CHAPTER 7

# Laplace Transforms

## 7.1 Introduction: A Mixing Problem

**Figure 7.1 depicts a mixing problem with valved input feeders. At time $t = 0$, valve $A$ is opened, delivering 6 L/min of a brine solution containing 0.04 kg of salt per liter. At $t = 10$ min, valve $A$ is closed and valve $B$ is opened, delivering 6 L/min of brine at a concentration of 0.02 kg/L. Initially, 30 kg of salt are dissolved in 1000 L of water in the tank. The exit valve $C$, which empties the tank at 6 L/min, maintains the contents of the tank at constant volume. Assuming the solution is kept well stirred, determine the amount of salt in the tank at all times $t > 0$.**



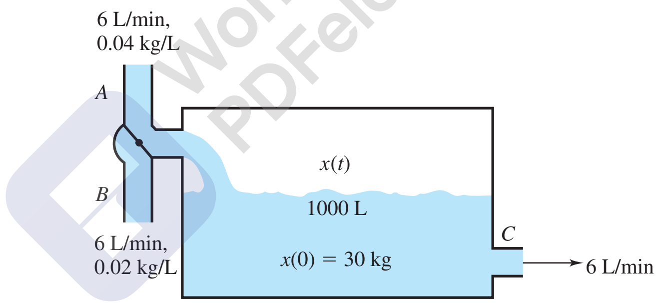


**Figure 7.1** Mixing tank with valve $A$ open

We analyzed a simpler version of this problem in Example 1 of Section 3.2. Let $x(t)$ be the amount of salt (in kilograms) in the tank at time $t$. Then of course $x(t)/1000$ is the concentration, in kilograms per liter. The salt content is depleted at the rate $(6 \text{ L/min}) \times (x(t)/1000 \text{ kg/L}) = 3x(t)/500$ kg/min through the exit valve. Simultaneously, it is enriched through valves $A$ and $B$ at the rate $g(t)$, given by

$$\textbf{(1)} \qquad g(t) = \begin{cases} 0.04 \text{ kg/L} \times 6 \text{ L/min} = 0.24 \text{ kg/min}\,, & 0 < t < 10 \text{ (valve } A)\,, \\ 0.02 \text{ kg/L} \times 6 \text{ L/min} = 0.12 \text{ kg/min}\,, & t > 10 \text{ (valve } B)\,. \end{cases}$$

Thus, $x(t)$ changes at a rate

$$\frac{d}{dt}x(t) = g(t) - \frac{3x(t)}{500}\,,$$

or

$$\textbf{(2)} \qquad \frac{dx}{dt} + \frac{3}{500}x = g(t)\,,$$

with initial condition

**(3)** $$x(0) = 30 .$$

To solve the initial value problem (2)–(3) using the techniques of Chapter 4, we would have to break up the time interval $(0, \infty)$ into two subintervals $(0, 10)$ and $(10, \infty)$. On these subintervals, the nonhomogeneous term $g(t)$ is constant, and the method of undetermined coefficients could be applied to equation (2) to determine general solutions for each subinterval, each containing one arbitrary constant (in the associated homogeneous solutions). The initial condition (3) would fix this constant for $0 < t < 10$, but then we would need to evaluate $x(10)$ and use it to reset the constant in the general solution for $t > 10$.

Our purpose here is to illustrate a new approach using Laplace transforms. As we will see, this method offers several advantages over the previous techniques. For one thing, it is much more convenient in solving initial value problems for linear constant coefficient equations when the forcing term contains jump discontinuities.

The **Laplace transform** of a function $f(t)$, defined on $[0, \infty)$, is given by[†]

**(4)** $$F(s) \coloneqq \int_0^{\infty} e^{-st} f(t)\, dt .$$

Thus we multiply $f(t)$ by $e^{-st}$ and integrate with respect to $t$ from 0 to $\infty$. This takes a function of $t$ and produces a function of $s$.

In this chapter we'll scrutinize many of the details on this "exchange of functions," but for now let's simply state the main advantage of executing the transform. *The Laplace transform replaces linear constant coefficient differential equations in the t-domain by (simpler) algebraic equations in the s-domain!* In particular, if $X(s)$ is the Laplace transform of $x(t)$, then the transform of $x'(t)$ is simply $sX(s) - x(0)$. Therefore, the information in the differential equation (2) and initial condition (3) is transformed from the $t$-domain to the $s$-domain simply as

| | ***t*-Domain** | ***s*-Domain** |
|---|---|---|
| **(5)** | $x'(t) + \frac{3}{500}x(t) = g(t) , \quad x(0) = 30 ;$ | $sX(s) - 30 + \frac{3}{500}X(s) = G(s) ,$ |

where $G(s)$ is the Laplace transform of $g(t)$. (Notice that we have taken certain linearity properties for granted, such as the fact that the transform preserves sums and multiplications by constants.) We can find $X(s)$ in the $s$-domain without solving any differential equations: the solution is simply

**(6)** $$X(s) = \frac{30}{s + 3/500} + \frac{G(s)}{s + 3/500} .$$

For this procedure to be useful, there has to be an easy way to convert from the $t$-domain to the $s$-domain and vice versa. There are, in fact, tables and theorems that facilitate this conversion in many useful circumstances. We'll see, for instance, that the transform of $g(t)$, despite its unpleasant piecewise specification in equation (1), is given by the single formula

$$G(s) = \frac{0.24}{s} - \frac{0.12}{s}e^{-10s} ,$$

and as a consequence the transform of $x(t)$ equals

$$X(s) = \frac{30}{s + 3/500} + \frac{0.24}{s(s + 3/500)} - \frac{0.12e^{-10s}}{s(s + 3/500)} .$$

[†]*Historical Footnote:* The Laplace transform was first introduced by Pierre Laplace in 1779 in his research on probability. G. Doetsch helped develop the use of Laplace transforms to solve differential equations. His work in the 1930s served to justify the operational calculus procedures earlier used by Oliver Heaviside.

Again by table lookup (and a little theory), we can deduce that

$$x(t) = 200 - 170e^{-3t/500} - 100 \cdot \begin{cases} 0\,, & t \leq 10\,, \\ \left[1 - e^{-3(t-10)/500}\right], & t \geq 10\,. \end{cases} \tag{7}$$

See Figure 7.2.

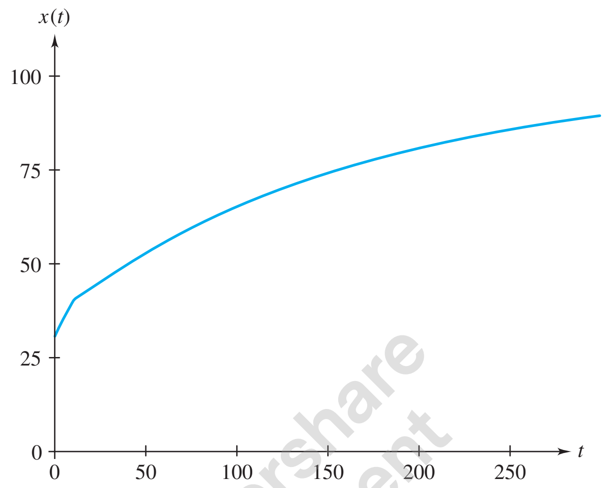


**Figure 7.2** Solution to mixing tank example

Note that to arrive at (7) we did not have to take derivatives of trial solutions, break up intervals, or evaluate constants through initial data. The Laplace transform machinery replaces all of these operations by basic algebra: addition, subtraction, multiplication, division—and, of course, the judicious use of the table. Figure 7.3 depicts the advantages of the transform method.

Unfortunately, the Laplace transform method is less helpful with equations containing variable coefficients or nonlinear equations (and sometimes determining inverse transforms can be a Herculean task!). But it is ideally suited for many problems arising in applications. Thus, we devote the present chapter to this important topic.

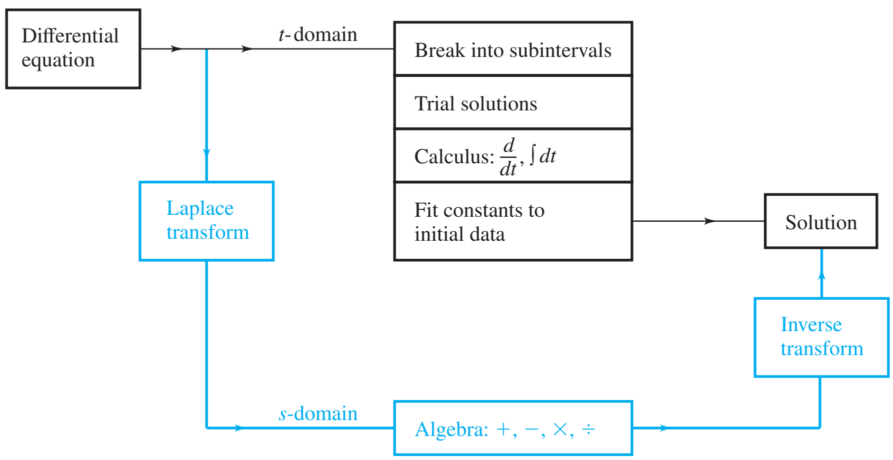


**Figure 7.3** Comparison of solution methods

# 7.2 Definition of the Laplace Transform

In earlier chapters we studied differential operators. These operators took a function and mapped or transformed it (via differentiation) into another function. The Laplace transform, which is an integral operator, is another such transformation.

**Laplace Transform**

**Definition 1.** Let $f(t)$ be a function on $[0, \infty)$. The **Laplace transform** of $f$ is the function $F$ defined by the integral

$$\text{(1)} \qquad F(s) := \int_0^\infty e^{-st} f(t)\,dt\,.$$

The domain of $F(s)$ is all the values of $s$ for which the integral in (1) exists.† The Laplace transform of $f$ is denoted by both $F$ and $\mathscr{L}\{f\}$.

Notice that the integral in (1) is an **improper** integral. More precisely,

$$\int_0^\infty e^{-st} f(t)\,dt := \lim_{N\to\infty} \int_0^N e^{-st} f(t)\,dt$$

whenever the limit exists.



**Example 1** Determine the Laplace transform of the constant function $f(t) = 1, t \geq 0$.

**Solution** Using the definition of the transform, we compute

$$F(s) = \int_0^\infty e^{-st} \cdot 1\,dt = \lim_{N\to\infty} \int_0^N e^{-st}\,dt$$

$$= \lim_{N\to\infty} \frac{-e^{-st}}{s}\Bigg|_{t=0}^{t=N} = \lim_{N\to\infty} \left[\frac{1}{s} - \frac{e^{-sN}}{s}\right].$$

Since $e^{-sN} \to 0$ when $s > 0$ is fixed and $N \to \infty$, we get

$$F(s) = \frac{1}{s} \qquad \text{for} \qquad s > 0\,.$$

When $s \leq 0$, the integral $\int_0^\infty e^{-st}\,dt$ diverges. (Why?) Hence $F(s) = 1/s$, with the domain of $F(s)$ being all $s > 0$. ◆

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†We treat $s$ as real-valued, but in certain applications $s$ may be a complex variable. For a detailed treatment of complex-valued Laplace transforms, see *Complex Variables and the Laplace Transform for Engineers,* by Wilbur R. LePage (Dover Publications, New York, 2010), or *Fundamentals of Complex Analysis with Applications to Engineering and Science* (3rd ed.), by E. B. Saff and A. D. Snider (Pearson Education, Boston, MA, 2003).

**Example 2** Determine the Laplace transform of $f(t) = e^{at}$, where $a$ is a constant.

**Solution** Using the definition of the transform,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt \\ &= \lim_{N\to\infty}\int_0^N e^{-(s-a)t}\,dt = \lim_{N\to\infty}\frac{-e^{-(s-a)t}}{s-a}\Big|_0^N \\ &= \lim_{N\to\infty}\left[\frac{1}{s-a} - \frac{e^{-(s-a)N}}{s-a}\right] \\ &= \frac{1}{s-a} \quad \text{for} \quad s > a\,. \end{aligned}$$

Again, if $s \le a$ the integral diverges, and hence the domain of $F(s)$ is all $s > a$. ◆

It is comforting to note from Example 2 that the transform of the constant function $f(t) = 1 = e^{0t}$ is $1/(s-0) = 1/s$, which agrees with the solution in Example 1.



