Chapter Four

Laplace Transform

Let $f(t)$ be a function of t. Then the ***Laplace Transform*** of $f(t)$ is

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

Laplace Transform for some Functions

- $f(t) = 1$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \int_0^\infty (1).e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = -\frac{1}{s}\left[e^{-\infty} - e^0\right] = \frac{1}{s}$$

So, $\mathcal{L}\{k\} = \dfrac{k}{s}$

- $f(t) = e^{at}$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at}.e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = -\frac{1}{s-a}e^{-(s-a)t}\Big|_0^\infty$$

$$\mathcal{L}\{e^{at}\} = -\frac{1}{s-a}(0-1) = \frac{1}{s-a}$$

- $\cos(at)$, $\sin(at)$

$$e^{jat} = \cos(at) + j\sin(at)$$

$$\mathcal{L}\{e^{jat}\} = \mathcal{L}\{\cos(at)\} + j\mathcal{L}\{\sin(at)\}$$

$$\mathcal{L}\{e^{jat}\} = \frac{1}{s-ja} \times \frac{s+ja}{s+ja} = \frac{s+ja}{s^2+a^2}$$

$$\mathcal{L}\{e^{jat}\} = \frac{s}{s^2+a^2} + j\frac{a}{s^2+a^2}$$

By comparison $\Rightarrow$ $\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2+a^2}$ & $\mathcal{L}\{\sin(at)\} = \dfrac{a}{s^2+a^2}$

- $f(t)=\sinh(at)=\frac{1}{2}\left(e^{at}-e^{-at}\right)$

$$\mathcal{L}\{\sinh(at)\}=\frac{1}{2}\left(\frac{1}{s-a}-\frac{1}{s+a}\right)=\frac{1}{2}\frac{s+a-s+a}{s^2-a^2}=\frac{a}{s^2-a^2}$$

- $f(t)=\cosh(at)=\frac{1}{2}\left(e^{at}+e^{-at}\right)$

$$\mathcal{L}\{\cosh(at)\}=\frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)=\frac{1}{2}\frac{s+a+s-a}{s^2-a^2}=\frac{s}{s^2-a^2}$$

- $f(t)=t$

$$\mathcal{L}\{t\}=\int_0^\infty t.e^{-st}\,dt$$

$$u=t \quad \Rightarrow \quad du=dt\,, \quad dv=e^{-st}dt \quad \Rightarrow \quad v=-\frac{1}{s}e^{-st}$$

$$\mathcal{L}\{t\}=\frac{-t}{s}e^{-st}\Big|_0^\infty+\int_0^\infty\frac{1}{s}e^{-st}dt=0-0-\frac{1}{s^2}e^{-st}\Big|_0^\infty=\frac{-1}{s^2}\left(e^{-\infty}-e^0\right)$$

$$\mathcal{L}\{t\}=\frac{1}{s^2}$$

In general, $\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}}$

- $f(t)=u(t)$

$$\mathcal{L}\{u(t)\}=\int_0^\infty u(t)e^{-st}dt=\int_0^\infty(1)e^{-st}dt=\frac{-1}{s}e^{-st}\Big|_0^\infty=\frac{1}{s}$$

- $f(t)=u_a(t)$

$$\mathcal{L}\{u_a(t)\} = \int_0^\infty u_a(t)e^{-st}dt = \int_a^\infty (1)e^{-st}dt = \frac{-1}{s}e^{-st}\Big|_a^\infty = \frac{e^{-as}}{s}$$

Laplace Transform Properties

1) Linearity

If $F_1(s) = \mathcal{L}\{f_1(t)\}$ and $F_2(s) = \mathcal{L}\{f_2(t)\}$ then

$$\mathcal{L}\{C_1 f_1(t) + C_2 f_2(t)\} = C_1 F_1(s) + C_2 F_2(s)$$

Example

$$\mathcal{L}\{4t^2 - 3\cos(2t) + 5e^{-t}\} = 4\times\frac{2!}{s^3} - 3\times\frac{s}{s^2+4} + 5\times\frac{1}{s+1}$$

$$= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}$$

2) Shifting Property

- If $F(s) = \mathcal{L}\{f(t)\}$ then

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

- If $F(s) = \mathcal{L}\{f(t)\}$ then

$$\mathcal{L}\{f(t-a)\} = F(s)e^{-as}$$

Example

$$\mathcal{L}\{e^{-t}\cos(2t)\}$$

Here, $f(t) = \cos(2t)$ & $a = -1$, then $F(s) = \dfrac{s}{s^2+4}$ and

$$\mathcal{L}\{e^{-t}\cos(2t)\} = F(s+1) = \frac{s+1}{(s+1)^2+4}$$

3) Derivative Property

If $F(s) = \mathcal{L}\{f(t)\}$ then

- $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
- $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

4) Integral Property

If $F(s) = \mathcal{L}\{f(t)\}$ then

$$\mathcal{L}\left\{\int_0^t f(u)du\right\} = \frac{F(s)}{s}$$

Example

$$\mathcal{L}\left\{\int_0^t \sin(2u)du\right\}$$

Here, $f(t) = \sin(2t)$ then $F(s) = \dfrac{2}{s^2+4}$ and

$$\mathcal{L}\left\{\int_0^t \sin(2u)du\right\} = \frac{F(s)}{s} = \frac{2}{s(s^2+4)}$$

5) Multiplication by t^n

If $F(s) = \mathcal{L}\{f(t)\}$ then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^nF(s)}{ds^n}$$

Example

$\mathcal{L}\{t^2 \sin(t)\}$

Here, $f(t) = \sin(t)$ then $F(s) = \dfrac{1}{s^2+1}$

$$\mathcal{L}\{t^2 \sin(t)\} = (-1)^2 \frac{d^2}{ds^2}\left\{\frac{1}{s^2+1}\right\}$$

$$\frac{d}{ds}\left\{\frac{1}{s^2+1}\right\} = \frac{-2s}{(s^2+1)^2}$$

$$\frac{d^2}{ds^2}\left\{\frac{1}{s^2+1}\right\} = \frac{(s^2+1)^2 \times (-2) + 2s \times 2(s^2+1)(2s)}{(s^2+1)^4} = \frac{(s^2+1)\left[(-2)(s^2+1) + 8s^2\right]}{(s^2+1)^4}$$

$$= \frac{6s^2-2}{(s^2+1)^3}$$

6) Division by t

If $F(s) = \mathcal{L}\{f(t)\}$ then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$$

