Chapter Three

Partial Differential Equations (PDE)

Functions of Independent Variables

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, ..., x_n)$. A ***real-valued function*** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, ..., x_n)$$

to each element in D. The set D is the function's ***domain***. The set of w-values taken on by f is the function's ***range***. The symbol w is the ***dependent variable*** of f, and f is said to be a function of the n ***independent variables*** x_1 to x_n. We also call the x_j's the function's ***input variables*** and call w the function's ***output variable***.

Example

The value of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(3,0,4)$ is

$$f(3,0,4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5$$

Domains and Ranges

Example

Function	Domain	Range
$w = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$w = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$w = \sin xy$	Both x & y $(-\infty, +\infty)$ or Entire plane	$[-1, +1]$

Function	Domain	Range
$w=\sqrt{x^2+y^2+z^2}$	Entire space	$[0,\infty)$
$w=\dfrac{1}{x^2+y^2+z^2}$	$(x,y,z)\neq(0,0,0)$	$(0,\infty)$
$w=xy\ln z$	Half-space $z>0$	$(-\infty,\infty)$

Partial Derivatives

- The **partial derivative of** $f(x,y)$ **with respect to** x at the point (x_0,y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}=\frac{d}{dx}f(x,y_0)=f_x=\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

provided the limit exists.

- The **partial derivative of** $f(x,y)$ **with respect to** y at the point (x_0,y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}=\frac{d}{dy}f(x_0,y)=f_y=\lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$$

provided the limit exists.

Example

Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4,-5)$ if

$$f(x,y)=x^2+3xy+y-1$$

Solution

To find $\partial f/\partial x$, we treat y as a constant and differentiate with respect to x

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(x^2+3xy+y-1)=2x+3y+0-0=2x+3y$$

The values of $\partial f/\partial x$ at $(4,-5)$ is $2(4)+3(-5)=-7$

To find $\partial f / \partial y$, we treat x as a constant and differentiate with respect to y

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3x + 1 - 0 = 3x + 1$$

The values of $\partial f / \partial y$ at $(4,-5)$ is $3(4) + 1 = 13$

Example

Find $\partial f / \partial y$ if $f(x, y) = y\sin(xy)$

Solution

We treat x as a constant and f as a product of y and $\sin(xy)$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y\sin(xy)\right) = y\frac{\partial}{\partial y}\sin(xy) + \sin(xy)\frac{\partial}{\partial y}(y)$$

$$= \left(y\cos(xy)\right)\frac{\partial}{\partial y}(xy) + \sin(xy) = xy\cos(xy) + \sin(xy)$$

Example

Find f_x and f_y if $f(x, y) = \dfrac{2y}{y + \cos x}$

Solution

We treat f as a quotient

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y\sin x}{(y + \cos x)^2}$$

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right) = \frac{(y+\cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{\partial y}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(2) - 2y(1)}{(y+\cos x)^2} = \frac{2\cos x}{(y+\cos x)^2}$$

Example

Find $\partial z/\partial x$ for $yz - \ln z = x + y$

Solution

We differentiate both sides of the equation with respect to x, holding y constant and treating z as a differentiable function of x

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}(\ln z) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(y)$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1 \qquad \Rightarrow \qquad \frac{\partial z}{\partial x} = \frac{z}{yz-1}$$

Example

If x, y and z are independent variables and

$$f(x,y,z) = x\sin(y+3z)$$

then $$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[x\sin(y+3z)] = x\frac{\partial}{\partial z}\sin(y+3z)$$

$$= x\cos(y+3z)\frac{\partial}{\partial z}(y+3z) = 3x\cos(y+3z)$$

Second Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}, \qquad \frac{\partial^2 f}{\partial y^2} = f_{yy}, \qquad \frac{\partial^2 f}{\partial x \partial y} = f_{yx}, \qquad \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right),$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$ Differentiate first with respect to y, then with respect to x.

$$f_{yx} = \left(f_y\right)_x$$ Means the same thing

The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout a region containing a point (a,b) and are all continuous at (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Example

If $f(x, y) = x\cos y + ye^x$, find

$$\frac{\partial^2 f}{\partial x^2}, \qquad \frac{\partial^2 f}{\partial y \partial x}, \qquad \frac{\partial^2 f}{\partial y^2}, \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x \partial y},$$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^x) = \cos y + ye^x,$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x\cos y+ye^{x})=-x\sin y+e^{x}, \qquad \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=-\sin y+e^{x}$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=-x\cos y$$

