2x}$$

Since $F(x) = 5e^x$ then let $y_p = Ae^x \Rightarrow y'_p = Ae^x \Rightarrow y''_p = Ae^x$

Substituting into the differential equation $y'' - 3y' + 2y = 5e^x$ we get

$$Ae^x - 3Ae^x + 2Ae^x = 5e^x$$

$$0 = 5e^x \qquad \textbf{\textit{(Wrong Answer)}}$$

The trouble can be traced to the fact that e^x is already a solution in the homogeneous equation $y_h = C_1 e^x + C_2 e^{2x}$.

The appropriate way is to modify the particular solution to replace Ae^x by

$$y_p = Axe^x$$

$$y'_p = Axe^x + Ae^x$$

$$y''_p = Axe^x + Ae^x + Ae^x = Axe^x + 2Ae^x$$

Substituting into the differential equation $y'' - 3y' + 2y = 5e^x$ we get

$$\left(Axe^x + 2Ae^x\right) - 3\left(Axe^x + Ae^x\right) + 2Axe^x = 5e^x$$

$$-Ae^x = 5e^x$$

$$\Rightarrow \qquad A = -5$$

So, $$y_p = -5xe^x$$

The complete solution (general solution) is

$$y = C_1e^x + C_2e^{2x} - 5xe^x$$

Example

Solve the equation

(a) $y'' - 6y' + 9y = e^{3x}$, (b) $y'' - y' = 5e^x - \sin(2x)$

(c) $y'' - y' - 2y = 4x^3$

Solution

(a) The homogeneous solution y_h can be found using the reduced equation

$$y'' - 6y' + 9y = 0$$

The characteristic equation is

$$D^2 - 6D + 9 = 0$$

$$(D-3)^2 = 0$$

The roots are $$r_1 = r_2 = 3$$

$$y_h = (C_1x + C_2)e^{3x}$$

Since $F(x) = e^{3x}$ then let $y_p = Ae^{3x}$. But, Ae^{3x} is similar to the second term of the homogeneous solution so, let $y_p = Axe^{3x}$. Again Axe^{3x} is also similar to the first term of the homogeneous solution. Finally, let

$$y_p = Ax^2e^{3x} \quad \Rightarrow \quad y_p' = 3Ax^2e^{3x} + 2Axe^{3x}$$

$$y_p'' = \left(9Ax^2e^{3x} + 6Axe^{3x}\right) + \left(6Axe^{3x} + 2Ae^{3x}\right)$$
$$= 9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}$$

Substituting into the differential equation $y'' - 6y' + 9y = e^{3x}$ we get

$$\left(9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}\right) - 6\left(3Ax^2e^{3x} + 2Axe^{3x}\right) + 9Ax^2e^{3x} = e^{3x}$$
$$2Ae^{3x} = e^{3x}$$
$$\Rightarrow \quad 2A = 1$$
$$\Rightarrow \quad A = \frac{1}{2}$$

So, $$y_p = \frac{1}{2}x^2e^{3x}$$

The general solution is $$y = (C_1x + C_2)e^{3x} + \frac{1}{2}x^2e^{3x}$$

(b) The homogeneous solution y_h can be found using the reduced equation

$$y'' - y' = 0$$

The characteristic equation is

$$D^2 - D = 0$$
$$D(D-1) = 0$$

The roots are $r_1 = 1$, and $r_2 = 0$

$$y_h = C_1e^x + C_2$$

Since $F(x) = 5e^x - \sin(2x)$ then let $y_p = Ae^x + B\cos(2x) + C\sin(2x)$. But, Ae^x is similar to the first term of the homogeneous solution so, let