Example

$$\mathcal{L}\left\{\frac{\sin(t)}{t}\right\}$$

Here, $f(t) = \sin(t)$ then $F(s) = \dfrac{1}{s^2+1}$ and

$$\mathcal{L}\left\{\frac{\sin(t)}{t}\right\} = \int_s^\infty \frac{1}{u^2+1} du = \tan^{-1}(u)\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}(s) = \tan^{-1}\left(\frac{1}{s}\right)$$

7) Initial-Value Property

If $F(s)=\mathcal{L}\{f(t)\}$ then

$$\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)$$

8) Final-Value Property

If $F(s)=\mathcal{L}\{f(t)\}$ then

$$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$$

Gamma Function

The gamma function $\Gamma(n)$ can be defined as

$$\Gamma(n)=\int_0^\infty t^{n-1}e^{-t}dt$$

Important Properties of the Gamma Function

- $\Gamma(n+1)=n\Gamma(n)$
- $\Gamma(n+1)=n!$ for $n=0,1,2,\ldots.$
- $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

Example

- $\Gamma(n+1)=\int_0^\infty t^n e^{-t}dt$

$$u=t^n \;\Rightarrow\; du=n.t^{n-1}dt,\; dv=e^{-t}dt \;\Rightarrow\; v=-e^{-t}$$

$$\Gamma(n+1)=-e^{-t}t^n\Big|_0^\infty+n\int_0^\infty t^{n-1}e^{-t}dt=n\Gamma(n)$$

- $\Gamma(1)=\int_{0}^{\infty} e^{-t}(1)dt=-e^{-t}\Big|_{0}^{\infty}=-\left(e^{-\infty}-e^{0}\right)=1$

  So, $\Gamma(1)=1$

  $\Gamma(2)=1\times\Gamma(1)=1$

  $\Gamma(3)=2\times\Gamma(2)=2$

  $\Gamma(4)=3\times\Gamma(3)=3\times 2=3!$

  $\Gamma(n+1)=n!$

- $\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{1}{2}\sqrt{\pi}$

- $\Gamma\left(\frac{7}{2}\right)=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)=\frac{5}{2}\times\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{15}{8}\sqrt{\pi}$

- $\mathcal{L}\{t^{n}\}=\frac{n!}{s^{n+1}}=\frac{\Gamma(n+1)}{s^{n+1}}$

- $\mathcal{L}\{t^{3/2}\}=\frac{\Gamma\left(\frac{5}{2}\right)}{s^{5/2}}=\frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{5/2}}=\frac{\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{5/2}}=\frac{\frac{3}{4}\sqrt{\pi}}{s^{5/2}}$

Laplace Transform of Periodic Functions

If $f(t)$ is a periodic function with a period of $T>0$ such that $f(t+T)=f(t)$ then

$$\mathcal{L}\{f(t)\}=\frac{\int_{0}^{T} f(t)e^{-st}dt}{1-e^{-sT}}$$

Evaluation of Integrals

If $F(s) = \mathcal{L}\{f(t)\}$ then

$$\int_0^\infty f(t)e^{-st}dt = F(s) \quad \Rightarrow \quad \int_0^\infty f(t)dt = F(0)$$

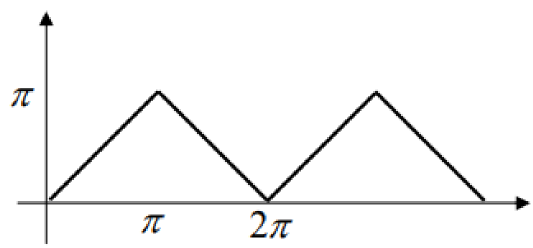
This can be used in finding a lot of integrals. For example

$$\int_0^\infty e^{-st}\sin(t)dt = \mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}$$

$$\int_0^\infty e^{-2t}\sin(t)dt = \mathcal{L}\{\sin(t)\}\Big|_{s=2} = \frac{1}{(2)^2+1} = \frac{1}{5}$$

Example

Find the Laplace transform for the the signal shown:



Solution

This is a periodic signal with a period of $T = 2\pi$.

$$f(t) = \begin{cases} t & 0 < t < \pi \\ 2\pi - t & \pi < t < 2\pi \end{cases}$$

$$F(s) = \frac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}} = \frac{F_1(s)}{1-e^{-2\pi s}}$$

$$F_1(s) = \int_0^\pi t\,e^{-st}\,dt + \int_\pi^{2\pi}(2\pi - t)e^{-st}dt$$

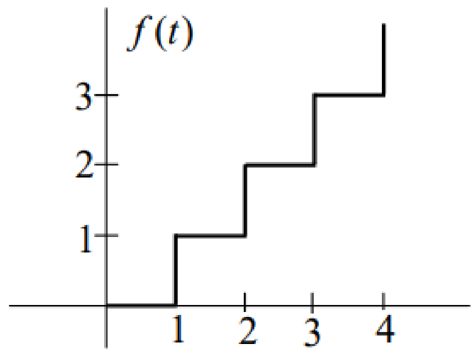
$$= \int_0^\pi t\,e^{-st}\,dt + \int_\pi^{2\pi} 2\pi\,e^{-st}\,dt - \int_\pi^{2\pi} t\,e^{-st}\,dt$$

$$=\left.\frac{-te^{-st}}{s}\right|_0^{\pi}+\int_0^{\pi}\frac{e^{-st}}{s}dt-\left.\frac{2\pi}{s}e^{-st}\right|_{\pi}^{2\pi}+\left.\frac{te^{-st}}{s}\right|_{\pi}^{2\pi}-\int_{\pi}^{2\pi}\frac{e^{-st}}{s}dt$$

and so on.