Partial Derivatives of Higher Order

Example

Find f_{yxyz} if $f(x,y,z)=1-2xy^2z+x^2y$

Solution

We first differentiate with respect to the variable y, then x, then y again, and finally with respect to z

$$f_y=-4xyz+x^2, \qquad f_{yx}=-4yz+2x, \qquad f_{yxy}=-4z, \qquad f_{yxyz}=-4$$

Exercises

Find the Partial Derivatives of the functions with respect to each variable

1) $f(x,y)=2x^2-3y-4$ **Ans.** $f_x=4x,\ f_y=-3$

2) $f(x,y)=(x^2-1)(y+2)$ **Ans.** $f_x=2x(y+2),\ f_y=x^2-1$

3) $f(x,y)=(xy-1)^2$ **Ans.** $f_x=2y(xy-1), f_y=2x(xy-1)$

4) $f(x,y)=\sqrt{x^2+y^2}$ **Ans.** $f_x=\frac{x}{\sqrt{x^2+y^2}}, f_y=\frac{y}{\sqrt{x^2+y^2}}$

5) $f(x,y)=\dfrac{1}{x+y}$ **Ans.** $f_x=\dfrac{-1}{(x+y)^2},\ f_y=\dfrac{-1}{(x+y)^2}$

6) $f(x,y)=\dfrac{x+y}{xy-1}$ **Ans.** $f_x=\dfrac{-y^2-1}{(xy-1)^2},\ f_y=\dfrac{-x^2-1}{(xy-1)^2}$

7) $f(x,y)=e^{(x+y+1)}$ **Ans.** $f_x=e^{(x+y+1)},\ f_y=e^{(x+y+1)}$

8) $f(x,y)=\ln(x+y)$ **Ans.** $f_x=\dfrac{1}{x+y},\ f_y=\dfrac{1}{x+y}$

9) $f(x,y)=\sin^2(x-3y)$ **Ans.** $f_x=2\sin(x-3y)\cos(x-3y)$,

$f_y=-6\sin(x-3y)\cos(x-3y)$

10) $f(x,y)=x^y$ **Ans.** $f_x=yx^{y-1},\ f_y=x^y\ln(x)$

11) $f(x,y)=\int_x^y g(t)dt$, **Ans.** $f_x=-g(x),\ f_y=g(y)$

(g continuous for all t)

12) $f(x,y,z)=1+xy^2-2z^2$ **Ans.** $f_x=y^2,\ f_y=2xy,\ f_z=-4z$

13) $f(x,y,z)=x-\sqrt{y^2+z^2}$ **Ans.** $f_x=1,\ f_y=-y\left(y^2+z^2\right)^{-1/2}$,

$f_z=-z\left(y^2+z^2\right)^{-1/2}$

14) $f(x,y,z)=\sin^{-1}(xyz)$ **Ans.** $f_x=\dfrac{yz}{\sqrt{1-x^2y^2z^2}}$,

$f_y=\dfrac{xz}{\sqrt{1-x^2y^2z^2}}$,

$f_z=\dfrac{xy}{\sqrt{1-x^2y^2z^2}}$

15) $f(x,y,z)=\ln(x+2y+3z)$ ***Ans.*** $f_x=\dfrac{1}{x+2y+3z},\ f_y=\dfrac{2}{x+2y+3z},$

$$f_z=\frac{3}{x+2y+3z}$$

16) $f(x,y,z)=e^{-\left(x^2+y^2+z^2\right)}$ ***Ans.*** $f_x=-2xe^{-\left(x^2+y^2+z^2\right)},$

$$f_y=-2ye^{-\left(x^2+y^2+z^2\right)},$$

$$f_z=-2ze^{-\left(x^2+y^2+z^2\right)}$$

17) $f(x,y,z)=\tanh(x+2y+3z)$ ***Ans.*** $f_x=sech^2(x+2y+3z),$

$$f_y=2sech^2(x+2y+3z),$$

$$f_z=3sech^2(x+2y+3z)$$

18) $f(t,\alpha)=\cos(2\pi t-\alpha)$ ***Ans.*** $f_t=-2\pi\sin(2\pi t-\alpha),$

$$f_\alpha=\sin(2\pi t-\alpha)$$

$$h_\rho=\sin(\varphi)\cos(\theta),$$

19) $h(\rho,\varphi,\theta)=\rho\sin(\varphi)\cos(\theta)$ ***Ans.*** $h_\varphi=\rho\cos(\varphi)\cos(\theta),$

$$h_\theta=-\rho\sin(\varphi)\sin(\theta)$$

Find the second order Partial Derivatives of the functions with respect to each variable