$$y_p = Axe^x + B\cos(2x) + C\sin(2x)$$

$$y_p' = Axe^x + Ae^x - 2B\sin(2x) + 2C\cos(2x)$$

$$y_p'' = Axe^x + Ae^x + Ae^x - 4B\cos(2x) - 4C\sin(2x)$$

$$= Axe^x + 2Ae^x - 4B\cos(2x) - 4C\sin(2x)$$

Substituting into the differential equation $y'' - y' = 5e^x - \sin(2x)$ we get

$$\left(Axe^x + 2Ae^x - 4B\cos(2x) - 4C\sin(2x)\right)$$
$$-\left(Axe^x + Ae^x - 2B\sin(2x) + 2C\cos(2x)\right) = 5e^x - \sin(2x)$$

$$Ae^x - (4B + 2C)\cos(2x) + (2B - 4C)\sin(2x) = 5e^x - \sin(2x)$$

$\Rightarrow$ $$A = 5,\quad (4B + 2C) = 0,\quad (2B - 4C) = -1$$

or $$A = 5,\qquad B = -\frac{1}{10},\qquad C = \frac{1}{5}$$

So, $$y_p = 5xe^x - \frac{1}{10}\cos(2x) + \frac{1}{5}\sin(2x)$$

The general solution is

$$y = y_h + y_p = C_1e^x + C_2 + 5xe^x - \frac{1}{10}\cos(2x) + \frac{1}{5}\sin(2x)$$

(c) The homogeneous solution y_h can be found using the reduced equation

$$y'' - y' - 2y = 0$$

The characteristic equation is

$$D^2 - D - 2 = 0$$

$$(D-2)(D+1)=0$$

The roots are $r_1 = 2$, and $r_2 = -1$

$$y_h = C_1 e^{2x} + C_2 e^{-x}$$

Since $F(x) = 4x^3$ then let

$$y_p = Ax^3 + Bx^2 + Cx + D \quad \Rightarrow \quad y_p' = 3Ax^2 + 2Bx + C$$

$$y_p'' = 6Ax + 2B$$

Substituting into the differential equation $y'' - y' - 2y = 4x^3$ we get

$$6Ax + 2B - \left(3Ax^2 + 2Bx + C\right) - 2\left(Ax^3 + Bx^2 + Cx + D\right) = 4x^3$$

$$-2Ax^3 - (3A + 2B)x^2 + (6A - 2B - 2C)x + (2B - C - 2D) = 4x^3$$

$$\Rightarrow \quad A = -2$$

$$3A + 2B = 0 \quad \Rightarrow \quad 3(-2) + 2B = 0 \quad \Rightarrow \quad B = 3$$

$$6A - 2B - 2C = 0 \quad \Rightarrow \quad 6(-2) - 2(3) - 2C = 0 \quad \Rightarrow \quad C = -9$$

$$2B - C - 2D = 0 \quad \Rightarrow \quad 2(3) - (-9) - 2D = 0 \quad \Rightarrow \quad D = \frac{15}{2}$$

So, $y_p = -2x^3 + 3x^2 - 9x + 7.5$

The general solution is

$$y = C_1 e^{2x} + C_2 e^{-x} - 2x^3 + 3x^2 - 9x + 7.5$$

Example

- $y'' = 9x^2 + 2x - 1$

  $D^2 = 0 \;\Rightarrow\; r_1 = r_2 = 0 \;\Rightarrow\; y_h = C_1 x + C_2$

  $y_p = x^2\left(Ax^2 + Bx + C\right)$

- $y'' - y' = x$

  $D^2 - D = 0$

  $D(D-1) = 0 \;\Rightarrow\; r_1 = 0$ and $r_2 = 1 \Rightarrow\; y_h = C_1 + C_2 e^x$

  $y_p = x\left(Ax + B\right)$

- $y'' - 5y = 3e^x - 2x + 1$

  $D^2 - 5 = 0$

  $(D - \sqrt{5})(D + \sqrt{5}) = 0 \;\Rightarrow\; r_1 = \sqrt{5}$ and $r_2 = -\sqrt{5}$