Example

Express the function $f(t)$ in terms of the unit step function. Then find its Laplace transform.



Solution

$$f(t)=\begin{cases}0 & 0<t<1\\1 & 1<t<2\\2 & 2<t<3\\3 & 3<t<4\\4 & 4<t<5\\ \vdots & \vdots\end{cases}$$

$$f(t)=u_1(t)-u_2(t)+2\big(u_2(t)-u_3(t)\big)+3\big(u_3(t)-u_4(t)\big)+4\big(u_4(t)-u_5(t)\big)+\ldots.$$

$$f(t)=u_1(t)+u_2(t)+u_3(t)+u_4(t)+\ldots.$$

$$\mathcal{L}\{f(t)\}=\frac{e^{-s}}{s}+\frac{e^{-2s}}{s}+\frac{e^{-3s}}{s}+\frac{e^{-4s}}{s}+\ldots$$

$$=\frac{e^{-s}}{s}\left(1+e^{-s}+e^{-2s}+e^{-3s}+\ldots\right)=\frac{e^{-s}}{s}\cdot\frac{1}{1-e^{-s}}$$

Exercises

Find the Laplace Transform of the following functions

1) $f(t) = t\cos(\omega t)$ **Ans.** $F(s) = \dfrac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$

2) $f(t) = t\sin(\omega t)$ **Ans.** $F(s) = \dfrac{2\omega s}{(s^2 + \omega^2)^2}$

3) $f(t) = t\cosh(a t)$ **Ans.** $F(s) = \dfrac{s^2 + a^2}{(s^2 - a^2)^2}$

4) $f(t) = t\sinh(a t)$ **Ans.** $F(s) = \dfrac{2a s}{(s^2 - a^2)^2}$

5) $f(t) = 2te^t$ **Ans.** $F(s) = \dfrac{2}{(s-1)^2}$

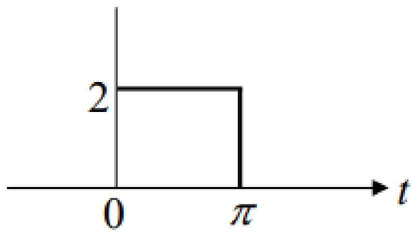
6) $f(t) = e^{-2t}\cos(t)$ **Ans.** $F(s) = \dfrac{s+2}{(s+2)^2 + 1}$

7) $f(t) = e^{-\alpha t}\left(A\cos(\beta t) + B\sin(\beta t)\right)$ **Ans.** $F(s) = \dfrac{A(s+\alpha) + B\beta}{(s+\alpha)^2 + \beta^2}$

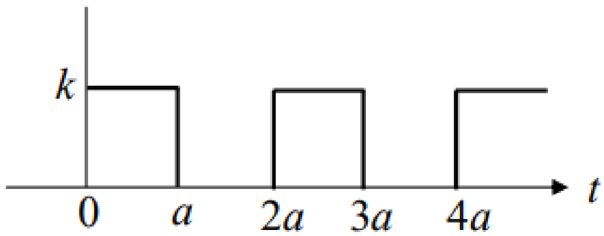
8) $f(t) = (t-\pi)u_\pi(t)$ **Ans.** $F(s) = \dfrac{e^{-\pi s}}{s^2}$

9) $f(t) = tu_2(t)$ **Ans.** $F(s) = \left(\dfrac{1}{s^2} + \dfrac{2}{s}\right)e^{-2s}$

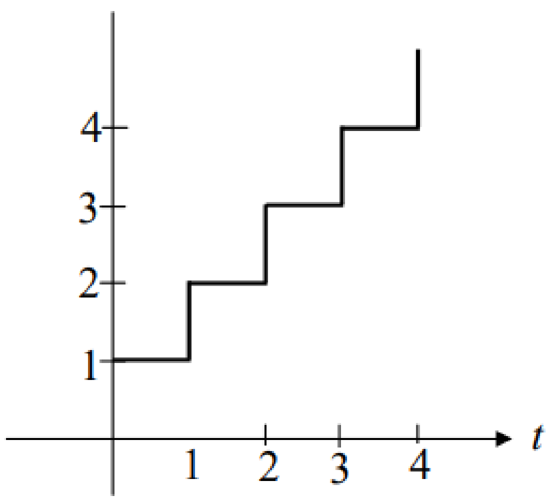
10) $f(t) = u_\pi(t)\sin(t)$ **Ans.** $F(s) = \dfrac{-e^{-\pi s}}{s^2 + 1}$

11) 

Ans. $F(s) = \dfrac{2(1 - e^{-\pi s})}{s}$

12) 

Ans. $F(s) = \dfrac{k}{s(1 + e^{-as})}$

13) 

Ans. $F(s) = \dfrac{1}{s(1 + e^{-s})}$

14) $f(t) = t, \quad 0 < t < 2$ ***Ans.*** $F(s) = \dfrac{1}{s^2} - \dfrac{e^{-2s}}{s^2} - \dfrac{2e^{-2s}}{s}$

15) $f(t) = K\sin(\omega t) \quad 0 < t < \pi/\omega$ ***Ans.*** $F(s) = \dfrac{K\omega(1 + e^{-\pi s/\omega})}{(s^2 + \omega^2)}$

16) $f(t) = K\cos(\omega t) \quad 0 < t < 2\pi/\omega$ ***Ans.*** $F(s) = \dfrac{K\,s\,(1 - e^{-2\pi s/\omega})}{(s^2 + \omega^2)}$

Find the Laplace Transform of the following periodic functions

1) $f(t)=\pi - t, \quad (0<t<2\pi)$ ***Ans.*** $F(s)=\dfrac{\pi s - 1 + (\pi s + 1)e^{-2\pi s}}{s^2(1-e^{-2\pi s})}$

2) $f(t)=4\pi^2 - t^2, \quad (0<t<2\pi)$ ***Ans.*** $F(s)=\dfrac{(4\pi^2 s^2 - 2)e^{2\pi s} + 4\pi s + 2}{s^3(e^{2\pi s}-1)}$