**Example 3** Find $\mathscr{L}\{\sin bt\}$, where $b$ is a nonzero constant.

**Solution** We need to compute

$$\mathscr{L}\{\sin bt\}(s) = \int_0^\infty e^{-st}\sin bt\,dt = \lim_{N\to\infty}\int_0^N e^{-st}\sin bt\,dt\,.$$

Referring to the table of integrals at the back of the book, we see that

$$\begin{aligned} \mathscr{L}\{\sin bt\}(s) &= \lim_{N\to\infty}\left[\frac{e^{-st}}{s^2+b^2}(-s\sin bt - b\cos bt)\Big|_0^N\right] \\ &= \lim_{N\to\infty}\left[\frac{b}{s^2+b^2} - \frac{e^{-sN}}{s^2+b^2}(s\sin bN + b\cos bN)\right] \\ &= \frac{b}{s^2+b^2} \quad \text{for} \quad s > 0 \end{aligned}$$

(since for such $s$ we have $\lim_{N\to\infty} e^{-sN}(s\sin bN + b\cos bN) = 0$; see Problem 32). ◆

**Example 4** Determine the Laplace transform of

$$f(t) = \begin{cases} 2\,, & 0 < t < 5\,, \\ 0\,, & 5 < t < 10\,, \\ e^{4t}\,, & 10 < t\,. \end{cases}$$

**Solution** Since $f(t)$ is defined by a different formula on different intervals, we begin by breaking up the integral in (1) into three separate parts.† Thus,

$$\begin{aligned}
F(s) &= \int_0^{\infty} e^{-st} f(t)\,dt \\
&= \int_0^{5} e^{-st} \cdot 2\,dt + \int_5^{10} e^{-st} \cdot 0\,dt + \int_{10}^{\infty} e^{-st} e^{4t}\,dt \\
&= 2\int_0^{5} e^{-st}\,dt + \lim_{N\to\infty} \int_{10}^{N} e^{-(s-4)t}\,dt \\
&= \frac{2}{s} - \frac{2e^{-5s}}{s} + \lim_{N\to\infty}\left[\frac{e^{-10(s-4)}}{s-4} - \frac{e^{-(s-4)N}}{s-4}\right] \\
&= \frac{2}{s} - \frac{2e^{-5s}}{s} + \frac{e^{-10(s-4)}}{s-4} \qquad \text{for} \quad s > 4\,. \; \blacklozenge
\end{aligned}$$

Notice that the function $f(t)$ of Example 4 has jump discontinuities at $t = 5$ and $t = 10$. These values are reflected in the exponential terms $e^{-5s}$ and $e^{-10s}$ that appear in the formula for $F(s)$. We'll make this connection more precise when we discuss the unit step function in Section 7.6.

An important property of the Laplace transform is its **linearity.** That is, the Laplace transform $\mathscr{L}$ is a linear operator.

## Linearity of the Transform

**Theorem 1.** Let $f, f_1$, and $f_2$ be functions whose Laplace transforms exist for $s > \alpha$ and let $c$ be a constant. Then, for $s > \alpha$,

**(2)** $$\mathscr{L}\{f_1 + f_2\} = \mathscr{L}\{f_1\} + \mathscr{L}\{f_2\}\,,$$

**(3)** $$\mathscr{L}\{cf\} = c\mathscr{L}\{f\}\,.$$

**Proof.** Using the linearity properties of integration, we have for $s > \alpha$

$$\begin{aligned}
\mathscr{L}\{f_1 + f_2\}(s) &= \int_0^{\infty} e^{-st}[f_1(t) + f_2(t)]\,dt \\
&= \int_0^{\infty} e^{-st} f_1(t)\,dt + \int_0^{\infty} e^{-st} f_2(t)\,dt \\
&= \mathscr{L}\{f_1\}(s) + \mathscr{L}\{f_2\}(s)\,.
\end{aligned}$$

Hence, equation (2) is satisfied. In a similar fashion, we see that

$$\begin{aligned}
\mathscr{L}\{cf\}(s) &= \int_0^{\infty} e^{-st}[cf(t)]\,dt = c\int_0^{\infty} e^{-st} f(t)\,dt \\
&= c\mathscr{L}\{f\}(s)\,. \; \blacklozenge
\end{aligned}$$

† Notice that $f(t)$ is not defined at the points $t = 0$, 5, and 10. Nevertheless, the integral in (1) is still meaningful and unaffected by the function's values at finitely many points.

**Example 5** Determine $\mathscr{L}\{11 + 5e^{4t} - 6\sin 2t\}$ .

**Solution** From the linearity property, we know that the Laplace transform of the sum of any finite number of functions is the sum of their Laplace transforms. Thus,

$$\begin{aligned}\mathscr{L}\{11 + 5e^{4t} - 6\sin 2t\} &= \mathscr{L}\{11\} + \mathscr{L}\{5e^{4t}\} + \mathscr{L}\{-6\sin 2t\}\\ &= 11\mathscr{L}\{1\} + 5\mathscr{L}\{e^{4t}\} - 6\mathscr{L}\{\sin 2t\}\ .\end{aligned}$$

In Examples 1, 2, and 3, we determined that

$$\mathscr{L}\{1\}(s) = \frac{1}{s}\,, \qquad \mathscr{L}\{e^{4t}\}(s) = \frac{1}{s-4}\,, \qquad \mathscr{L}\{\sin 2t\}(s) = \frac{2}{s^2+2^2}\,.$$

Using these results, we find

$$\begin{aligned}\mathscr{L}\{11 + 5e^{4t} - 6\sin 2t\}(s) &= 11\left(\frac{1}{s}\right) + 5\left(\frac{1}{s-4}\right) - 6\left(\frac{2}{s^2+4}\right)\\ &= \frac{11}{s} + \frac{5}{s-4} - \frac{12}{s^2+4}\,.\end{aligned}$$

Since $\mathscr{L}\{1\}, \mathscr{L}\{e^{4t}\}$, and $\mathscr{L}\{\sin 2t\}$ are all defined for $s > 4$, so is the transform $\mathscr{L}\{11 + 5e^{4t} - 6\sin 2t\}$. ◆



Table 7.1 lists the Laplace transforms of some of the elementary functions. You should become familiar with these, since they are frequently encountered in solving linear differential equations with constant coefficients. The entries in the table can be derived from the definition of the Laplace transform. A more elaborate table of transforms is given on the inside back cover of this book.

**TABLE 7.1 Brief Table of Laplace Transforms**

| $f(t)$ | $F(s) = \mathscr{L}\{f\}(s)$ |
|---|---|
| $1$ | $\frac{1}{s}\,, \quad s > 0$ |
| $e^{at}$ | $\frac{1}{s-a}\,, \quad s > a$ |
| $t^n\,, \quad n = 1, 2, \ldots$ | $\frac{n!}{s^{n+1}}\,, \quad s > 0$ |
| $\sin bt$ | $\frac{b}{s^2+b^2}\,, \quad s > 0$ |
| $\cos bt$ | $\frac{s}{s^2+b^2}\,, \quad s > 0$ |
| $e^{at}t^n\,, \quad n = 1, 2, \ldots$ | $\frac{n!}{(s-a)^{n+1}}\,, \quad s > a$ |
| $e^{at}\sin bt$ | $\frac{b}{(s-a)^2+b^2}\,, \quad s > a$ |
| $e^{at}\cos bt$ | $\frac{s-a}{(s-a)^2+b^2}\,, \quad s > a$ |

**Example 6** Use Table 7.1 to determine $\mathscr{L}\{5t^2e^{-3t} - e^{12t}\cos 8t\}$ .

**Solution** From the table,

$$\mathscr{L}\{t^2e^{-3t}\} = \frac{2!}{[s-(-3)]^{2+1}} = \frac{2}{(s+3)^3} \quad \text{for } s > -3,$$

and

$$\mathscr{L}\{e^{12t}\cos 8t\} = \frac{s-12}{(s-12)^2+8^2} \quad \text{for } s > 12 .$$

Therefore, by linearity,

$$\mathscr{L}\{5t^2e^{-3t} - e^{12t}\cos 8t\} = \frac{10}{(s+3)^3} - \frac{s-12}{(s-12)^2+64} \quad \text{for } s > 12 . \blacklozenge$$

## Existence of the Transform

There are functions for which the improper integral in (1) fails to converge for any value of $s$. For example, this is the case for the function $f(t) = 1/t$, which grows too fast near zero. Likewise, no Laplace transform exists for the function $f(t) = e^{t^2}$, which increases too rapidly as $t \to \infty$. Fortunately, the set of functions for which the Laplace transform is defined includes many of the functions that arise in applications involving linear differential equations. We now discuss some properties that will (collectively) ensure the existence of the Laplace transform.

A function $f(t)$ on $[a, b]$ is said to have a **jump discontinuity** at $t_0 \in (a, b)$ if $f(t)$ is discontinuous at $t_0$, but the one-sided limits

$$\lim_{t \to t_0^-} f(t) \quad \text{and} \quad \lim_{t \to t_0^+} f(t)$$

exist as finite numbers. We have encountered jump discontinuities in Example 4 (page 354) and in the input to the mixing tank in Section 7.1 (page 350). If the discontinuity occurs at an endpoint, $t_0 = a$ (or $b$), a jump discontinuity occurs if the one-sided limit of $f(t)$ as $t \to a^+ (t \to b^-)$ exists as a finite number. We can now define piecewise continuity.

### Piecewise Continuity

**Definition 2.** A function $f(t)$ is said to be **piecewise continuous on a finite interval** $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

A function $f(t)$ is said to be **piecewise continuous on** $[\mathbf{0}, \boldsymbol{\infty})$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

**Example 7** Show that

$$f(t) = \begin{cases} t, & 0 < t < 1, \\ 2, & 1 < t < 2, \\ (t-2)^2, & 2 \le t \le 3, \end{cases}$$

whose graph is sketched in Figure 7.4 (on page 358), is piecewise continuous on $[0, 3]$.

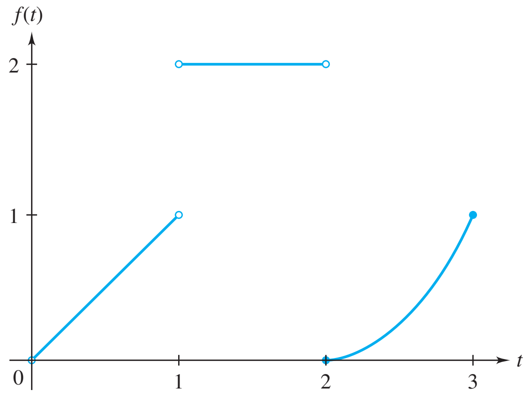



**Figure 7.4** Graph of $f(t)$ in Example 7

**Solution** From the graph of $f(t)$ we see that $f(t)$ is continuous on the intervals (0, 1), (1, 2), and (2, 3]. Moreover, at the points of discontinuity, $t = 0$, 1, and 2, the function has jump discontinuities, since the one-sided limits exist as finite numbers. In particular, at $t = 1$, the left-hand limit is 1 and the right-hand limit is 2. Therefore $f(t)$ is piecewise continuous on $[0, 3]$. ◆

Observe that the function $f(t)$ of Example 4 on page 354 is piecewise continuous on $[0, \infty)$ because it is piecewise continuous on every finite interval of the form $[0, N]$, with $N > 0$. In contrast, the function $f(t) = 1/t$ is not piecewise continuous on any interval containing the origin, since it has an "infinite jump" at the origin (see Figure 7.5).