  $y_h = C_1 e^{\sqrt{5}x} + C_2 e^{-\sqrt{5}x}$

  $y_p = Ae^x + Bx + C$

- $y'' - 4y' + 3y = e^{3x} + 2$

  $D^2 - 4D + 3 = 0$

  $(D-3)(D-1) = 0 \;\Rightarrow\; r_1 = 3$ and $r_2 = 1 \Rightarrow\; y_h = C_1 e^{3x} + C_2 e^x$

  $y_p = Axe^{3x} + B$

- $y'' + y = 6e^{x} + 6\cos(x)$

  $D^2 + 1 = 0 \Rightarrow r_1 = j$ and $r_2 = -j \Rightarrow \alpha = 0,\ \beta = 1$

  $y_h = C_1\cos(x) + C_2\sin(x)$

  $y_p = Ae^{3x} + x\big(B\cos(x) + C\sin(x)\big)$

- $y'' - 2y' + y = xe^{x}$

  $D^2 - 2D + 1 = 0$

  $(D-1)^2 = 0 \Rightarrow r_1 = r_2 = 1 \Rightarrow y_h = (C_1x + C_2)e^{x}$

  $y_p = (Ax + B)(x^2e^{x})$

- $y'' + y = x^2\sin(2x)$

  $D^2 + 1 = 0 \Rightarrow r_1 = j$ and $r_2 = -j \Rightarrow \alpha = 0,\ \beta = 1$

  $y_h = C_1\cos(x) + C_2\sin(x)$

  $y_p = (Ax^2 + Bx + C)\ (\cos(2x) + \sin(2x))$

Notes:

To find the roots of an equation $x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$

- r is a root of $f(x)$ if $f(r) = 0$.
- r is a repeated root of $f(x)$ if $f'(r) = 0$.
- If r is a root then r must be a factor of a_n.
- If r is a root then $f(x)$ is divided by $(x - r)$.

Example

- $x^3 + 4x^2 - 3x - 18 = 0$

  Factors of 18 are: $(\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6, \quad \pm 9, \quad \pm 18)$

  $f(1) = (1)^3 + 4(1)^2 - 3(1) - 18 = -16 \neq 0$

  $f(-1) = (-1)^3 + 4(-1)^2 - 3(-1) - 18 = -12 \neq 0$

  $f(2) = (2)^3 + 4(2)^2 - 3(2) - 18 = 0 \quad \Rightarrow \quad r_1 = 2.$

  $f'(x) = 3x^2 + 8x - 3$

  $f'(2) = 3(2)^2 + 8(2) - 3 = 25 \neq 0 \quad \Rightarrow \quad r_1 = 2$ is not a repeated root.

$$
\begin{array}{r|rrrr}
 & x^2 & +6x & +9 & \\
\hline
x \quad -2 & x^3 & +4x^2 & -3x & -18 \\
 & \mp x^3 & \pm 2x^2 & & \\
\hline
 & & 6x^2 & -3x & \\
 & & \mp 6x^2 & \pm 12x & \\
\hline
 & & & 9x & -18 \\
 & & & \mp 9x & \pm 18 \\
\hline
 & & & 0 & 0
\end{array}
$$

$x^2 + 6x + 9 = 0 \quad \Rightarrow \quad (x+3)^2 = 0 \quad \Rightarrow \quad r_2 = r_3 = -3.$

Higher Order Differential Equation

A general differential equation can be put in the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = F(x)$$

Homogeneous Higher Order Differential Equation

It is homogeneous if $F(x) = 0$

Example

$y''' - 6y'' + 11y' - 6y = 0$

$D^3 - 6D^2 + 11D - 6 = 0$

Factors of 6 are: $(\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6)$.