3) $f(t)=e^t, \quad (0<t<2\pi)$ ***Ans.*** $F(s)=\dfrac{e^{2\pi(1-s)} - 1}{(1-s)(1-e^{-2\pi s})}$

4) $f(t)=\begin{cases} t & 0<t<\pi \\ 0 & \pi<t<2\pi \end{cases}$ ***Ans.*** $F(s)=\dfrac{\dfrac{1}{s^2}(1-e^{-\pi s}) - \dfrac{\pi}{s}e^{-\pi s}}{(1-e^{-2\pi s})}$

5) $f(t)=\begin{cases} t & 0<t<\pi \\ \pi - t & \pi<t<2\pi \end{cases}$ ***Ans.*** $F(s)=\dfrac{\dfrac{\pi}{s}e^{-\pi s}(e^{-\pi s}-1) + \dfrac{1}{s^2}(e^{-\pi s}-1)^2}{(1-e^{-2\pi s})}$

Inverse Laplace Transform

$F(s)$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$	$F(s)$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
$\frac{1}{s}$	1	$\frac{1}{s^2+a^2}$	$\frac{\sin(at)}{a}$
$\frac{1}{s^2}$	t	$\frac{s}{s^2+a^2}$	$\cos(at)$
$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$	$\frac{1}{s^2-a^2}$	$\frac{\sinh(at)}{a}$
$\frac{1}{s-a}$	e^{at}	$\frac{s}{s^2-a^2}$	$\cosh(at)$

Example

Find $f(t)$ if

(a) $F(s) = \dfrac{5}{s+3}$, (b) $F(s) = \dfrac{s+1}{s^2+1}$, (c) $F(s) = \dfrac{1}{(s+25)^2}$,

(d) $F(s) = \dfrac{s+2}{(s+2)^2+1}$, (e) $F(s) = \dfrac{s}{(s-1)^2-4}$, (f) $F(s) = \dfrac{1}{s^2(s^2+1)}$,

(g) $F(s) = \dfrac{4}{s^2+2s+10}$

Solution

(a) $f(t)=5e^{-3t}$

(b) $F(s)=\dfrac{s+1}{s^2+1}=\dfrac{s}{s^2+1}+\dfrac{1}{s^2+1} \quad \Rightarrow \quad f(t)=\cos(t)+\sin(t)$

(c) Using the shifting property $\mathcal{L}\{e^{at}f(t)\}=F(s-a)$ then $\mathcal{L}^{-1}\{F(s-a)\}=e^{at}f(t)$.

Here, we have $F(s)=\dfrac{1}{s^2}$ with $a=-25$. Since, $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\}=t$ then

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+25)^2}\right\}=t\cdot e^{-25t}$$

(d) Here, we have a shifting of -2 with $F_1(s)=\dfrac{s}{s^2+1}$. So, $f_1(t)=\cos(t)$ and

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\}=f(t)=\cos(t)\cdot e^{-2t}$$

(e) $F(s)=\dfrac{s}{(s-1)^2-4}=\dfrac{s-1+1}{(s-1)^2-4}=\dfrac{s-1}{(s-1)^2-4}+\dfrac{1}{(s-1)^2-4}$

So, $f(t)=e^t\cosh(2t)+\dfrac{1}{2}e^t\sinh(2t)$

(f) We know that $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin(t)$ and using the property of division by s which means an integration in time domain, we get

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+1}\right\}=\int_0^t \sin(u)du=1-\cos(t)$$

Again using the same property we get

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s^2+1}\right\}=\int_0^t (1-\cos(u))du=t-\sin(t)$$

(g) $F(s)=\frac{4}{s^2+2s+10}=\frac{4}{s^2+2s+1-1+10}=\frac{4}{(s+1)^2+9}$

This is $F_1(s)=\frac{4}{s^2+9}$ with a shifting of $a=-1$. So, $f_1(t)=\frac{4}{3}\sin(3t)$ and $f(t)=\frac{4}{3}e^{-t}\sin(3t)$

Solution of Inverse Using Partial Fraction Method

Example

Find $f(t)$ if $F(s)=\frac{3s+7}{s^2-2s-3}$

Solution

$$\frac{3s+7}{(s+1)(s-3)}=\frac{A}{s+1}+\frac{B}{s-3}$$

$$\Rightarrow \quad 3s+7=A(s-3)+B(s+1)$$

First Method

$$3s+7=As-3A+Bs+B \quad \Rightarrow \quad A+B=3$$

$$\Rightarrow \quad -3A+B=7$$

Solving these two equations we get $A=-1$ and $B=4$

Second Method

$$3s+7=A(s-3)+B(s+1)$$

At $s=-1$ we get $-3+7=-4A \Rightarrow A=-1$

At $s=3$ we get $9+7=4B \Rightarrow B=4$

Third Method

$$A=\left.\frac{3s+7}{(s-3)}\right|_{s=-1}=\frac{3(-1)+7}{-1-3}=\frac{4}{-4}=-1$$

$$B=\left.\frac{3s+7}{(s+1)}\right|_{s=3}=\frac{3(3)+7}{3+1}=\frac{16}{4}=4$$

$$F(s)=\frac{3s+7}{(s+1)(s-3)}=\frac{-1}{s+1}+\frac{4}{s-3}$$

So $$f(t)=-e^{-t}+4e^{3t}$$

Example

Find $f(t)$ if $F(s)=\dfrac{3s+1}{(s-1)(s^2+1)}$

Solution

$$\frac{3s+1}{(s-1)(s^2+1)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+1}$$

$$A=\left.\frac{3s+1}{s^2+1}\right|_{s=1}=\frac{3(1)+1}{(1)^2+1}=\frac{4}{2}=2$$

So $\dfrac{3s+1}{(s-1)(s^2+1)}=\dfrac{2}{s-1}+\dfrac{Bs+C}{s^2+1} \Rightarrow 3s+1=2(s^2+1)+(Bs+C)(s-1)$

$$3s+1=2s^2+2+Bs^2-Bs+Cs-C$$

$$3s+1=(2+B)s^2+(C-B)s+2-C \Rightarrow 2+B=0 \Rightarrow B=-2$$

$$\Rightarrow C-B=3 \Rightarrow C=1$$

$$F(s)=\frac{3s+1}{(s-1)(s^2+1)}=\frac{2}{s-1}+\frac{-2s+1}{s^2+1}$$

$$F(s)=\frac{2}{s-1}-\frac{2s}{s^2+1}+\frac{1}{s^2+1}$$

So $\quad f(t)=2e^t-2\cos(t)+\sin(t)$

Note:

If $f(t)$ has the form of $\dfrac{K}{(s-s_i)^n}$ then the partial fraction of it will be

$$\frac{K}{(s-s_i)^n}=\frac{C_1}{(s-s_i)}+\frac{C_2}{(s-s_i)^2}+\ldots\ldots+\frac{C_{n-1}}{(s-s_i)^{n-1}}+\frac{C_n}{(s-s_i)^n}$$

$$C_n = F(s)(s-s_i)^n\Big|_{s=s_i} \qquad C_{n-1} = \frac{1}{1!}\frac{d}{ds}\left[F(s)(s-s_i)^n\right]\Bigg|_{s=s_i}$$

$$C_{n-2} = \frac{1}{2!}\frac{d^2}{ds^2}\left[F(s)(s-s_i)^n\right]\Bigg|_{s=s_i}$$

or in general $$C_{n-k} = \frac{1}{k!}\frac{d^k}{ds^k}\left[F(s)(s-s_i)^n\right]\Bigg|_{s=s_i}$$

Example

Find $f(t)$ if $F(s) = \dfrac{s-1}{(s+1)^3}$

Solution

$$\frac{s-1}{(s+1)^3} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$C = s-1\Big|_{s=-1} = -2$$

$$B = \frac{1}{1!}\frac{d}{ds}(s-1)\Bigg|_{s=-1} = 1$$

$$A = \frac{1}{2!}\frac{d^2}{ds^2}(s-1)\Bigg|_{s=-1} = \frac{1}{2}\frac{d}{ds}(1)\Bigg|_{s=-1} = 0$$

So $F(s) = \dfrac{1}{(s+1)^2} - \dfrac{2}{(s+1)^3}$

$$\Rightarrow\ f(t) = e^{-t}\left(t - t^2\right)$$

Example

$$F(s)=\frac{-4s}{(s^2+4)^2}=\frac{As+B}{(s^2+4)}+\frac{Cs+D}{(s^2+4)^2} \qquad (H.W)$$

Another Solution

$$\int\frac{-4s}{(s^2+4)^2}ds=-2\int\frac{2s}{(s^2+4)^2}ds$$

$$=\frac{-2}{-1}\left(\frac{1}{s^2+4}\right)=\sin(2t)$$

$$\Rightarrow\ f(t)=-t\sin(2t)$$

Example

- $F(s)=\tan^{-1}(s) \qquad \Rightarrow \qquad F'(s)=\frac{1}{1+s^2}$

  $t\cdot f(t)\Leftrightarrow -\frac{d}{ds}[F(s)] \quad \Rightarrow \quad -t\cdot f(t)\Leftrightarrow\frac{1}{s^2+1}$

  $-t\cdot f(t)=\sin(t) \qquad \Rightarrow \qquad f(t)=-\frac{\sin(t)}{t}$

- $F(s)=\ln(s^2+2) \qquad \Rightarrow \qquad F'(s)=\frac{2s}{s^2+2}$

  $-t\cdot f(t)=2\cos\left(\sqrt{2}t\right) \quad \Rightarrow \quad f(t)=\frac{-2}{t}\cos\left(\sqrt{2}t\right)$

- $F(s)=\dfrac{e^{-4s}}{s-2}$

We know that $\mathcal{L}\{g(t-a)u_a(t)\}=G(s)e^{-as}$. Here $a=4$

$$\frac{1}{s-2}\Leftrightarrow e^{2t}=g(t) \qquad \Rightarrow \qquad g(t-4)=e^{2(t-4)}$$

$$f(t)=e^{2(t-4)}u_4(t)=e^{2(t-4)}u(t-4)$$

Exercises

Find the Inverse Laplace Transform of the following functions

1) $F(s)=\dfrac{1}{s^2+s}$ ***Ans.*** $f(t)=1-e^{-t}$

2) $F(s)=\dfrac{1}{s^3+4s}$ ***Ans.*** $f(t)=(1+\cos(2t))/4$

3) $F(s)=\dfrac{1}{s}\left(\dfrac{s-a}{s+a}\right)$ ***Ans.*** $f(t)=2e^{-at}-1$

4) $F(s)=\dfrac{8}{s^4-4s^2}$ ***Ans.*** $f(t)=\sinh(2t)-2t$

5) $F(s)=\dfrac{1}{s^4-2s^3}$ ***Ans.*** $f(t)=\left(e^{2t}-1-2t-2t^2\right)/8$

6) $F(s)=\dfrac{1}{s^2}\left(\dfrac{s+1}{s^2+1}\right)$ ***Ans.*** $f(t)=1+t-\cos(t)-\sin(t)$

7) $F(s)=\dfrac{1}{s^2+s/2}$ **Ans.** $f(t)=2-2e^{-t/2}$

8) $F(s)=\dfrac{1}{s^3-ks^2}$ **Ans.** $f(t)=\dfrac{1}{k^2}\left(e^{kt}-1\right)-\dfrac{t}{k}$

9) $F(s)=\dfrac{5}{s^3-5s}$ **Ans.** $f(t)=\cosh\left(\sqrt{5}\,t\right)-1$

10) $F(s)=\dfrac{1}{s^4-4s^2}$ **Ans.** $f(t)=\dfrac{1}{8}\sinh(2t)-\dfrac{1}{4}t$

11) $F(s)=\dfrac{n\pi}{(s+2)^2+n^2\pi^2}$ **Ans.** $f(t)=e^{-2t}\sin(n\pi t)$

12) $F(s)=\dfrac{s}{(s+3)^2+1}$ **Ans.** $f(t)=e^{-3t}\left(\cos(t)-3\sin(t)\right)$