1) $f(x,y)=x+y+xy$ ***Ans.*** $f_{xx}=0,\ f_{yy}=0,\ f_{xy}=1$

2) $g(x,y) = x^2 y + \cos(y) + y\sin(x)$

Ans. $g_{xx} = 2y - y\sin(x)$,

$g_{yy} = -\cos(y)$,

$g_{xy} = 2x + \cos(x)$

3) $r(x,y) = \ln(x+y)$

Ans. $r_{xx} = \dfrac{-1}{(x+y)^2}$,

$r_{yy} = \dfrac{-1}{(x+y)^2}$,

$r_{xy} = \dfrac{-1}{(x+y)^2}$

Find the mixed Partial Derivatives for the following functions

1) $w = \ln(2x+3y)$

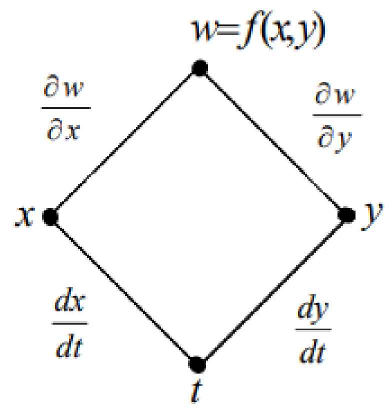
2) $w = e^x + x\ln(y) + y\ln(x)$

3) $w = xy^2 + x^2y^3 + x^3y^4$

4) $w = x\sin(y) + y\sin(x) + xy$

Chain Rule

If $w = f(x, y)$ has continuous partial derivatives f_x and f_y and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Example

Use the Chain Rule to find the derivative of

$$w = xy$$

with respect to t along the path

$$x = \cos(t) \quad \& \quad y = \sin(t)$$

What is the derivative's value at $t = \pi / 2$?

Solution

We apply the Chain Rule to find dw/dt as follows

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$=\frac{\partial}{\partial x}(xy)\times\frac{d}{dt}(\cos(t))+\frac{\partial}{\partial y}(xy)\times\frac{d}{dt}(\sin(t))$$

$$=y\times(-\sin(t))+x\times(\cos(t))$$

$$=(\sin(t))\times(-\sin(t))+(\cos(t))\times(\cos(t))$$

$$=-\sin^2(t)+\cos^2(t)$$

$$=\cos(2t)$$

We can check the result with a more direct calculation as a function of t

$$w=xy=\cos(t).\sin(t)=\frac{1}{2}\sin(2t)$$

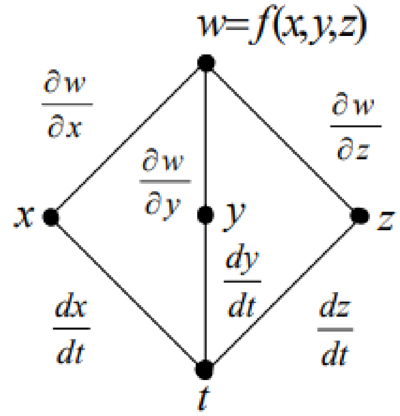
So, $$\frac{dw}{dt}=\frac{d}{dt}\left(\frac{1}{2}\sin(2t)\right)=\frac{1}{2}\times 2\cos(2t)=\cos(2t)$$

In either case, at a given value of t,

$$\left(\frac{dw}{dt}\right)_{t=\pi/2}=\cos\left(2\times\frac{\pi}{2}\right)=\cos\pi=-1$$

Chain Rule for Functions of Three Independent Variables

Here we have three routes from w to t instead of two, but finding dw/dt is still the same. Read each route, multiplying derivatives along the way; then add.



$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example

Find dw/dt if

$$w=xy+z, \qquad x=\cos(t), \qquad y=\sin(t), \qquad z=t$$

What is the derivative's value at $t=0$?

Solution

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

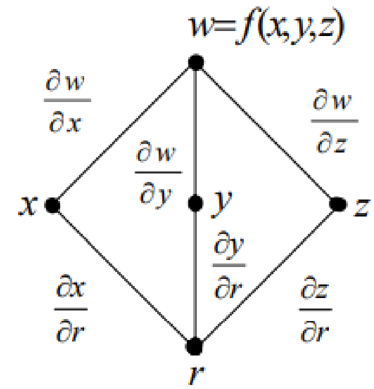
$$=(y)(-\sin(t))+(x)(\cos(t))+(1)(1)$$

$$=(\sin(t))(-\sin(t))+(\cos(t))(\cos(t))+1$$

$$=-\sin^2(t)+\cos^2(t)+1=1+\cos(2t)$$

$$\left(\frac{dw}{dt}\right)_{t=0}=1+\cos(0)=2$$

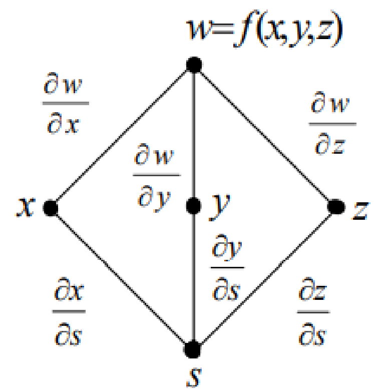
Chain Rule for Two Independent Variables and Three Intermediate Variables