A function that is piecewise continuous on a *finite* interval is necessarily integrable over that interval. However, piecewise continuity on $[0, \infty)$ is not enough to guarantee the existence (as a finite number) of the improper integral over $[0, \infty)$; we also need to consider the

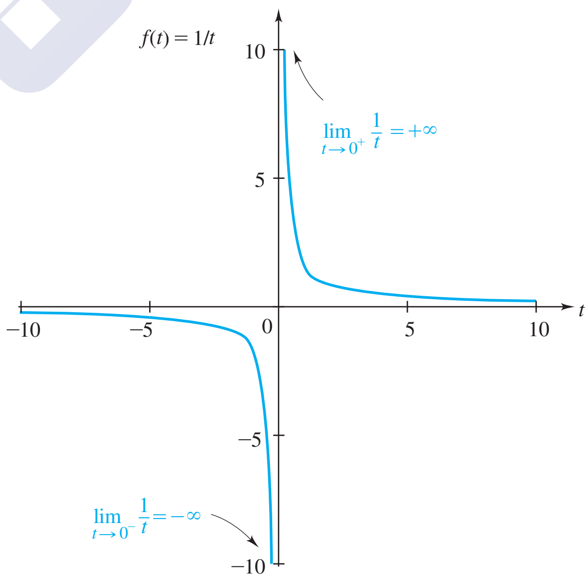



**Figure 7.5** Infinite jump at origin

growth of the integrand for large $t$. Roughly speaking, we'll show that the Laplace transform of a piecewise continuous function exists, provided the function does not grow "faster than an exponential."

## Exponential Order $\alpha$

**Definition 3.** A function $f(t)$ is said to be of **exponential order $\alpha$** if there exist positive constants $T$ and $M$ such that

$$|f(t)| \le Me^{\alpha t}, \qquad \text{for all } t \ge T. \tag{4}$$

For example, $f(t) = e^{5t}\sin 2t$ is of exponential order $\alpha = 5$ since

$$|e^{5t}\sin 2t| \le e^{5t},$$

and hence (4) holds with $M = 1$ and $T$ any positive constant.

We use the phrase $f(t)$ *is of exponential order* to mean that for *some* value of $\alpha$, the function $f(t)$ satisfies the conditions of Definition 3; that is, $f(t)$ grows no faster than a function of the form $Me^{\alpha t}$. The function $e^{t^2}$ is *not* of exponential order. To see this, observe that

$$\lim_{t\to\infty} \frac{e^{t^2}}{e^{\alpha t}} = \lim_{t\to\infty} e^{t(t-\alpha)} = +\infty$$

for any $\alpha$. Consequently, $e^{t^2}$ grows faster than $e^{\alpha t}$ for every choice of $\alpha$.

The functions usually encountered in solving linear differential equations with constant coefficients (e.g., polynomials, exponentials, sines, and cosines) are both piecewise continuous and of exponential order. As we now show, the Laplace transforms of such functions exist for large enough values of $s$.

## Conditions for Existence of the Transform

**Theorem 2.** If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order $\alpha$, then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$.

**Proof.** We need to show that the integral

$$\int_0^\infty e^{-st} f(t)\, dt$$

converges for $s > \alpha$. We begin by breaking up this integral into two separate integrals:

$$\int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt, \tag{5}$$

where $T$ is chosen so that inequality (4) holds. The first integral in (5) exists because $f(t)$ and hence $e^{-st}f(t)$ are piecewise continuous on the interval $[0, T]$ for any fixed $s$. To see that the second integral in (5) converges, we use the **comparison test for improper integrals.**

Since $f(t)$ is of exponential order $\alpha$, we have for $t \ge T$

$$|f(t)| \le Me^{\alpha t},$$

and hence

$$|e^{-st}f(t)| = e^{-st}|f(t)| \le Me^{-(s-\alpha)t},$$

for all $t \ge T$. Now for $s > \alpha$.

$$\int_T^\infty Me^{-(s-\alpha)t}dt = M\int_T^\infty e^{-(s-\alpha)t}dt = \frac{Me^{-(s-\alpha)T}}{s-\alpha} < \infty .$$

Since $|e^{-st}f(t)| \le Me^{-(s-\alpha)t}$ for $t \ge T$ and the improper integral of the larger function converges for $s > \alpha$, then, by the comparison test, the integral

$$\int_T^\infty e^{-st}f(t)\,dt$$

converges for $s > \alpha$. Finally, because the two integrals in (5) exist, the Laplace transform $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$. ◆

## 7.2 EXERCISES

*In Problems 1–12, use Definition 1 to determine the Laplace transform of the given function.*

**1.** $t$

**2.** $t^2$

**3.** $e^{6t}$

**4.** $te^{3t}$

**5.** $\cos 2t$

**6.** $\cos bt$, $b$ a constant

**7.** $e^{2t}\cos 3t$

**8.** $e^{-t}\sin 2t$

**9.** $f(t) = \begin{cases} 0, & 0 < t < 2, \\ t, & 2 < t \end{cases}$

**10.** $f(t) = \begin{cases} 1-t, & 0 < t < 1, \\ 0, & 1 < t \end{cases}$

**11.** $f(t) = \begin{cases} \sin t, & 0 < t < \pi, \\ 0, & \pi < t \end{cases}$

**12.** $f(t) = \begin{cases} e^{2t}, & 0 < t < 3, \\ 1, & 3 < t \end{cases}$

*In Problems 13–20, use the Laplace transform table and the linearity of the Laplace transform to determine the following transforms.*

**13.** $\mathcal{L}\{6e^{-3t} - t^2 + 2t - 8\}$

**14.** $\mathcal{L}\{5 - e^{2t} + 6t^2\}$

**15.** $\mathcal{L}\{t^3 - te^t + e^{4t}\cos t\}$

**16.** $\mathcal{L}\{t^2 - 3t - 2e^{-t}\sin 3t\}$

**17.** $\mathcal{L}\{e^{3t}\sin 6t - t^3 + e^t\}$

**18.** $\mathcal{L}\{t^4 - t^2 - t + \sin\sqrt{2}t\}$

**19.** $\mathcal{L}\{t^4e^{5t} - e^t\cos\sqrt{7}t\}$

**20.** $\mathcal{L}\{e^{-2t}\cos\sqrt{3}t - t^2e^{-2t}\}$

*In Problems 21–28, determine whether $f(t)$ is continuous, piecewise continuous, or neither on $[0, 10]$ and sketch the graph of $f(t)$.*

**21.** $f(t) = \begin{cases} 1, & 0 \le t \le 1, \\ (t-2)^2, & 1 < t \le 10 \end{cases}$

**22.** $f(t) = \begin{cases} 0, & 0 \le t < 2, \\ t, & 2 \le t \le 10 \end{cases}$

**23.** $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ t-1, & 1 < t < 3, \\ t^2 - 4, & 3 < t \le 10 \end{cases}$

**24.** $f(t) = \dfrac{t^2 - 3t + 2}{t^2 - 4}$

**25.** $f(t) = \dfrac{t^2 - t - 20}{t^2 + 7t + 10}$

**26.** $f(t) = \dfrac{t}{t^2 - 1}$

**27.** $f(t) = \begin{cases} 1/t, & 0 < t < 1, \\ 1, & 1 \le t \le 2, \\ 1-t, & 2 < t \le 10 \end{cases}$

**28.** $f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \ne 0, \\ 1, & t = 0 \end{cases}$

**29.** Which of the following functions are of exponential order?

**(a)** $t^3\sin t$ **(b)** $100e^{49t}$ **(c)** $e^{t^3}$

**(d)** $t\ln t$ **(e)** $\cosh(t^2)$ **(f)** $\dfrac{1}{t^2+1}$

**(g)** $\sin(t^2) + t^4e^{6t}$ **(h)** $3 - e^{t^2} + \cos 4t$

**(i)** $\exp\{t^2/(t+1)\}$ **(j)** $\sin(e^{t^2}) + e^{\sin t}$

**30.** For the transforms $F(s)$ in Table 7.1, what can be said about $\lim_{s\to\infty} F(s)$?

**31.** Thanks to Euler's formula (page 166) and the algebraic properties of complex numbers, several of the entries of Table 7.1 can be derived from a single formula; namely,

(6) $$\mathcal{L}\left\{e^{(a+ib)t}\right\}(s) = \frac{s-a+ib}{(s-a)^2+b^2}, \qquad s>a.$$

**(a)** By computing the integral in the definition of the Laplace transform on page 353 with $f(t) = e^{(a+ib)t}$, show that

$$\mathcal{L}\left\{e^{(a+ib)t}\right\}(s) = \frac{1}{s-(a+ib)}, \qquad s>a.$$

**(b)** Deduce (6) from part (a) by showing that

$$\frac{1}{s-(a+ib)} = \frac{s-a+ib}{(s-a)^2+b^2}.$$

**(c)** By equating the real and imaginary parts in formula (6), deduce the last two entries in Table 7.1.

**32.** Prove that for fixed $s>0$, we have

$$\lim_{N\to\infty} e^{-sN}(s\sin bN + b\cos bN) = 0.$$

**33.** Prove that if $f$ is piecewise continuous on $[a,b]$ and $g$ is continuous on $[a,b]$, then the product $fg$ is piecewise continuous on $[a,b]$.

# 7.3 Properties of the Laplace Transform

In the previous section, we defined the Laplace transform of a function $f(t)$ as

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)\,dt.$$

Using this definition to get an explicit expression for $\mathcal{L}\{f\}$ requires the evaluation of the improper integral—frequently a tedious task! We have already seen how the linearity property of the transform can help relieve this burden. In this section we discuss some further properties of the Laplace transform that simplify its computation. These new properties will also enable us to use the Laplace transform to solve initial value problems.

## Translation in s

**Theorem 3.** If the Laplace transform $\mathcal{L}\{f\}(s) = F(s)$ exists for $s>\alpha$, then

(1) $$\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)$$

for $s>\alpha+a$.

**Proof.** We simply compute

$$\begin{aligned}\mathcal{L}\{e^{at}f(t)\}(s) &= \int_0^\infty e^{-st}e^{at}f(t)\,dt \\ &= \int_0^\infty e^{-(s-a)t}f(t)\,dt \\ &= F(s-a). \ \blacklozenge\end{aligned}$$

Theorem 3 illustrates the effect on the Laplace transform of multiplication of a function $f(t)$ by $e^{at}$.

**Example 1** Determine the Laplace transform of $e^{at}\sin bt$.

**Solution** In Example 3 in Section 7.2, page 354, we found that

$$\mathscr{L}\{\sin bt\}(s) = F(s) = \frac{b}{s^2+b^2}.$$

Thus, by the translation property of $F(s)$, we have

$$\mathscr{L}\{e^{at}\sin bt\}(s) = F(s-a) = \frac{b}{(s-a)^2+b^2}. \quad ◆$$

## Laplace Transform of the Derivative

**Theorem 4.** Let $f(t)$ be continuous on $[0, \infty)$ and $f'(t)$ be piecewise continuous on $[0, \infty)$, with both of exponential order $\alpha$. Then, for $s > \alpha$,

**(2)** $$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0).$$

**Proof.** Since $\mathscr{L}\{f'\}$ exists, we can use integration by parts [with $u = e^{-st}$ and $dv = f'(t)dt$] to obtain

(3) $$\begin{aligned}\mathscr{L}\{f'\}(s) &= \int_0^\infty e^{-st}f'(t)\,dt = \lim_{N\to\infty}\int_0^N e^{-st}f'(t)\,dt \\ &= \lim_{N\to\infty}\left[e^{-st}f(t)\Big|_0^N + s\int_0^N e^{-st}f(t)\,dt\right] \\ &= \lim_{N\to\infty} e^{-sN}f(N) - f(0) + s\lim_{N\to\infty}\int_0^N e^{-st}f(t)\,dt \\ &= \lim_{N\to\infty} e^{-sN}f(N) - f(0) + s\mathscr{L}\{f\}(s).\end{aligned}$$

To evaluate $\lim_{N\to\infty} e^{-sN}f(N)$, we observe that since $f(t)$ is of exponential order $\alpha$, there exists a constant $M$ such that for $N$ large,

$$|e^{-sN}f(N)| \le e^{-sN}Me^{\alpha N} = Me^{-(s-\alpha)N}.$$

Hence, for $s > \alpha$,

$$0 \le \lim_{N\to\infty}|e^{-sN}f(N)| \le \lim_{N\to\infty}Me^{-(s-\alpha)N} = 0,$$

so

$$\lim_{N\to\infty} e^{-sN}f(N) = 0$$

for $s > \alpha$. Equation (3) now reduces to

$$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0). \quad ◆$$

Using induction, we can extend the last theorem to higher-order derivatives of $f(t)$. For example,

$$\begin{aligned}\mathscr{L}\{f''\}(s) &= s\mathscr{L}\{f'\}(s) - f'(0) \\ &= s[s\mathscr{L}\{f\}(s) - f(0)] - f'(0),\end{aligned}$$

which simplifies to

$$\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0) .$$

In general, we obtain the following result.