13) $F(s)=\dfrac{2(e^{-2s}-e^{-4s})}{s}$ **Ans.** $f(t)=\begin{cases}2 & 2<t<4\\ 0 & elsewhere\end{cases}$

14) $F(s)=\dfrac{e^{-as}}{s^2}$ **Ans.** $f(t)=\begin{cases}t-a & t>a\\ 0 & t<a\end{cases}$

15) $F(s)=\dfrac{(e^{-s}+e^{-2s}-3e^{-3s}+e^{-6s})}{s^2}$ **Ans.** $f(t)=\begin{cases} t-1 & 1<t<2 \\ 2t-3 & 2<t<3 \\ 6-t & 3<t<6 \\ 0 & elsewhere \end{cases}$

16) $F(s)=\dfrac{se^{-\pi s}}{s^2+4}$ **Ans.** $f(t)=\begin{cases} \cos(2t) & t>\pi \\ 0 & elsewhere \end{cases}$

17) $F(s)=\dfrac{e^{-\pi s}}{s^2+2s+2}$ **Ans.** $f(t)=\begin{cases} -e^{\pi-t}\sin(t) & t>\pi \\ 0 & elsewhere \end{cases}$

18) $F(s)=\dfrac{s+12}{s^2+4s}$ **Ans.** $f(t)=3-2e^{-4t}$

19) $F(s)=\dfrac{3s}{s^2+2s-8}$ **Ans.** $f(t)=2e^{-4t}+e^{2t}$

20) $F(s)=\dfrac{3s^2-2s-1}{(s-3)(s^2+1)}$ **Ans.** $f(t)=2e^{3t}+\cos(t)+\sin(t)$

21) $F(s)=\dfrac{10-4s}{(s-2)^2}$ **Ans.** $f(t)=e^{2t}(2t-4)$

22) $F(s)=\dfrac{s^3+3s^2-s-3}{(s^2+2s+5)^2}$ **Ans.** $f(t)=e^{-t}\left(\cos(2t)-2t\sin(2t)\right)$

23) $F(s)=\dfrac{s^3-7s^2+14s-9}{(s-1)^2(s-2)^3}$ **Ans.** $f(t)=te^{t}-0.5t^2e^{2t}$

Solution of Differential Equation Using Laplace Transform

Here, we use the derivative property as follows:

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

Example

Solve the following differential equation using Laplace transform

(a) $y'' + 2y' + y = t$ with $y(0) = 0$, and $y'(0) = 1$

(b) $y' + y = 4$ with $y(0) = 0$

(c) $y' + y = \sin(t)$ with $y(0) = 1$

(d) $ty' + y = t$

(e) $ty'' - ty' + y = 0$ with $y(0) = 0$, and $y'(0) = 1$

Solution

(a) Taking the Laplace transform of the two sides, we get

$$\left(s^2Y(s) - sy(0) - y'(0)\right) + 2\left(sY(s) - y(0)\right) + Y(s) = \frac{1}{s^2}$$

$$\left(s^2Y(s) - s(0) - 1\right) + 2\left(sY(s) - 0\right) + Y(s) = \frac{1}{s^2}$$

$$s^2Y(s) - 1 + 2sY(s) + Y(s) = \frac{1}{s^2}$$

$$Y(s)\left(s^2 + 2s + 1\right) = \frac{1}{s^2} + 1$$

$$Y(s)\left(s^2 + 2s + 1\right) = \frac{1 + s^2}{s^2}$$

$$Y(s)=\frac{1+s^2}{s^2\left(s^2+2s+1\right)}=\frac{1+s^2}{s^2(s+1)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{(s+1)}+\frac{D}{(s+1)^2}$$

$$B=\left.\frac{1+s^2}{(s+1)^2}\right|_{s=0}=1$$

$$A=\frac{d}{ds}\left[\frac{1+s^2}{(s+1)^2}\right]\Bigg|_{s=0}=\left.\frac{(s+1)^2(2s)-2\left(1+s^2\right)(s+1)}{(s+1)^4}\right|_{s=0}=\frac{-2}{1}=-2$$

$$D=\left.\frac{1+s^2}{s^2}\right|_{s=-1}=2$$

$$C=\frac{d}{ds}\left[\frac{1+s^2}{s^2}\right]\Bigg|_{s=-1}=\left.\frac{s^2(2s)-\left(1+s^2\right)(2s)}{s^4}\right|_{s=-1}=\frac{-2+4}{1}=2$$

$$Y(s)=\frac{-2}{s}+\frac{1}{s^2}+\frac{2}{s+1}+\frac{2}{(s+1)^2}\quad\Rightarrow\quad y(t)=-2+t+2e^{-t}+2t\cdot e^{-t}$$

(b) Taking the Laplace transform of the two sides, we get

$$sY(s)-y(0)+Y(s)=\frac{4}{s}\quad\Rightarrow\quad sY(s)-0+Y(s)=\frac{4}{s}$$

$$Y(s)(s+1)=\frac{4}{s}\qquad\Rightarrow\quad Y(s)=\frac{4}{s(s+1)}$$

$$\frac{4}{s(s+1)}=\frac{A}{s}+\frac{B}{s+1}$$

$$A=\left.\frac{4}{(s+1)}\right|_{s=0}=4,\quad\text{and}\quad B=\left.\frac{4}{s}\right|_{s=-1}=-4$$

$$Y(s)=\frac{4}{s}-\frac{4}{s+1}\quad\Rightarrow\quad y(t)=4-4e^{-t}$$

(c) Taking the Laplace transform of the two sides, we get

$$sY(s) - y(0) + Y(s) = \frac{1}{s^2+1} \quad \Rightarrow \quad Y(s)(s+1) = \frac{1}{s^2+1} + 1$$

$$Y(s)(s+1) = \frac{1+s^2+1}{s^2+1} \quad \Rightarrow \quad Y(s) = \frac{s^2+2}{(s+1)(s^2+1)}$$

$$\frac{s^2+2}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$A = \left.\frac{s^2+2}{(s^2+1)}\right|_{s=-1} = \frac{(-1)^2+2}{(-1)^2+1} = \frac{3}{2}$$

$$s^2+2 = A(s^2+1) + (Bs+C)(s+1)$$

$$s^2+2 = (A+B)s^2 + (B+C)s + A + C$$

$$1 = A + B \Rightarrow B = 1 - \frac{3}{2} = -\frac{1}{2}, \quad B + C = 0 \Rightarrow C = \frac{1}{2}$$

$$Y(s) = \frac{3/2}{s+1} + \frac{(-1/2)s + (1/2)}{s^2+1}$$

$$Y(s) = \frac{3}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1} \quad \Rightarrow \quad y(t) = \frac{3}{2}e^{-t} - \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)$$