$f(1) = (1)^3 - 6(1)^2 + 11(1) - 6 = 0 \quad \Rightarrow \quad r_1 = 1.$

$f'(D) = 3D^2 - 12D + 11$

$f'(1) = 3(1)^2 - 12(1) + 11 = 2 \neq 0 \quad \Rightarrow \quad r_1 = 1$ is not a repeated root.

$$
\begin{array}{rr|rrrr}
 & & D^2 & -5D & +6 & \\
\hline
D & -1 & D^3 & -6D^2 & +11D & -6 \\
 & & \mp D^3 & \pm D^2 & & \\
\hline
 & & & -5D^2 & +11D & \\
 & & & \pm 5D^2 & \mp 5D & \\
\hline
 & & & & 6D & -6 \\
 & & & & \mp 6D & \pm 6 \\
\hline
 & & & & 0 & 0
\end{array}
$$

$D^2 - 5D + 6 = 0$

$(D-2)(D-3) = 0 \quad \Rightarrow \quad r_2 = 2$ and $r_3 = 3$

$$y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x} + C_3 e^{r_3 x}$$

$$y_h = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x}$$

Example

$$y''' - 6y'' + 2y' + 36y = 0 \quad \Rightarrow \quad D^3 - 6D^2 + 2D + 36 = 0$$

Factors of 36 are: $(\pm 1, \quad \pm 2, \quad \pm 3, \quad . \quad . \quad .)$

$$f(1) = (1)^3 - 6(1)^2 + 2(1) + 36 = 33 \neq 0$$

$$f(-1) = (-1)^3 - 6(-1)^2 + 2(-1) + 36 = 27 \neq 0$$

$$f(2) = (2)^3 - 6(2)^2 + 2(2) + 36 = 24 \neq 0$$

$$f(-2) = (-2)^3 - 6(-2)^2 + 2(-2) + 36 = 0 \quad \Rightarrow \quad r_1 = -2$$

$$f'(D) = 3D^2 - 12D + 2$$

$f'(-2) = 3(-2)^2 - 12(-2) + 2 = 38 \neq 0 \quad \Rightarrow \quad r_1 = -2$ is not a repeated root.

$$\begin{array}{r|rrrr} & D^2 & -8D & +18 & \\ \hline D \quad +2 & D^3 & -6D^2 & +2D & +36 \\ & \mp D^3 & \mp 2D^2 & & \\ \hline & & -8D^2 & +2D & \\ & & \pm 8D^2 & \pm 16D & \\ \hline & & & 18D & +36 \\ & & & \mp 18D & \mp 36 \\ \hline & & & 0 & 0 \end{array}$$

$$D^2 - 8D + 18 = 0$$

$$r_{2,3} = \frac{-B \mp \sqrt{B^2 - 4AC}}{2A} = \frac{8 \mp \sqrt{(8)^2 - 4(1)(18)}}{2(1)} = \frac{8 \mp \sqrt{64 - 72}}{2}$$

$$r_{2,3} = 4 \mp j\sqrt{2} \quad \Rightarrow \quad r_2 = 4 + j\sqrt{2} \quad \& \quad r_3 = 4 - j\sqrt{2}$$

$$\Rightarrow \quad \alpha = 4 \quad \& \quad \beta = \sqrt{2}$$

$$y_h = C_1 e^{r_1 x} + e^{\alpha x}\left(C_2 \cos(\beta x) + C_3 \sin(\beta x)\right)$$

$$y_h = C_1 e^{-2x} + e^{4x}\left(C_2 \cos\left(\sqrt{2}x\right) + C_3 \sin\left(\sqrt{2}x\right)\right)$$