## Laplace Transform of Higher-Order Derivatives

**Theorem 5.** Let $f(t), f'(t), \ldots, f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0, \infty)$, with all these functions of exponential order $\alpha$. Then, for $s > \alpha$,

**(4)** $$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) .$$

The last two theorems shed light on the reason why the Laplace transform is such a useful tool in solving initial value problems. Roughly speaking, they tell us that by using the Laplace transform we can replace "differentiation with respect to $t$" with "multiplication by $s$," thereby converting a differential equation into an algebraic one. This idea is explored in Section 7.5. For now, we show how Theorem 4 can be helpful in computing a Laplace transform.

**Example 2** Using Theorem 4 and the fact that

$$\mathcal{L}\{\sin bt\}(s) = \frac{b}{s^2 + b^2} ,$$

determine $\mathcal{L}\{\cos bt\}$ .

**Solution** Let $f(t) := \sin bt$. Then $f(0) = 0$ and $f'(t) = b \cos bt$. Substituting into equation (2), we have

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0) ,$$
$$\mathcal{L}\{b \cos bt\}(s) = s\mathcal{L}\{\sin bt\}(s) - 0 ,$$
$$b\mathcal{L}\{\cos bt\}(s) = \frac{sb}{s^2 + b^2} .$$

Dividing by $b$ gives

$$\mathcal{L}\{\cos bt\}(s) = \frac{s}{s^2 + b^2} . \ \blacklozenge$$

**Example 3** Prove the following identity for continuous functions $f(t)$ (assuming the transforms exist):

**(5)** $$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(s) .$$

Use it to verify the solution to Example 2.

**Solution** Define the function $g(t)$ by the integral

$$g(t) := \int_0^t f(\tau)\,d\tau .$$

Observe that $g(0) = 0$ and $g'(t) = f(t)$. Thus, if we apply Theorem 4 to $g(t)$ [instead of $f(t)$], equation (2) on page 362 reads

$$\mathcal{L}\{f(t)\}(s) = s\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}(s) - 0\,,$$

which is equivalent to equation (5).

Now since

$$\sin bt = \int_0^t b\cos b\tau\,d\tau\,,$$

equation (5) predicts

$$\mathcal{L}\{\sin bt\}(s) = \frac{1}{s}\mathcal{L}\{b\cos bt\}(s) = \frac{b}{s}\mathcal{L}\{\cos bt\}(s)\,.$$

This identity is indeed valid for the transforms in Example 2. ◆

Another question arises concerning the Laplace transform. If $F(s)$ is the Laplace transform of $f(t)$, is $F'(s)$ also a Laplace transform of some function of $t$? The answer is yes:

$$F'(s) = \mathcal{L}\{-tf(t)\}(s)\,.$$

In fact, the following more general assertion holds.



## Derivatives of the Laplace Transform

**Theorem 6.** Let $F(s) = \mathcal{L}\{f\}(s)$ and assume $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order $\alpha$. Then, for $s > \alpha$,

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^nF}{ds^n}(s)\,. \tag{6}$$

**Proof.** Consider the identity

$$\frac{dF}{ds}(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt\,.$$

Because of the assumptions on $f(t)$, we can apply a theorem from advanced calculus (sometimes called **Leibniz's rule**) to interchange the order of integration and differentiation:

$$\frac{dF}{ds}(s) = \int_0^\infty \frac{d}{ds}(e^{-st})f(t)\,dt$$
$$= -\int_0^\infty e^{-st}tf(t)\,dt = -\mathcal{L}\{tf(t)\}(s)\,.$$

Thus,

$$\mathcal{L}\{tf(t)\}(s) = (-1)\frac{dF}{ds}(s)\,.$$

The general result (6) now follows by induction on $n$. ◆

A consequence of the above theorem is that if $f(t)$ is piecewise continuous and of exponential order, then its transform $F(s)$ has derivatives of all orders.

**Example 4** Determine $\mathscr{L}\{t\sin bt\}$.

**Solution** We already know that

$$\mathscr{L}\{\sin bt\}(s) = F(s) = \frac{b}{s^2+b^2}.$$

Differentiating $F(s)$, we obtain

$$\frac{dF}{ds}(s) = \frac{-2bs}{(s^2+b^2)^2}.$$

Hence, using formula (6), we have

$$\mathscr{L}\{t\sin bt\}(s) = -\frac{dF}{ds}(s) = \frac{2bs}{(s^2+b^2)^2}. \quad \blacklozenge$$

For easy reference, Table 7.2 lists some of the basic properties of the Laplace transform derived so far.

**TABLE 7.2 Properties of Laplace Transforms**

| |
|---|
| $\mathscr{L}\{f+g\} = \mathscr{L}\{f\} + \mathscr{L}\{g\}$. |
| $\mathscr{L}\{cf\} = c\mathscr{L}\{f\}$ for any constant $c$. |
| $\mathscr{L}\{e^{at}f(t)\}(s) = \mathscr{L}\{f\}(s-a)$. |
| $\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0)$. |
| $\mathscr{L}\{f''\}(s) = s^2\mathscr{L}\{f\}(s) - sf(0) - f'(0)$. |
| $\mathscr{L}\{f^{(n)}\}(s) = s^n\mathscr{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$. |
| $\mathscr{L}\{t^nf(t)\}(s) = (-1)^n\frac{d^n}{ds^n}(\mathscr{L}\{f\}(s))$. |



## 7.3 EXERCISES

*In Problems 1–20, determine the Laplace transform of the given function using Table 7.1 on page 356 and the properties of the transform given in Table 7.2. [Hint: In Problems 12–20, use an appropriate trigonometric identity.]*

**1.** $t^2 + e^t\sin 2t$

**2.** $3t^2 - e^{2t}$

**3.** $e^{-t}\cos 3t + e^{6t} - 1$

**4.** $3t^4 - 2t^2 + 1$

**5.** $2t^2e^{-t} - t + \cos 4t$

**6.** $e^{-2t}\sin 2t + e^{3t}t^2$

**7.** $(t-1)^4$

**8.** $(1+e^{-t})^2$

**9.** $e^{-t}t\sin 2t$

**10.** $te^{2t}\cos 5t$

**11.** $\cosh bt$

**12.** $\sin 3t\cos 3t$

**13.** $\sin^2 t$

**14.** $e^{7t}\sin^2 t$

**15.** $\cos^3 t$

**16.** $t\sin^2 t$

**17.** $\sin 2t\sin 5t$

**18.** $\cos nt\cos mt$, $m \neq n$

**19.** $\cos nt\sin mt$, $m \neq n$

**20.** $t\sin 2t\sin 5t$

**21.** Given that $\mathscr{L}\{\cos bt\}(s) = s/(s^2+b^2)$, use the translation property to compute $\mathscr{L}\{e^{at}\cos bt\}$.

**22.** Starting with the transform $\mathscr{L}\{1\}(s) = 1/s$, use formula (6) for the derivatives of the Laplace transform to show that $\mathscr{L}\{t\}(s) = 1/s^2$, $\mathscr{L}\{t^2\}(s) = 2!/s^3$, and, by using induction, that $\mathscr{L}\{t^n\}(s) = n!/s^{n+1}$, $n = 1, 2, \ldots$.

**23.** Use Theorem 4 on page 362 to show how entry 32 follows from entry 31 in the Laplace transform table on the inside back cover of the text.

**24.** Show that $\mathscr{L}\{e^{at}t^n\}(s) = n!/(s-a)^{n+1}$ in two ways:

**(a)** Use the translation property for $F(s)$.

**(b)** Use formula (6) for the derivatives of the Laplace transform.

**25.** Use formula (6) to help determine

**(a)** $\mathcal{L}\{t\cos bt\}$ . **(b)** $\mathcal{L}\{t^2\cos bt\}$ .

**26.** Let $f(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order.

**(a)** Show that there exist constants $K$ and $\alpha$ such that

$$|f(t)| \le Ke^{\alpha t} \qquad \text{for all } t \ge 0 .$$

**(b)** By using the definition of the transform and estimating the integral with the help of part (a), prove that

$$\lim_{s\to\infty} \mathcal{L}\{f\}(s) = 0 .$$

**27.** Let $f(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order $\alpha$ and assume $\lim_{t\to 0^+}[f(t)/t]$ exists. Show that

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(u)\,du ,$$

where $F(s) = \mathcal{L}\{f\}(s)$. [*Hint:* First show that $\frac{d}{ds}\mathcal{L}\{f(t)/t\}(s) = -F(s)$ and then use the result of Problem 26.]

**28.** Verify the identity in Problem 27 for the following functions. (Use the table of Laplace transforms on the inside back cover.)

**(a)** $f(t) = t^5$ **(b)** $f(t) = t^{3/2}$

**29.** The **transfer function** of a linear system is defined as the ratio of the Laplace transform of the output function $y(t)$ to the Laplace transform of the input function $g(t)$, when all initial conditions are zero. If a linear system is governed by the differential equation

$$y''(t) + 6y'(t) + 10y(t) = g(t) , \qquad t > 0 ,$$

use the linearity property of the Laplace transform and Theorem 5 on page 363 on the Laplace transform of higher-order derivatives to determine the transfer function $H(s) = Y(s)/G(s)$ for this system.

**30.** Find the transfer function, as defined in Problem 29, for the linear system governed by

$$y''(t) + 5y'(t) + 6y(t) = g(t) , \qquad t > 0 .$$

**31. Translation in *t*.** Show that for $c > 0$, the translated function

$$g(t) = \begin{cases} 0 , & 0 < t < c , \\ f(t-c) , & c < t \end{cases}$$

has Laplace transform

$$\mathcal{L}\{g\}(s) = e^{-cs}\mathcal{L}\{f\}(s) .$$

*In Problems 32–35, let $g(t)$ be the given function $f(t)$ translated to the right by c units. Sketch $f(t)$ and $g(t)$ and find $\mathcal{L}\{g(t)\}(s)$. (See Problem 31.)*

**32.** $f(t) \equiv 1 , \quad c = 2$

**33.** $f(t) = t , \quad c = 1$

**34.** $f(t) = \sin t , \quad c = \pi$

**35.** $f(t) = \sin t , \quad c = \pi/2$

**36.** Use equation (5) to provide another derivation of the formula $\mathcal{L}\{t^n\}(s) = n!/s^{n+1}$. [*Hint*: Start with $\mathcal{L}\{1\}(s) = 1/s$ and use induction.]

**37. Initial Value Theorem.** Apply the relation

(7) $$\mathcal{L}\{f'\}(s) = \int_0^\infty e^{-st}f'(t)\,dt = s\mathcal{L}\{f\}(s) - f(0)$$

to argue that for any function $f(t)$ whose derivative is piecewise continuous and of exponential order on $[0, \infty)$,

$$f(0) = \lim_{s\to\infty} s\mathcal{L}\{f\}(s) .$$

**38.** Verify the initial value theorem (Problem 37) for the following functions. (Use the table of Laplace transforms on the inside back cover.)