(d) Taking the Laplace transform of the two sides, we get

$$\frac{-d}{ds}\left(sY(s) - y(0)\right) + Y(s) = \frac{1}{s^2} \quad \Rightarrow \quad \frac{-d}{ds}\left(sY(s) - 0\right) + Y(s) = \frac{1}{s^2}$$

$$\frac{-d}{ds}\left(sY(s)\right) + Y(s) = \frac{1}{s^2} \quad \Rightarrow \quad -\left(sY'(s) + Y(s)\right) + Y(s) = \frac{1}{s^2}$$

$$-sY'(s) = \frac{1}{s^2} \quad \Rightarrow \quad Y'(s) = \frac{-1}{s^3} \quad \Rightarrow \quad \frac{dY(s)}{ds} = \frac{-1}{s^3}$$

$$dY(s)=\frac{-1}{s^3}ds \quad \Rightarrow \quad Y(s)=\int\frac{-1}{s^3}ds \Rightarrow \quad Y(s)=\frac{1}{2s^2} \quad \Rightarrow \quad y(t)=\frac{1}{2}t$$

(e) Taking the Laplace transform of the two sides, we get

$$\frac{-d}{ds}\left(s^2Y(s)-sy(0)-y'(0)\right)-\frac{-d}{ds}\left(sY(s)-y(0)\right)+Y(s)=0$$

$$\frac{-d}{ds}\left(s^2Y(s)-0-1\right)+\frac{d}{ds}\left(sY(s)-0\right)+Y(s)=0$$

$$\frac{-d}{ds}\left(s^2Y(s)-1\right)+\frac{d}{ds}\left(sY(s)\right)+Y(s)=0$$

$$-\left(s^2Y'(s)+Y(s)\times(2s)\right)+\left(sY'(s)+Y(s)\right)+Y(s)=0$$

$$\left(-s^2+s\right)Y'(s)+(-2s+2)Y(s)=0 \quad \Rightarrow \quad \left(-s^2+s\right)Y'(s)=(2s-2)Y(s)$$

$$Y'(s)=\frac{dY(s)}{ds}=\frac{2s-2}{-s^2+s}Y(s) \qquad \Rightarrow \quad \frac{dY(s)}{Y(s)}=\frac{2(s-1)}{-s(s-1)}ds$$

$$\int\frac{dY(s)}{Y(s)}=\int\frac{2}{-s}ds \quad \Rightarrow \quad \ln(Y(s))=-2\ln(s) \quad \Rightarrow \quad \ln(Y(s))=\ln(s^{-2})$$

$$\ln(Y(s))=\ln\left(\frac{1}{s^2}\right) \quad \Rightarrow \qquad Y(s)=\frac{1}{s^2} \qquad \Rightarrow \quad y(t)=t$$

Example

Solve the following differential equations

(a) $y_1'=-y_2 \qquad y_1(0)=1$

$y_2'=y_1 \qquad y_2(0)=0$

(b) $\dfrac{dx}{dt}=2x-3y \qquad x(0)=8$

$\dfrac{dy}{dt}=y-2x \qquad y(0)=3$

Solution

(a) Taking the Laplace transform of the two equations, we get

$$sY_1(s) - y_1(0) = -Y_2(s) \quad \Rightarrow \quad sY_1(s) - 1 = -Y_2(s) \quad \Rightarrow \quad sY_1(s) + Y_2(s) = 1$$

$$sY_2(s) - y_2(0) = Y_1(s) \quad \Rightarrow \quad sY_2(s) - 0 = Y_1(s) \quad \Rightarrow \quad -Y_1(s) + sY_2(s) = 0$$

$$\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Y_1(s) = \frac{\begin{vmatrix} 1 & 1 \\ 0 & s \end{vmatrix}}{\begin{vmatrix} s & 1 \\ -1 & s \end{vmatrix}} = \frac{s}{s^2+1} \quad \Rightarrow \quad y_1(t) = \cos(t)$$

$$Y_2(s) = \frac{\begin{vmatrix} s & 1 \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} s & 1 \\ -1 & s \end{vmatrix}} = \frac{1}{s^2+1} \quad \Rightarrow \quad y_2(t) = \sin(t)$$

(b) Taking the Laplace transform of the two equations, we get

$$sX(s) - x(0) = 2X(s) - 3Y(s)$$

$$sX(s) - 8 = 2X(s) - 3Y(s)$$

$$(s-2)X(s) + 3Y(s) = 8 \qquad . \quad . \quad . \quad (1)$$

$$sY(s) - y(0) = Y(s) - 2X(s)$$

$$sY(s) - 3 = Y(s) - 2X(s)$$

$$2X(s) + (s-1)Y(s) = 3 \qquad . \quad . \quad . \quad (2)$$

$$\begin{bmatrix} s-2 & 3 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$X(s)=\frac{\begin{vmatrix}8 & 3\\ 3 & s-1\end{vmatrix}}{\begin{vmatrix}s-2 & 3\\ 2 & s-1\end{vmatrix}}=\frac{8(s-1)-3\times 3}{(s-2)(s-1)-3\times 2}=\frac{8s-8-9}{s^2-3s+2-6}=\frac{8s-17}{s^2-3s-4}$$