Example

$$y''' + 3y'' + 3y' + y = 0$$

$$D^3 + 3D^2 + 3D + 1 = 0$$

Factors of 1 are: ± 1

$$f(1) = (1)^3 + 3(1)^2 + 3(1) + 1 = 8 \neq 0$$

$$f(-1) = (-1)^3 + 3(-1)^2 + 3(-1) + 1 = 0 \quad \Rightarrow \quad r_1 = -1$$

$$f'(D) = 3D^2 + 6D + 3$$

$$f'(-1) = 3(-1)^2 + 6(-1) + 3 = 0 \quad \Rightarrow \quad r_2 = -1$$

$$f''(D) = 6D + 6$$

$$f''(-1) = 6(-1) + 6 = 0 \quad \Rightarrow \quad r_3 = -1$$

$$y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x} + C_3 e^{r_3 x}$$

$$y_h = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x}$$

Example

$$y^{(4)} + 8y'' + 16y = 0$$

$$D^4 + 8D^2 + 16 = 0$$

$$\left(D^2 + 4\right)^2 = 0$$

$$r_{1,2}^2 = -4 \quad \Rightarrow \quad r_{1,2} = \pm j2 \quad \Rightarrow \quad \alpha_1 = 0 \ \ \& \ \ \beta_1 = 2$$

$$r_{3,4}^2 = -4 \quad \Rightarrow \quad r_{3,4} = \pm j2 \quad \Rightarrow \quad \alpha_2 = 0 \ \ \& \ \ \beta_2 = 2$$

$$y_h = e^{\alpha_1 x}\left(C_1 \cos(\beta_1 x) + C_2 \sin(\beta_1 x)\right) + e^{\alpha_2 x}\left(C_3 \cos(\beta_2 x) + C_4 \sin(\beta_2 x)\right)$$

$$y_h = \left(C_1 \cos(2x) + C_2 \sin(2x)\right) + x\left(C_3 \cos(2x) + C_4 \sin(2x)\right)$$

Example

$y^{(4)}+8y'''+24y''+32y'+16y=0$

$D^4+8D^3+24D^2+32D+16=0$

Factors of 16 are: $(\pm 1,\quad \pm 2,\quad \pm 4,\quad \pm 8,\quad \pm 16)$

$f(-1)=(-1)^4+8(-1)^3+24(-1)^2+32(-1)+16=1\neq 0$

$f(-2)=(-2)^4+8(-2)^3+24(-2)^2+32(-2)+16=0 \qquad \Rightarrow \qquad r_1=-2$

$f'(D)=4D^3+24D^2+48D+32$

$f'(-2)=4(-2)^3+24(-2)^2+48(-2)+32=0 \qquad \Rightarrow \qquad r_2=-2$

$f''(D)=12D^2+48D+48$

$f''(-2)=12(-2)^2+48(-2)+48=0 \qquad \Rightarrow \qquad r_3=-2$

$f'''(D)=24D+48$

$f'''(-2)=24(-2)+48=0 \qquad \Rightarrow \qquad r_4=-2$

$y_h=C_1e^{r_1x}+C_2e^{r_2x}+C_3e^{r_3x}+C_4e^{r_4x}$

$y_h=C_1e^{-2x}+C_2xe^{-2x}+C_3x^2e^{-2x}+C_4x^3e^{-2x}$

Non-homogeneous Higher Order Differential Equation

A differential equation that has the form

$$a_ny^{(n)}+a_{n-1}y^{(n-1)}+\ldots+a_1y'+a_0y=F(x)$$

The solution is

$$y=y_h+y_p$$

Example

$$y''' + y' = \sec(x)$$

First of all we find the homogeneous solution, i.e.,

$$y''' + y' = 0$$

$$D^3 + D = 0$$

$$D(D^2 + 1) = 0 \quad \Rightarrow \quad r_1 = 0,$$

$$\Rightarrow \quad r_{2,3} = \pm j \quad \Rightarrow \quad \alpha = 0 \;\&\; \beta = 1$$

$$y_h = C_1 e^{r_1 x} + e^{\alpha x}\left(C_2 \sin(\beta x) + C_3 \cos(\beta x)\right)$$