**(a)** 1 **(b)** $e^t$ **(c)** $e^{-t}$ **(d)** $\cos t$
**(e)** $\sin t$ **(f)** $t^2$ **(g)** $t\cos t$

# 7.4 Inverse Laplace Transform

In Section 7.2 we defined the Laplace transform as an integral operator that maps a function $f(t)$ into a function $F(s)$. In this section we consider the problem of finding the function $f(t)$ when we are given the transform $F(s)$. That is, we seek an **inverse mapping** for the Laplace transform.

To see the usefulness of such an inverse, let's consider the simple initial value problem

(1) $$y'' - y = -t ; \qquad y(0) = 0 , \qquad y'(0) = 1 .$$

If we take the transform of both sides of equation (1) and use the linearity property of the transform, we find

$$\mathcal{L}\{y''\}(s) - Y(s) = -\frac{1}{s^2} ,$$

where $Y(s) := \mathscr{L}\{y\}(s)$. We know the initial values of the solution $y(t)$, so we can use Theorem 5, page 363, on the Laplace transform of higher-order derivatives to express

$$\mathscr{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 1 .$$

Substituting for $\mathscr{L}\{y''\}(s)$ yields

$$s^2Y(s) - 1 - Y(s) = -\frac{1}{s^2} .$$

Solving this algebraic equation for $Y(s)$ gives

$$Y(s) = \frac{1 - \left(\frac{1}{s^2}\right)}{s^2 - 1} = \frac{s^2 - 1}{s^2(s^2 - 1)} = \frac{1}{s^2} .$$

We now recall that $\mathscr{L}\{t\}(s) = 1/s^2$, and since $Y(s) = \mathscr{L}\{y\}(s)$, we have

$$\mathscr{L}\{y\}(s) = 1/s^2 = \mathscr{L}\{t\}(s) .$$

It therefore seems reasonable to conclude that $y(t) = t$ is the solution to the initial value problem (1). A quick check confirms this!

Notice that in the above procedure, a crucial step is to determine $y(t)$ from its Laplace transform $Y(s) = 1/s^2$. As we noted, $y(t) = t$ is such a function, but it is *not* the only function whose Laplace function is $1/s^2$. For example, the transform of

$$g(t) := \begin{cases} t, & t \neq 6 , \\ 0, & t = 6 \end{cases}$$

is also $1/s^2$. This is because the transform is an integral, and integrals are not affected by changing a function's values at isolated points. The significant difference between $y(t)$ and $g(t)$ as far as we are concerned is that $y(t)$ is continuous on $[0, \infty)$, whereas $g(t)$ is not. Naturally, we prefer to work with continuous functions, since solutions to differential equations are continuous. Fortunately, it can be shown that if two different functions have the same Laplace transform, at most one of them can be continuous.[†] With this in mind we give the following definition.



## Inverse Laplace Transform

**Definition 4.** Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies

**(2)** $\quad \mathscr{L}\{f\} = F ,$

then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and employ the notation $f = \mathscr{L}^{-1}\{F\}$.

In case every function $f(t)$ satisfying (2) is discontinuous (and hence not a solution of a differential equation), one could choose any one of them to be the inverse transform; the distinction among them has no physical significance. [Indeed, two *piecewise* continuous functions satisfying (2) can only differ at their points of discontinuity.]

[†]For this result and further properties of the Laplace transform and its inverse, we refer you to *Operational Mathematics,* 3rd ed., by R. V. Churchill (McGraw-Hill, New York, 1971).

Naturally the Laplace transform tables will be a great help in determining the inverse Laplace transform of a given function $F(s)$.

**Example 1** Determine $\mathscr{L}^{-1}\{F\}$, where

**(a)** $F(s) = \dfrac{2}{s^3}$. **(b)** $F(s) = \dfrac{3}{s^2+9}$. **(c)** $F(s) = \dfrac{s-1}{s^2-2s+5}$.

**Solution** To compute $\mathscr{L}^{-1}\{F\}$, we refer to the Laplace transform table on page 356.

**(a)** $\mathscr{L}^{-1}\left\{\dfrac{2}{s^3}\right\}(t) = \mathscr{L}^{-1}\left\{\dfrac{2!}{s^3}\right\}(t) = t^2$

**(b)** $\mathscr{L}^{-1}\left\{\dfrac{3}{s^2+9}\right\}(t) = \mathscr{L}^{-1}\left\{\dfrac{3}{s^2+3^2}\right\}(t) = \sin 3t$

**(c)** $\mathscr{L}^{-1}\left\{\dfrac{s-1}{s^2-2s+5}\right\}(t) = \mathscr{L}^{-1}\left\{\dfrac{s-1}{(s-1)^2+2^2}\right\}(t) = e^t\cos 2t$

In part (c) we used the technique of completing the square to rewrite the denominator in a form that we could find in the table. ◆

In practice, we do not always encounter a transform $F(s)$ that exactly corresponds to an entry in the second column of the Laplace transform table. To handle more complicated functions $F(s)$, we use properties of $\mathscr{L}^{-1}$, just as we used properties of $\mathscr{L}$. One such tool is the linearity of the inverse Laplace transform, a property that is inherited from the linearity of the operator $\mathscr{L}$.



### Linearity of the Inverse Transform

**Theorem 7.** Assume that $\mathscr{L}^{-1}\{F\}$, $\mathscr{L}^{-1}\{F_1\}$, and $\mathscr{L}^{-1}\{F_2\}$ exist and are continuous on $[0, \infty)$ and let $c$ be any constant. Then

(3) $$\mathscr{L}^{-1}\{F_1+F_2\} = \mathscr{L}^{-1}\{F_1\} + \mathscr{L}^{-1}\{F_2\}\ ,$$

(4) $$\mathscr{L}^{-1}\{cF\} = c\mathscr{L}^{-1}\{F\}\ .$$

The proof of Theorem 7 is outlined in Problem 37. We illustrate the usefulness of this theorem in the next example.

**Example 2** Determine $\mathscr{L}^{-1}\left\{\dfrac{5}{s-6} - \dfrac{6s}{s^2+9} + \dfrac{3}{2s^2+8s+10}\right\}$.

**Solution** We begin by using the linearity property. Thus,

$$\mathscr{L}^{-1}\left\{\frac{5}{s-6} - \frac{6s}{s^2+9} + \frac{3}{2(s^2+4s+5)}\right\}$$
$$= 5\mathscr{L}^{-1}\left\{\frac{1}{s-6}\right\} - 6\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{3}{2}\mathscr{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\}.$$

Referring to the Laplace transform tables, we see that

$$\mathscr{L}^{-1}\left\{\frac{1}{s-6}\right\}(t) = e^{6t} \qquad \text{and} \qquad \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}(t) = \cos 3t\,.$$

This gives us the first two terms. To determine $\mathscr{L}^{-1}\{1/(s^2+4s+5)\}$, we complete the square of the denominator to obtain $s^2+4s+5 = (s+2)^2+1$. We now recognize from the tables that

$$\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2+1^2}\right\}(t) = e^{-2t}\sin t\,.$$

Hence,

$$\mathscr{L}^{-1}\left\{\frac{5}{s-6} - \frac{6s}{s^2+9} + \frac{3}{2s^2+8s+10}\right\}(t) = 5e^{6t} - 6\cos 3t + \frac{3e^{-2t}}{2}\sin t\,. \ ◆$$

**Example 3** Determine $\mathscr{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\}$.

**Solution** The $(s+2)^4$ in the denominator suggests that we work with the formula

$$\mathscr{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\}(t) = e^{at}t^n\,.$$

Here we have $a=-2$ and $n=3$, so $\mathscr{L}^{-1}\{6/(s+2)^4\}(t) = e^{-2t}t^3$. Using the linearity property, we find

$$\mathscr{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\}(t) = \frac{5}{6}\mathscr{L}^{-1}\left\{\frac{3!}{(s+2)^4}\right\}(t) = \frac{5}{6}e^{-2t}t^3\,. \ ◆$$

**Example 4** Determine $\mathscr{L}^{-1}\left\{\frac{3s+2}{s^2+2s+10}\right\}$.

**Solution** By completing the square, the quadratic in the denominator can be written as

$$s^2+2s+10 = s^2+2s+1+9 = (s+1)^2+3^2\,.$$

The form of $F(s)$ now suggests that we use one or both of the formulas

$$\mathscr{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\}(t) = e^{at}\cos bt\,,$$

$$\mathscr{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\}(t) = e^{at}\sin bt\,.$$

In this case, $a=-1$ and $b=3$. The next step is to express

$$\frac{3s+2}{s^2+2s+10} = A\frac{s+1}{(s+1)^2+3^2} + B\frac{3}{(s+1)^2+3^2}\,, \tag{5}$$

where $A$, $B$ are constants to be determined. Multiplying both sides of (5) by $s^2+2s+10$ leaves

$$3s+2 = A(s+1)+3B = As+(A+3B)\,,$$

which is an identity between two polynomials in $s$. Equating the coefficients of like terms gives

$$A = 3\,, \qquad A+3B = 2\,,$$

so $A = 3$ and $B = -1/3$. Finally, from (5) and the linearity property, we find

$$\mathcal{L}^{-1}\left\{\frac{3s+2}{s^2+2s+10}\right\}(t) = 3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\}(t) - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+3^2}\right\}(t)$$
$$= 3e^{-t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t\,. \quad ◆$$

Given the choice of finding the inverse Laplace transform of

$$F_1(s) = \frac{7s^2+10s-1}{s^3+3s^2-s-3}$$

or of

$$F_2(s) = \frac{2}{s-1} + \frac{1}{s+1} + \frac{4}{s+3}\,,$$

which would you select? No doubt $F_2(s)$ is the easier one. Actually, the two functions $F_1(s)$ and $F_2(s)$ are identical. This can be checked by combining the simple fractions that form $F_2(s)$. Thus, if we are faced with the problem of computing $\mathcal{L}^{-1}$ of a rational function such as $F_1(s)$, we will first express it, as we did $F_2(s)$, as a sum of simple rational functions. This is accomplished by the **method of partial fractions.**

We briefly review this method. Recall from calculus that a rational function of the form $P(s)/Q(s)$, where $P(s)$ and $Q(s)$ are polynomials with the degree of $P$ less than the degree of $Q$, has a partial fraction expansion whose form is based on the linear and quadratic factors of $Q(s)$. (We assume the coefficients of the polynomials to be real numbers.) There are three cases to consider:

1. Nonrepeated linear factors.
2. Repeated linear factors.
3. Quadratic factors.



## 1. Nonrepeated Linear Factors

If $Q(s)$ can be factored into a product of distinct linear factors,

$$Q(s) = (s-r_1)(s-r_2)\cdots(s-r_n)\,,$$

where the $r_i$'s are all distinct real numbers, then the partial fraction expansion has the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-r_1} + \frac{A_2}{s-r_2} + \cdots + \frac{A_n}{s-r_n}\,,$$

where the $A_i$'s are real numbers. There are various ways of determining the constants $A_1, \ldots, A_n$. In the next example, we demonstrate two such methods.

**Example 5** Determine $\mathcal{L}^{-1}\{F\}$, where

$$F(s) = \frac{7s-1}{(s+1)(s+2)(s-3)}\,.$$

**Solution** We begin by finding the partial fraction expansion for $F(s)$. The denominator consists of three distinct linear factors, so the expansion has the form

$$\text{(6)} \qquad \frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3}\,,$$

where $A$, $B$, and $C$ are real numbers to be determined.