Suppose that $w=f(x,y,z)$, $x=g(r,s)$, $y=h(r,s)$, $z=k(r,s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$



Example

Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w=x+2y+z^2,\qquad x=\frac{r}{s},\qquad y=r^2+\ln s,\qquad z=2r$$

Solution

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}=(1)\left(\frac{1}{s}\right)+(2)(2r)+(2z)(2)$$

$$=\frac{1}{s}+4r+4(2r)=\frac{1}{s}+12r$$

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}=(1)\left(-\frac{r}{s^2}\right)+(2)\left(\frac{1}{s}\right)+(2z)(0)$$

$$=\frac{2}{s}-\frac{r}{s^2}$$

Example

Express $\partial w / \partial r$ and $\partial w / \partial s$ in terms of r and s if

$$w = x^2 + y^2, \qquad x = r - s, \qquad y = r + s$$

Solution

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = (2x)(1) + (2y)(1)$$

$$= 2(r - s) + 2(r + s) \;\; = 4r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = (2x)(-1) + (2y)(1)$$

$$= -2(r - s) + 2(r + s) \;\; = 4s$$

Implicit Differentiation

Suppose that $F(x, y) = 0$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example

Find dy/dx if $y^2 - x^2 = \sin xy$

Solution

Take $F(x, y) = y^2 - x^2 - \sin xy$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy} = \frac{2x + y\cos xy}{2y - x\cos xy}$$

Exercises

Find dw/dt at the given value for the following functions

1) $w = x^2 + y^2,\ x = \cos(t),\ y = \sin(t)$, at $t = \pi$ **Ans.** $\left.\dfrac{dw}{dt}\right|_{t=\pi} = 0$

2) $w = \dfrac{x}{z} + \dfrac{y}{z}, x = \cos^2(t), y = \sin^2(t), z = \dfrac{1}{t}$, at $t = 3$ **Ans.** $\left.\dfrac{dw}{dt}\right|_{t=3} = 1$

3) $w = 2ye^x - \ln(z),\ x = \ln(t^2+1),\ y = \tan^{-1}(t), z = e^t$, at $t = 1$ **Ans.** $\left.\dfrac{dw}{dt}\right|_{t=1} = \pi + 1$

Answer the following questions:

1) ***Find*** $\partial z/\partial u$ ***and*** $\partial z/\partial v$ ***for*** $z = 4e^x \ln(y),\ x = \ln(u\cos(v)),\ y = u\sin(v)$ ***at the point*** $(u,v) = (2, \pi/4)$.

Ans. $z_u = \sqrt{2}(\ln 2 + 2),\ z_v = -2\sqrt{2}(\ln 2 - 2)$

2) ***Find*** $\partial w/\partial u$ ***and*** $\partial w/\partial v$ ***for*** $w = xy + yz + xz,\ \ x = u + v,\ \ y = u - v,\ z = uv$ ***at the point*** $(u,v) = (1/2, 1)$.

Ans. $z_u = 3,\ z_v = -\dfrac{3}{2}$

3) ***Find*** $\partial u/\partial x$, $\partial u/\partial y$ ***and*** $\partial u/\partial z$ ***for*** $u=\dfrac{p-q}{q-r}$, $p=x+y+z$, $q=x-y+z$, $r=x+y-z$ ***at the point*** $(x,y,z)=(\sqrt{3},2,1)$

Ans. $u_x=0,\ u_y=1,\ u_z=-2$

4) ***Find*** $\partial w/\partial r$ ***if*** $w=(x+y+z)^2$, $x=r-s$, $y=\cos(r+s)$, $z=\sin(r+s)$ ***at the point*** $(r,s)=(1,-1)$

Ans. 12

5) ***Find*** $\partial w/\partial v$ ***if*** $w=x^2+(y/x)$, $x=u-2v+1$, ***and*** $y=2u+v-2$, ***at the point*** $(u,v)=(0,0)$

Ans. -7

6) ***Find*** $\partial z/\partial u$ ***and*** $\partial z/\partial v$ ***if*** $z=5\tan^{-1}(x)$ ***and*** $x=e^u+\ln v$ ***at the point*** $(u,v)=(\ln 2,1)$

Ans. $z_u=2,\ z_v=1$

Find dy/dx ***at the given point for the following functions***

1) $x^3-2y^2+xy=0$, $(1,1)$ ***Ans.*** $4/3$

2) $x^2+xy+y^2-7=0$, $(1,2)$ ***Ans.*** $-4/5$