$$X(s)=\frac{8s-17}{(s+1)(s-4)}=\frac{A}{s+1}+\frac{B}{s-4}$$

$$A=\left.\frac{8s-17}{s-4}\right|_{s=-1}=\frac{8(-1)-17}{-1-4}=\frac{-25}{-5}=5$$

$$B=\left.\frac{8s-17}{s+1}\right|_{s=4}=\frac{8(4)-17}{4+1}=\frac{15}{5}=3$$

$$X(s)=\frac{5}{s+1}+\frac{3}{s-4}\quad\Rightarrow\quad x(t)=5e^{-t}+3e^{4t}$$

$$Y(s)=\frac{\begin{vmatrix}s-2 & 8\\ 2 & 3\end{vmatrix}}{\begin{vmatrix}s-2 & 3\\ 2 & s-1\end{vmatrix}}=\frac{3(s-2)-8\times 2}{(s-2)(s-1)-3\times 2}=\frac{3s-6-16}{s^2-3s+2-6}=\frac{3s-22}{s^2-3s-4}$$

$$Y(s)=\frac{3s-22}{(s+1)(s-4)}=\frac{C}{s+1}+\frac{D}{s-4}$$

$$C=\left.\frac{3s-22}{s-4}\right|_{s=-1}=\frac{3(-1)-22}{-1-4}=\frac{-25}{-5}=5$$

$$D=\left.\frac{3s-22}{s+1}\right|_{s=4}=\frac{3(4)-22}{4+1}=\frac{-10}{5}=-2$$

$$Y(s)=\frac{5}{s+1}-\frac{2}{s-4}\quad\Rightarrow\quad y(t)=5e^{-t}-2e^{4t}$$

Exercises

Find the solution of the following Differential Equations

1) $4y'' + \pi^2 y = 0$, $y(0) = 2$, $y'(0) = 0$.

2) $y'' + \omega^2 y = 0$, $y(0) = A$, $y'(0) = B$.

3) $y'' + 2y' - 8y = 0$, $y(0) = 1$, $y'(0) = 8$.

4) $y'' - 2y' - 3y = 0$, $y(0) = 1$, $y'(0) = 7$.

5) $y'' - ky' = 0$, $y(0) = 2$, $y'(0) = k$.

6) $y'' + ky' - 2k^2 y = 0$, $y(0) = 2$, $y'(0) = 2k$.

7) $y' + 4y = 0$, $y(0) = 2.8$

8) $y' + \frac{1}{2}y = 17\sin(2t)$, $y(0) = -1$.

9) $y'' - y' - 6y = 0$, $y(0) = 6$, $y'(0) = 13$.

10) $y'' - \frac{1}{4}y = 0$, $y(0) = 4$, $y'(0) = 0$.

11) $y'' - 4y' + 4y = 0$, $y(0) = 2.1$, $y'(0) = 3.9$

12) $y'' + 2y' + 2y = 0$, $y(0) = 1$, $y'(0) = -3$.

13) $y'' + 7y' + 12y = 21e^{3t}$, $y(0) = 3.5$, $y'(0) = -10$.

14) $y'' + 9y = 10e^{-t}$, $y(0) = 0$, $y'(0) = 0$.

15) $y'' + 3y' + 2.25y = 9t^3 + 64$, $y(0) = 1$, $y'(0) = 31.5$

16) $y'' - 6y' + 5y = 29\cos(2t)$, $y(0) = 3.2$, $y'(0) = 6.2$

17) $y'' + 2y' + 2y = 0$, $y(0) = 0$, $y'(0) = 1$.

18) $y'' + 2y' + 17y = 0$, $y(0) = 0$, $y'(0) = 12$.

19) $y'' - 4y' + 5y = 0,$ $y(0) = 1,$ $y'(0) = 2.$

20) $9y'' - 6y' + y = 0,$ $y(0) = 3,$ $y'(0) = 1.$

21) $y'' - 2y' + 10y = 0,$ $y(0) = 3,$ $y'(0) = 3.$

22) $4y'' - 4y' + 37y = 0,$ $y(0) = 3,$ $y'(0) = 1.5$

23) $4y'' - 8y' + 5y = 0,$ $y(0) = 0,$ $y'(0) = 1.$

24) $y'' + y' + 1.25y = 0,$ $y(0) = 1,$ $y'(0) = -0.5$

25) $y'' + y = 2\cos(t),$ $y(0) = 2,$ $y'(0) = 0.$

26) $y'' - 4y' + 3y = 0,$ $y(0) = 3,$ $y'(0) = 7.$

27) $y'' + 2y' + y = e^{-2t},$ $y(0) = 0,$ $y'(0) = 0.$

28) $y'' + 2y' - 3y = 10\sinh(2t),$ $y(0) = 0,$ $y'(0) = 4.$

29) $y'' + 25y = 10(\cos(5t) - 2\sin(5t)),$ $y(0) = 1,$ $y'(0) = 2.$

30) $y_1' = -y_2,\ y_2' = y_1,$ $y_1(0) = 1,$ $y_2(0) = 0.$

31) $y_1' + y_2 = 2\cos(t),\ y_2' + y_1 = 0,$ $y_1(0) = 0,$ $y_2(0) = 1.$

32) $y_1' + y_2' = 2\sinh(t),\ y_2' + y_3' = e^t,$ $y_1(0) = y_2(0) = 1,$ $y_3(0) = 0.$

$y_1' + y_3' = 2e^t + e^{-t}$

33) $-2y_1' + y_2' + y_3' = 0,\ y_1' + y_2' = 4t + 2,$ $y_1(0) = y_2(0) = y_3(0) = 0$

$y_2' + y_3 = t^2 + 2$

34) $y_1'' = y_1 + 3y_2, y_2'' = 4y_1 - 4e^t,\ y_1(0) = 2, y_1'(0) = 3,\ y_2(0) = 1, y_2'(0) = 2.$

35) $y_1'' + y_2 = -5\cos(2t),\quad y_2'' + y_1 = 5\cos(2t),\quad y_1(0) = y_1'(0) = 1,\quad y_2(0) = -1,$

$y_2'(0) = 1.$