One procedure that works for all partial fraction expansions is first to multiply the expansion equation by the denominator of the given rational function. This leaves us with two identical polynomials. Equating the coefficients of $s^k$ leads to a system of linear equations that we can solve to determine the unknown constants. In this example, we multiply (6) by $(s+1)(s+2)(s-3)$ and find

**(7)** $$7s-1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)\,,^{\dagger}$$

which reduces to

$$7s-1 = (A+B+C)s^2 + (-A-2B+3C)s + (-6A-3B+2C)\,.$$

Equating the coefficients of $s^2$, $s$, and 1 gives the system of linear equations

$$\begin{aligned} A+B+C &= 0\,,\\ -A-2B+3C &= 7\,,\\ -6A-3B+2C &= -1\,. \end{aligned}$$

Solving this system yields $A=2$, $B=-3$, and $C=1$. Hence,

**(8)** $$\frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3}\,.$$

An alternative method for finding the constants $A$, $B$, and $C$ from (7) is to choose three values for $s$ and substitute them into (7) to obtain three linear equations in the three unknowns. If we are careful in our choice of the values for $s$, the system is easy to solve. In this case, equation (7) obviously simplifies if $s=-1, -2$, or 3. Putting $s=-1$ gives

$$\begin{aligned} -7-1 &= A(1)(-4) + B(0) + C(0)\,,\\ -8 &= -4A\,. \end{aligned}$$

Hence $A=2$. Next, setting $s=-2$ gives

$$\begin{aligned} -14-1 &= A(0) + B(-1)(-5) + C(0)\,,\\ -15 &= 5B\,, \end{aligned}$$

and so $B=-3$. Finally, letting $s=3$, we similarly find that $C=1$. In the case of nonrepeated linear factors, the alternative method is easier to use.

Now that we have obtained the partial fraction expansion (8), we use linearity to compute

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{3}{s+2} + \frac{1}{s-3}\right\}(t)\\ &= 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t)\\ &\quad + \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}(t)\\ &= 2e^{-t} - 3e^{-2t} + e^{3t}\,. \end{aligned}$$ ◆

†Rigorously speaking, equation (7) was derived for $s$ different from $-1$, $-2$, and 3, but by continuity it holds for these values as well.

## 2. Repeated Linear Factors

If $s-r$ is a factor of $Q(s)$ and $(s-r)^m$ is the highest power of $s-r$ that divides $Q(s)$, then the portion of the partial fraction expansion of $P(s)/Q(s)$ that corresponds to the term $(s-r)^m$ is

$$\frac{A_1}{s-r}+\frac{A_2}{(s-r)^2}+\cdots+\frac{A_m}{(s-r)^m},$$

where the $A_i$'s are real numbers.

**Example 6** Determine $\mathcal{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$.

**Solution** Since $s-1$ is a repeated linear factor with multiplicity two and $s+3$ is a nonrepeated linear factor, the partial fraction expansion has the form

$$\frac{s^2+9s+2}{(s-1)^2(s+3)}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s+3}.$$

We begin by multiplying both sides by $(s-1)^2(s+3)$ to obtain

$$s^2+9s+2=A(s-1)(s+3)+B(s+3)+C(s-1)^2. \tag{9}$$

Now observe that when we set $s=1$ (or $s=-3$), two terms on the right-hand side of (9) vanish, leaving a linear equation that we can solve for $B$ (or $C$). Setting $s=1$ in (9) gives

$$1+9+2=A(0)+4B+C(0),$$
$$12=4B,$$

and, hence, $B=3$. Similarly, setting $s=-3$ in (9) gives

$$9-27+2=A(0)+B(0)+16C$$
$$-16=16C.$$

Thus, $C=-1$. Finally, to find $A$, we pick a different value for $s$, say $s=0$. Then, since $B=3$ and $C=-1$, plugging $s=0$ into (9) yields

$$2=-3A+3B+C=-3A+9-1$$

so that $A=2$. Hence,

$$\frac{s^2+9s+2}{(s-1)^2(s+3)}=\frac{2}{s-1}+\frac{3}{(s-1)^2}-\frac{1}{s+3}. \tag{10}$$

We could also have determined the constants $A$, $B$, and $C$ by first rewriting equation (9) in the form

$$s^2+9s+2=(A+C)s^2+(2A+B-2C)s+(-3A+3B+C).$$

Then, equating the corresponding coefficients of $s^2$, $s$, and 1 and solving the resulting system, we again find $A=2$, $B=3$, and $C=-1$.

Now that we have derived the partial fraction expansion (10) for the given rational function, we can determine its inverse Laplace transform:

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}(t) &= \mathscr{L}^{-1}\left\{\frac{2}{s-1}+\frac{3}{(s-1)^2}-\frac{1}{s+3}\right\}(t)\\ &= 2\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}(t)+3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}(t)\\ &\quad -\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\}(t)\\ &= 2e^t+3te^t-e^{-3t}\,. \quad \blacklozenge\end{aligned}$$

## 3. Quadratic Factors

If $(s-\alpha)^2+\beta^2$ is a quadratic factor of $Q(s)$ that cannot be reduced to linear factors with real coefficients and $m$ is the highest power of $(s-\alpha)^2+\beta^2$ that divides $Q(s)$, then the portion of the partial fraction expansion that corresponds to $(s-\alpha)^2+\beta^2$ is

$$\frac{C_1s+D_1}{(s-\alpha)^2+\beta^2}+\frac{C_2s+D_2}{[(s-\alpha)^2+\beta^2]^2}+\cdots+\frac{C_ms+D_m}{[(s-\alpha)^2+\beta^2]^m}\,.$$

As we saw in Example 4, page 369, it is more convenient to express $C_is+D_i$ in the form $A_i(s-\alpha)+\beta B_i$ when we look up the Laplace transforms. So let's agree to write this portion of the partial fraction expansion in the equivalent form

$$\frac{A_1(s-\alpha)+\beta B_1}{(s-\alpha)^2+\beta^2}+\frac{A_2(s-\alpha)+\beta B_2}{[(s-\alpha)^2+\beta^2]^2}+\cdots+\frac{A_m(s-\alpha)+\beta B_m}{[(s-\alpha)^2+\beta^2]^m}\,.$$

**Example 7** Determine $\mathscr{L}^{-1}\left\{\dfrac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$.

**Solution** We first observe that the quadratic factor $s^2-2s+5$ is irreducible (check the sign of the discriminant in the quadratic formula). Next we write the quadratic in the form $(s-\alpha)^2+\beta^2$ by completing the square:

$$s^2-2s+5=(s-1)^2+2^2\,.$$

Since $s^2-2s+5$ and $s+1$ are nonrepeated factors, the partial fraction expansion has the form

$$\frac{2s^2+10s}{(s^2-2s+5)(s+1)}=\frac{A(s-1)+2B}{(s-1)^2+2^2}+\frac{C}{s+1}\,.$$

When we multiply both sides by the common denominator, we obtain

$$2s^2+10s=[A(s-1)+2B](s+1)+C(s^2-2s+5)\,. \tag{11}$$

In equation (11), let's put $s=-1$, 1, and 0. With $s=-1$, we find

$$\begin{aligned}2-10&=[A(-2)+2B](0)+C(8)\,,\\ -8&=8C\,,\end{aligned}$$

and, hence, $C = -1$. With $s = 1$ in (11), we obtain

$$2 + 10 = [A(0) + 2B](2) + C(4) ,$$

and since $C = -1$, the last equation becomes $12 = 4B - 4$. Thus $B = 4$. Finally, setting $s = 0$ in (11) and using $C = -1$ and $B = 4$ gives

$$\begin{aligned} 0 &= [A(-1) + 2B](1) + C(5) , \\ 0 &= -A + 8 - 5 , \\ A &= 3 . \end{aligned}$$

Hence, $A = 3$, $B = 4$, and $C = -1$ so that

$$\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} = \frac{3(s - 1) + 2(4)}{(s - 1)^2 + 2^2} - \frac{1}{s + 1} .$$

With this partial fraction expansion in hand, we can immediately determine the inverse Laplace transform:

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)}\right\}(t) &= \mathscr{L}^{-1}\left\{\frac{3(s - 1) + 2(4)}{(s - 1)^2 + 2^2} - \frac{1}{s + 1}\right\}(t) \\ &= 3\mathscr{L}^{-1}\left\{\frac{s - 1}{(s - 1)^2 + 2^2}\right\}(t) \\ &\quad + 4\mathscr{L}^{-1}\left\{\frac{2}{(s - 1)^2 + 2^2}\right\}(t) - \mathscr{L}^{-1}\left\{\frac{1}{s + 1}\right\}(t) \\ &= 3e^t \cos 2t + 4e^t \sin 2t - e^{-t} . \end{aligned}$$ ◆

In Section 7.8, we discuss a different method (involving convolutions) for computing inverse transforms that does not require partial fraction decompositions. Moreover, the convolution method is convenient in the case of a rational function with a repeated quadratic factor in the denominator. Other helpful tools are described in Problems 33–36 and 38–43.



## 7.4 EXERCISES

*In Problems 1–10, determine the inverse Laplace transform of the given function.*

**1.** $\dfrac{6}{(s - 1)^4}$

**2.** $\dfrac{2}{s^2 + 4}$

**3.** $\dfrac{s + 1}{s^2 + 2s + 10}$

**4.** $\dfrac{4}{s^2 + 9}$

**5.** $\dfrac{1}{s^2 + 4s + 8}$

**6.** $\dfrac{3}{(2s + 5)^3}$

**7.** $\dfrac{2s + 16}{s^2 + 4s + 13}$

**8.** $\dfrac{1}{s^5}$

**9.** $\dfrac{3s - 15}{2s^2 - 4s + 10}$

**10.** $\dfrac{s - 1}{2s^2 + s + 6}$

*In Problems 11–20, determine the partial fraction expansion for the given rational function.*

**11.** $\dfrac{s^2 - 26s - 47}{(s - 1)(s + 2)(s + 5)}$

**12.** $\dfrac{-s - 7}{(s + 1)(s - 2)}$

**13.** $\dfrac{-2s^2 - 3s - 2}{s(s + 1)^2}$

**14.** $\dfrac{-8s^2 - 5s + 9}{(s + 1)(s^2 - 3s + 2)}$

**15.** $\dfrac{8s - 2s^2 - 14}{(s + 1)(s^2 - 2s + 5)}$

**16.** $\dfrac{-5s - 36}{(s + 2)(s^2 + 9)}$

**17.** $\dfrac{3s + 5}{s(s^2 + s - 6)}$

**18.** $\dfrac{3s^2 + 5s + 3}{s^4 + s^3}$

**19.** $\dfrac{1}{(s-3)(s^2+2s+2)}$ **20.** $\dfrac{s}{(s-1)(s^2-1)}$

*In Problems 21–30, determine* $\mathscr{L}^{-1}\{F\}$.

**21.** $F(s) = \dfrac{6s^2-13s+2}{s(s-1)(s-6)}$

**22.** $F(s) = \dfrac{s+11}{(s-1)(s+3)}$

**23.** $F(s) = \dfrac{5s^2+34s+53}{(s+3)^2(s+1)}$

**24.** $F(s) = \dfrac{7s^2-41s+84}{(s-1)(s^2-4s+13)}$

**25.** $F(s) = \dfrac{7s^2+23s+30}{(s-2)(s^2+2s+5)}$

**26.** $F(s) = \dfrac{7s^3-2s^2-3s+6}{s^3(s-2)}$

**27.** $s^2F(s) - 4F(s) = \dfrac{5}{s+1}$

**28.** $s^2F(s) + sF(s) - 6F(s) = \dfrac{s^2+4}{s^2+s}$

**29.** $sF(s) + 2F(s) = \dfrac{10s^2+12s+14}{s^2-2s+2}$

**30.** $sF(s) - F(s) = \dfrac{2s+5}{s^2+2s+1}$

**31.** Determine the Laplace transform of each of the following functions:

**(a)** $f_1(t) = \begin{cases} 0, & t = 2, \\ t, & t \neq 2. \end{cases}$

**(b)** $f_2(t) = \begin{cases} 5, & t = 1, \\ 2, & t = 6, \\ t, & t \neq 1, 6. \end{cases}$

**(c)** $f_3(t) = t$.

Which of the preceding functions is the inverse Laplace transform of $1/s^2$?