$$y_h = C_1 + C_2 \sin(x) + C_3 \cos(x)$$

Now, we find the particular solution with $F(x) = \sec(x)$

$$y_p = v_1 u_1 + v_2 u_2 + v_3 u_3$$

$$u_1 = 1, \quad u_2 = \sin(x), \quad u_3 = \cos(x)$$

$$v_1 = \int \frac{D_1}{Det} dx, \quad v_2 = \int \frac{D_2}{Det} dx, \quad v_3 = \int \frac{D_3}{Det} dx$$

$$Det = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1' & u_2' & u_3' \\ u_1'' & u_2'' & u_3'' \end{vmatrix}, \qquad D_1 = \begin{vmatrix} 0 & u_2 & u_3 \\ 0 & u_2' & u_3' \\ F(x) & u_2'' & u_3'' \end{vmatrix},$$

$$D_2 = \begin{vmatrix} u_1 & 0 & u_3 \\ u_1' & 0 & u_3' \\ u_1'' & F(x) & u_3'' \end{vmatrix}, \qquad D_3 = \begin{vmatrix} u_1 & u_2 & 0 \\ u_1' & u_2' & 0 \\ u_1'' & u_2'' & F(x) \end{vmatrix}$$

$$Det = \begin{vmatrix} 1 & \sin(x) & \cos(x) \\ 0 & \cos(x) & -\sin(x) \\ 0 & -\sin(x) & -\cos(x) \end{vmatrix} = +1\begin{vmatrix} \cos(x) & -\sin(x) \\ -\sin(x) & -\cos(x) \end{vmatrix}$$

$$Det = -\cos^2(x) - \sin^2(x) = -\left(\cos^2(x) + \sin^2(x)\right) = -1$$

$$D_1 = \begin{vmatrix} 0 & \sin(x) & \cos(x) \\ 0 & \cos(x) & -\sin(x) \\ \sec(x) & -\sin(x) & -\cos(x) \end{vmatrix} = +\sec(x)\begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix}$$

$$= \sec(x)\left(-\sin^2(x) - \cos^2(x)\right) = \sec(x) \times (-1) = -\sec(x)$$

$$D_2 = \begin{vmatrix} 1 & 0 & \cos(x) \\ 0 & 0 & -\sin(x) \\ 0 & \sec(x) & -\cos(x) \end{vmatrix} = +1\begin{vmatrix} 0 & -\sin(x) \\ \sec(x) & -\cos(x) \end{vmatrix}$$

$$= \sin(x) \times \sec(x) = \tan(x)$$

$$D_3 = \begin{vmatrix} 1 & \sin(x) & 0 \\ 0 & \cos(x) & 0 \\ 0 & -\sin(x) & \sec(x) \end{vmatrix} = +1\begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & \sec(x) \end{vmatrix}$$

$$= \cos(x) \times \sec(x) = 1$$

$$v_1 = \int \frac{D_1}{D}\,dx = \int \sec(x)dx = \int \sec(x) \times \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\,dx$$

$$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\,dx = \ln\left|\sec(x) + \tan(x)\right|$$

$$v_2 = \int \frac{D_2}{D}dx = -\int \tan(x)dx = -\int \frac{\sin(x)}{\cos(x)}dx = \ln|\cos(x)|$$

$$v_3 = \int \frac{D_3}{D}dx = -\int dx = -x$$

$$y_p = v_1u_1 + v_2u_2 + v_3u_3$$
$$= \ln|\sec(x) + \tan(x)| + (\ln|\cos(x)|)\sin(x) - x\cos(x)$$

The general (complete) solution is

$$y = y_h + y_p$$
$$= C_1 + C_2\sin(x) + C_3\cos(x) + \ln|\sec(x) + \tan(x)| + (\ln|\cos(x)|)\sin(x) - x\cos(x)$$