**32.** Determine the Laplace transform of each of the following functions:

**(a)** $f_1(t) = \begin{cases} t, & t = 1, 2, 3, \ldots, \\ e^t, & t \neq 1, 2, 3, \ldots. \end{cases}$

**(b)** $f_2(t) = \begin{cases} e^t, & t \neq 5, 8, \\ 6, & t = 5, \\ 0, & t = 8. \end{cases}$

**(c)** $f_3(t) = e^t$.

Which of the preceding functions is the inverse Laplace transform of $1/(s-1)$?

*Theorem 6 in Section 7.3 on page 364 can be expressed in terms of the inverse Laplace transform as*

$$\mathscr{L}^{-1}\left\{\frac{d^nF}{ds^n}\right\}(t) = (-t)^n f(t),$$

*where* $f = \mathscr{L}^{-1}\{F\}$. *Use this equation in Problems 33–36 to compute* $\mathscr{L}^{-1}\{F\}$.

**33.** $F(s) = \ln\left(\dfrac{s+2}{s-5}\right)$ **34.** $F(s) = \ln\left(\dfrac{s-4}{s-3}\right)$

**35.** $F(s) = \ln\left(\dfrac{s^2+9}{s^2+1}\right)$ **36.** $F(s) = \arctan(1/s)$

**37.** Prove Theorem 7, page 368, on the linearity of the inverse transform. [*Hint:* Show that the right-hand side of equation (3) is a continuous function on $[0, \infty)$ whose Laplace transform is $F_1(s) + F_2(s)$.]

**38. Residue Computation.** Let $P(s)/Q(s)$ be a rational function with deg $P <$ deg $Q$ and suppose $s - r$ is a nonrepeated linear factor of $Q(s)$. Prove that the portion of the partial fraction expansion of $P(s)/Q(s)$ corresponding to $s - r$ is

$$\frac{A}{s-r},$$

where $A$ (called the **residue**) is given by the formula

$$A = \lim_{s\to r} \frac{(s-r)P(s)}{Q(s)} = \frac{P(r)}{Q'(r)}.$$

**39.** Use the residue computation formula derived in Problem 38 to determine quickly the partial fraction expansion for

$$F(s) = \frac{2s+1}{s(s-1)(s+2)}.$$

**40. Heaviside's Expansion Formula.**† Let $P(s)$ and $Q(s)$ be polynomials with the degree of $P(s)$ less than the degree of $Q(s)$. Let

$$Q(s) = (s-r_1)(s-r_2)\cdots(s-r_n),$$

where the $r_i$'s are distinct real numbers. Show that

$$\mathscr{L}^{-1}\left\{\frac{P}{Q}\right\}(t) = \sum_{i=1}^{n} \frac{P(r_i)}{Q'(r_i)} e^{r_i t}.$$

†*Historical Footnote:* This formula played an important role in the "operational solution" to ordinary differential equations developed by Oliver Heaviside in the 1890s.

**41.** Use Heaviside's expansion formula derived in Problem 40 to determine the inverse Laplace transform of

$$F(s) = \frac{3s^2 - 16s + 5}{(s+1)(s-3)(s-2)} .$$

**42. Complex Residues.** Let $P(s)/Q(s)$ be a rational function with deg $P <$ deg $Q$ and suppose $(s-\alpha)^2 + \beta^2$ is a nonrepeated quadratic factor of $Q$. (That is, $\alpha \pm i\beta$ are complex conjugate zeros of $Q$.) Prove that the portion of the partial fraction expansion of $P(s)/Q(s)$ corresponding to $(s-\alpha)^2 + \beta^2$ is

$$\frac{A(s-\alpha) + \beta B}{(s-\alpha)^2 + \beta^2} ,$$

where the **complex residue** $\beta B + i\beta A$ is given by the formula

$$\beta B + i\beta A = \lim_{s \to \alpha + i\beta} \frac{[(s-\alpha)^2 + \beta^2]P(s)}{Q(s)} .$$

(Thus we can determine $B$ and $A$ by taking the real and imaginary parts of the limit and dividing them by $\beta$.)

**43.** Use the residue formulas derived in Problems 38 and 42 to determine the partial fraction expansion for

$$F(s) = \frac{6s^2 + 28}{(s^2 - 2s + 5)(s+2)} .$$

# 7.5 Solving Initial Value Problems

Our goal is to show how Laplace transforms can be used to solve initial value problems for linear differential equations. Recall that we have already studied ways of solving such initial value problems in Chapter 4. These previous methods required that we first find a *general solution* of the differential equation and then use the initial conditions to determine the desired solution. As we will see, the method of Laplace transforms leads to the solution of the initial value problem *without* first finding a general solution.

Other advantages to the transform method are worth noting. For example, the technique can easily handle equations involving forcing functions having jump discontinuities, as illustrated in Section 7.1. Further, the method can be used for certain linear differential equations with variable coefficients, a special class of integral equations, systems of differential equations, and partial differential equations.



### Method of Laplace Transforms

To solve an initial value problem:

**(a)** Take the Laplace transform of both sides of the equation.
**(b)** Use the properties of the Laplace transform and the initial conditions to obtain an equation for the Laplace transform of the solution and then solve this equation for the transform.
**(c)** Determine the inverse Laplace transform of the solution by looking it up in a table or by using a suitable method (such as partial fractions) in combination with the table.

In step (a) we are tacitly assuming the solution is piecewise continuous on $[0, \infty)$ and of exponential order. Once we have obtained the inverse Laplace transform in step (c), we can verify that these tacit assumptions are satisfied.

**Example 1** Solve the initial value problem

(1) $$y'' - 2y' + 5y = -8e^{-t} ; \quad y(0) = 2 , \quad y'(0) = 12 .$$

**Solution** The differential equation in (1) is an identity between two functions of $t$. Hence equality holds for the Laplace transforms of these functions:

$$\mathscr{L}\{y'' - 2y' + 5y\} = \mathscr{L}\{-8e^{-t}\}\ .$$

Using the linearity property of $\mathscr{L}$ and the previously computed transform of the exponential function, we can write

$$\mathscr{L}\{y''\}(s) - 2\mathscr{L}\{y'\}(s) + 5\mathscr{L}\{y\}(s) = \frac{-8}{s+1}\ . \tag{2}$$

Now let $Y(s) := \mathscr{L}\{y\}(s)$. From the formulas for the Laplace transform of higher-order derivatives (see Section 7.3) and the initial conditions in (1), we find

$$\begin{aligned}\mathscr{L}\{y'\}(s) &= sY(s) - y(0) = sY(s) - 2\ ,\\ \mathscr{L}\{y''\}(s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 12\ .\end{aligned}$$

Substituting these expressions into (2) and solving for $Y(s)$ yields

$$\begin{aligned}[s^2Y(s) - 2s - 12] - 2[sY(s) - 2] + 5Y(s) &= \frac{-8}{s+1}\\ (s^2 - 2s + 5)Y(s) &= 2s + 8 - \frac{8}{s+1}\\ (s^2 - 2s + 5)Y(s) &= \frac{2s^2 + 10s}{s+1}\\ Y(s) &= \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s+1)}\ .\end{aligned}$$

Our remaining task is to compute the inverse transform of the rational function $Y(s)$. This was done in Example 7 of Section 7.4, page 373, where, using a partial fraction expansion, we found

$$y(t) = 3e^t\cos 2t + 4e^t\sin 2t - e^{-t}\ , \tag{3}$$

which is the solution to the initial value problem (1). ◆

As a quick check on the accuracy of our computations, the reader is advised to verify that the computed solution satisfies the given initial conditions.

The reader is probably questioning the wisdom of using the Laplace transform method to solve an initial value problem that can be easily handled by the methods discussed in Chapter 4. The objective of the first few examples in this section is simply to make the reader familiar with the Laplace transform procedure. We will see in Example 4 and in later sections that the method is applicable to problems that cannot be readily handled by the techniques discussed in the previous chapters.

**Example 2** Solve the initial value problem

$$y'' + 4y' - 5y = te^t\ ; \qquad y(0) = 1\ , \qquad y'(0) = 0\ . \tag{4}$$

**Solution** Let $Y(s) := \mathscr{L}\{y\}(s)$. Taking the Laplace transform of both sides of the differential equation in (4) gives

$$\mathscr{L}\{y''\}(s) + 4\mathscr{L}\{y'\}(s) - 5Y(s) = \frac{1}{(s-1)^2}\ . \tag{5}$$

Using the initial conditions, we can express $\mathscr{L}\{y'\}(s)$ and $\mathscr{L}\{y''\}(s)$ in terms of $Y(s)$. That is,

$$\mathscr{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 1\,,$$
$$\mathscr{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s\,.$$

Substituting back into (5) and solving for $Y(s)$ gives

$$[s^2Y(s) - s] + 4[sY(s) - 1] - 5Y(s) = \frac{1}{(s-1)^2}$$
$$(s^2 + 4s - 5)Y(s) = s + 4 + \frac{1}{(s-1)^2}$$
$$(s+5)(s-1)Y(s) = \frac{s^3 + 2s^2 - 7s + 5}{(s-1)^2}$$
$$Y(s) = \frac{s^3 + 2s^2 - 7s + 5}{(s+5)(s-1)^3}\,.$$

The partial fraction expansion for $Y(s)$ has the form

(6) $$\frac{s^3 + 2s^2 - 7s + 5}{(s+5)(s-1)^3} = \frac{A}{s+5} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)^3}\,.$$

Solving for the numerators, we ultimately obtain $A = 35/216$, $B = 181/216$, $C = -1/36$, and $D = 1/6$. Substituting these values into (6) gives

$$Y(s) = \frac{35}{216}\left(\frac{1}{s+5}\right) + \frac{181}{216}\left(\frac{1}{s-1}\right) - \frac{1}{36}\left(\frac{1}{(s-1)^2}\right) + \frac{1}{12}\left(\frac{2}{(s-1)^3}\right),$$

where we have written $D = 1/6 = (1/12)2$ to facilitate the final step of taking the inverse transform. From the tables, we now obtain

(7) $$y(t) = \frac{35}{216}e^{-5t} + \frac{181}{216}e^t - \frac{1}{36}te^t + \frac{1}{12}t^2e^t$$

as the solution to the initial value problem (4). ◆

**Example 3** Solve the initial value problem

(8) $$w''(t) - 2w'(t) + 5w(t) = -8e^{\pi - t}\,; \quad w(\pi) = 2\,, \quad w'(\pi) = 12\,.$$

**Solution** To use the method of Laplace transforms, we first move the initial conditions to $t = 0$. This can be done by setting $y(t) \coloneqq w(t + \pi)$. Then

$$y'(t) = w'(t+\pi)\,, \qquad y''(t) = w''(t+\pi)\,.$$

Replacing $t$ by $t + \pi$ in the differential equation in (8), we have

(9) $$w''(t+\pi) - 2w'(t+\pi) + 5w(t+\pi) = -8e^{\pi-(t+\pi)} = -8e^{-t}\,.$$

Substituting $y(t) = w(t+\pi)$ in (9), the initial value problem in (8) becomes

$$y''(t) - 2y'(t) + 5y(t) = -8e^{-t}\,; \qquad y(0) = 2\,, \qquad y'(0) = 12\,.$$

Because the initial conditions are now given at the origin, the Laplace transform method is applicable. In fact, we carried out the procedure in Example 1, page 376, where we found

(10) $$y(t) = 3e^t\cos 2t + 4e^t\sin 2t - e^{-t}\,.$$

Since $w(t + \pi) = y(t)$, then $w(t) = y(t - \pi)$. Hence, replacing $t$ by $t - \pi$ in (10) gives

$$\begin{aligned} w(t) = y(t-\pi) &= 3e^{t-\pi}\cos[2(t-\pi)] + 4e^{t-\pi}\sin[2(t-\pi)] - e^{-(t-\pi)} \\ &= 3e^{t-\pi}\cos 2t + 4e^{t-\pi}\sin 2t - e^{\pi - t} . \end{aligned}$$ ◆

Thus far we have applied the Laplace transform method only to linear equations with constant coefficients. Yet several important equations in mathematical physics involve linear equations whose coefficients are polynomials in $t$. To solve such equations using Laplace transforms, we apply Theorem 6, page 364, where we proved that

**(11)** $$\mathscr{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n \mathscr{L}\{f\}}{ds^n}(s) .$$

If we let $n = 1$ and $f(t) = y'(t)$, we find

$$\begin{aligned} \mathscr{L}\{ty'(t)\}(s) &= -\frac{d}{ds}\mathscr{L}\{y'\}(s) \\ &= -\frac{d}{ds}[sY(s) - y(0)] = -sY'(s) - Y(s) . \end{aligned}$$

Similarly, with $n = 1$ and $f(t) = y''(t)$, we obtain from (11)

$$\begin{aligned} \mathscr{L}\{ty''(t)\}(s) &= -\frac{d}{ds}\mathscr{L}\{y''\}(s) \\ &= -\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] \\ &= -s^2Y'(s) - 2sY(s) + y(0) . \end{aligned}$$

Thus, we see that for a linear differential equation in $y(t)$ whose coefficients are polynomials in $t$, the method of Laplace transforms will convert the given equation into a linear differential equation in $Y(s)$ whose coefficients are polynomials in $s$. Moreover, if the coefficients of the given equation are polynomials of degree 1 in $t$, then (regardless of the order of the given equation) the differential equation for $Y(s)$ is just a linear *first-order* equation. Since we know how to solve this first-order equation, the only serious obstacle we may encounter is obtaining the inverse Laplace transform of $Y(s)$. [This problem may be insurmountable, since the solution $y(t)$ may *not* have a Laplace transform.]