Example

$$y''' - y' = 4x^3 + 6x^2$$

The homogeneous solution is found by solving $y''' - y' = 0$

$$D^3 - D = 0 \qquad \Rightarrow \quad D(D^2 - 1) = 0$$

$$D(D-1)(D+1) = 0 \;\Rightarrow\; r_1 = 0, \quad r_2 = 1 \;\&\; r_3 = -1$$

$$y_h = C_1 + C_2e^x + C_3e^{-x}$$

To find the particular solution, let

$$y_p = x(Ax^3 + Bx^2 + Cx + D) = Ax^4 + Bx^3 + Cx^2 + Dx$$

$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + D \quad \Rightarrow \quad y_p'' = 12Ax^2 + 6Bx + 2C$$

$$y_p''' = 24Ax + 6B$$

$$(24Ax+6B)-(4Ax^3+3Bx^2+2Cx+D)=4x^3+6x^2$$

$$-4A=4 \quad \Rightarrow \quad A=-1$$

$$-3B=6 \quad \Rightarrow \quad B=-2$$

$$24A-2C=0 \quad \Rightarrow \quad 24(-1)-2C=0 \quad \Rightarrow \quad C=-12$$

$$6B-D=0 \quad \Rightarrow \quad 6(-2)-D=0 \quad \Rightarrow \quad D=-12$$

$$\Rightarrow \quad y_p=-x^4-2x^3-12x^2-12x$$

The general solution is

$$y=y_h+y_p=C_1+C_2e^x+C_3e^{-x}-x^4-2x^3-12x^2-12x$$

Example

$$y^{(4)}-8y''+16y=-18\sin(x)$$

The homogeneous solution is found using

$$y^{(4)}-8y''+16y=0 \quad \Rightarrow \quad D^4-8D^2+16=0$$

$$(D^2-4)^2=0 \quad \Rightarrow \quad r_{1,2}=\pm 2 \;\;\&\;\; r_{3,4}=\pm 2$$

$$y_h=C_1e^{2x}+C_2e^{-2x}+C_3xe^{2x}+C_4xe^{-2x}$$

To find the particular solution, let

$$y_p=A\cos(x)+B\sin(x) \quad \Rightarrow \quad y_p'=-A\sin(x)+B\cos(x)$$

$$y_p''=-A\cos(x)-B\sin(x) \quad \Rightarrow \quad y_p'''=A\sin(x)-B\cos(x)$$

$$y_p^{(4)}=A\cos(x)+B\sin(x)$$

$$(A\cos(x)+B\sin(x))-8(-A\cos(x)-B\sin(x))$$
$$+16(A\cos(x)+B\sin(x))=-18\sin(x)$$

$$(A+8A+16A)\cos(x)+(B+8B+16B)\sin(x)=-18\sin(x)$$

$$\Rightarrow\quad 25A=0\qquad\Rightarrow\quad A=0$$

$$\Rightarrow\quad 25B=-18\quad\Rightarrow\quad B=\frac{-18}{25}$$

$$\Rightarrow\ y_p=-\frac{18}{25}\sin(x)$$

$$y=y_h+y_p\ =C_1e^{2x}+C_2e^{-2x}+C_3xe^{2x}+C_4xe^{-2x}-\frac{18}{25}\sin(x)$$

Exercises

Find the solution of the following Differential Equations

1) $y''+y=3x^2$

2) $y''+2y'+y=x^2$

3) $y''+2y'+3y=27x$

4) $y''+y=-30\sin(4x)$

5) $y''+y=6\sin(x)$

6) $y''+4y'+3y=\sin(x)+2\cos(x)$

7) $y''+4y'+4y=18\cosh(x)$

8) $y''-2y'+2y=2e^x\cos(x)$

9) $y^{(4)}-5y''+4y=10\cos(x)$

10) $y''+y'-2y=3e^x$

11) $y''+y=x^2+x$

12) $y''-y=e^x$

13) $y''-2y'+y=e^x$

14) $y''+y'+y=x^4+4x^3+12x^2$

15) $y'''+2y''-y'-2y=1-4x^3$

16) $y''-2y'+2y=2e^x\cos(x)$