In illustrating the technique, we make use of the following fact. *If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order, then*

**(12)** $$\lim_{s \to \infty} \mathscr{L}\{f\}(s) = 0 .$$

(You may have already guessed this from the entries in Table 7.1, page 356.) An outline of the proof of (12) is given in Exercises 7.3, page 366, Problem 26.

**Example 4** Solve the initial value problem

**(13)** $$y'' + 2ty' - 4y = 1 , \qquad y(0) = y'(0) = 0 .$$

**Solution** Let $Y(s) = \mathscr{L}\{y\}(s)$ and take the Laplace transform of both sides of the equation in (13):

**(14)** $$\mathscr{L}\{y''\}(s) + 2\mathscr{L}\{ty'(t)\}(s) - 4Y(s) = \frac{1}{s} .$$

Using the initial conditions, we find

$$\mathscr{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$$

and

$$\mathscr{L}\{ty'(t)\}(s) = -\frac{d}{ds}\mathscr{L}\{y'\}(s)$$

$$= -\frac{d}{ds}[sY(s) - y(0)] = -sY'(s) - Y(s) .$$

Substituting these expressions into (14) gives

$$s^2Y(s) + 2[-sY'(s) - Y(s)] - 4Y(s) = \frac{1}{s}$$

$$-2sY'(s) + (s^2 - 6)Y(s) = \frac{1}{s}$$

**(15)** $$Y'(s) + \left(\frac{3}{s} - \frac{s}{2}\right)Y(s) = \frac{-1}{2s^2} .$$

Equation (15) is a linear first-order equation and has the integrating factor

$$\mu(s) = e^{\int (3/s - s/2)ds} = e^{\ln s^3 - s^2/4} = s^3 e^{-s^2/4}$$

(see Section 2.3). Multiplying (15) by $\mu(s)$, we obtain

$$\frac{d}{ds}\{\mu(s)Y(s)\} = \frac{d}{ds}\{s^3e^{-s^2/4}Y(s)\} = -\frac{s}{2}e^{-s^2/4} .$$

Integrating and solving for $Y(s)$ yields

$$s^3e^{-s^2/4}Y(s) = -\int \frac{s}{2}e^{-s^2/4}\,ds = e^{-s^2/4} + C$$

**(16)** $$Y(s) = \frac{1}{s^3} + C\frac{e^{s^2/4}}{s^3} .$$



Now if $Y(s)$ is the Laplace transform of a piecewise continuous function of exponential order, then it follows from equation (12) that

$$\lim_{s\to\infty} Y(s) = 0 .$$

For this to occur, the constant $C$ in equation (16) must be zero. Hence, $Y(s) = 1/s^3$, and taking the inverse transform gives $y(t) = t^2/2$. We can easily verify that $y(t) = t^2/2$ is the solution to the given initial value problem by substituting it into (13). ◆

We end this section with an application from **control theory.** Let's consider a servomechanism that models an automatic pilot. Such a mechanism applies a torque to the steering control shaft so that a plane or boat will follow a prescribed course. If we let $y(t)$ be the true direction (angle) of the craft at time $t$ and $g(t)$ be the desired direction at time $t$, then

$$e(t) := y(t) - g(t)$$

denotes the **error** or **deviation** between the desired direction and the true direction.

Let's assume that the servomechanism can measure the error $e(t)$ and feed back to the steering shaft a component of torque that is proportional to $e(t)$ but opposite in sign (see Figure 7.6 on page 381). Newton's second law, expressed in terms of torques, states that

**(moment of inertia) × (angular acceleration) = total torque.**

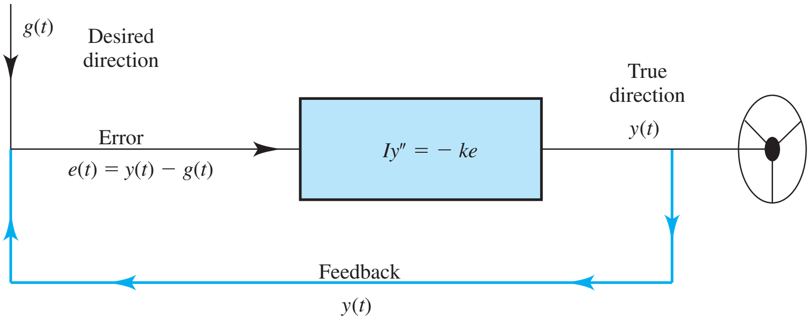



**Figure 7.6** Servomechanism with feedback

For the servomechanism described, this becomes

**(17)** $$Iy''(t) = -ke(t)\ ,$$

where $I$ is the moment of inertia of the steering shaft and $k$ is a positive proportionality constant.

**Example 5** Determine the error $e(t)$ for the automatic pilot if the steering shaft is initially at rest in the zero direction and the desired direction is given by $g(t) = at$, where $a$ is a constant.



**Solution** Based on the discussion leading to equation (17), a model for the mechanism is given by the initial value problem

**(18)** $$Iy''(t) = -ke(t)\ ; \qquad y(0) = 0\ , \qquad y'(0) = 0\ ,$$

where $e(t) = y(t) - g(t) = y(t) - at$. We begin by taking the Laplace transform of both sides of (18):

$$I\mathscr{L}\{y''\}(s) = -k\mathscr{L}\{e\}(s)$$

$$I[s^2Y(s) - sy(0) - y'(0)] = -kE(s)$$

**(19)** $$s^2IY(s) = -kE(s)\ ,$$

where $Y(s) = \mathscr{L}\{y\}(s)$ and $E(s) = \mathscr{L}\{e\}(s)$. Since

$$E(s) = \mathscr{L}\{y(t) - at\}(s) = Y(s) - \mathscr{L}\{at\}(s) = Y(s) - as^{-2}\ ,$$

we find from (19) that

$$s^2IE(s) + aI = -kE(s)\ .$$

Solving this equation for $E(s)$ gives

$$E(s) = -\frac{aI}{s^2I + k} = \frac{-a}{\sqrt{k/I}}\,\frac{\sqrt{k/I}}{s^2 + k/I}\ .$$

Hence, on taking the inverse Laplace transform, we obtain the error

**(20)** $$e(t) = -\frac{a}{\sqrt{k/I}}\sin\left(\sqrt{k/I}\,t\right)\ . \ ◆$$

As we can see from equation (20), the automatic pilot will oscillate back and forth about the desired course, always "oversteering" by the factor $a/\sqrt{k/I}$. Clearly, we can make the

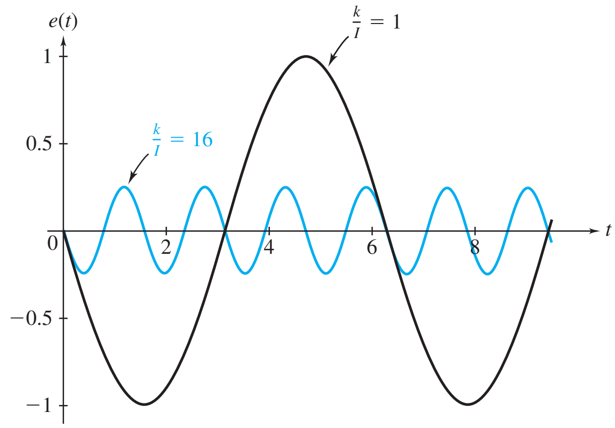



**Figure 7.7** Error for automatic pilot when $k/I = 1$ and when $k/I = 16$

error small by making $k$ large relative to $I$, but then the term $\sqrt{k/I}$ becomes large, causing the error to oscillate more rapidly. (See Figure 7.7.) As with vibrations, the oscillations or oversteering can be controlled by introducing a damping torque proportional to $e'(t)$ but opposite in sign (see Problem 40).



## 7.5 EXERCISES

*In Problems 1–14, solve the given initial value problem using the method of Laplace transforms.*

1. $y'' - 2y' + 5y = 0\,; \quad y(0) = 2\,, \quad y'(0) = 4$
2. $y'' - y' - 2y = 0\,; \quad y(0) = -2\,, \quad y'(0) = 5$
3. $y'' + 6y' + 9y = 0\,; \quad y(0) = -1\,, \quad y'(0) = 6$
4. $y'' + 6y' + 5y = 12e^t\,; \quad y(0) = -1\,, \quad y'(0) = 7$
5. $w'' + w = t^2 + 2\,; \quad w(0) = 1\,, \quad w'(0) = -1$
6. $y'' - 4y' + 5y = 4e^{3t}\,; \quad y(0) = 2\,, \quad y'(0) = 7$
7. $y'' - 7y' + 10y = 9\cos t + 7\sin t\,;$ $y(0) = 5\,, \quad y'(0) = -4$
8. $y'' + 4y = 4t^2 - 4t + 10\,;$ $y(0) = 0\,, \quad y'(0) = 3$
9. $z'' + 5z' - 6z = 21e^{t-1}\,;$ $z(1) = -1\,, \quad z'(1) = 9$
10. $y'' - 4y = 4t - 8e^{-2t}\,; \quad y(0) = 0\,, \quad y'(0) = 5$
11. $y'' - y = t - 2\,; \quad y(2) = 3\,, \quad y'(2) = 0$
12. $w'' - 2w' + w = 6t - 2\,;$ $w(-1) = 3\,; \quad w'(-1) = 7$
13. $y'' - y' - 2y = -8\cos t - 2\sin t\,;$ $y(\pi/2) = 1\,, \quad y'(\pi/2) = 0$
14. $y'' + y = t\,; \quad y(\pi) = 0\,, \quad y'(\pi) = 0$

*In Problems 15–24, solve for $Y(s)$, the Laplace transform of the solution $y(t)$ to the given initial value problem.*

15. $y'' - 3y' + 2y = \cos t\,; \quad y(0) = 0\,, \quad y'(0) = -1$
16. $y'' + 6y = t^2 - 1\,; \quad y(0) = 0\,, \quad y'(0) = -1$
17. $y'' + y' - y = t^3\,; \quad y(0) = 1\,, \quad y'(0) = 0$
18. $y'' - 2y' - y = e^{2t} - e^t\,; \quad y(0) = 1\,, \quad y'(0) = 3$
19. $y'' + 5y' - y = e^t - 1\,; \quad y(0) = 1\,, \quad y'(0) = 1$
20. $y'' + 3y = t^3\,; \quad y(0) = 0\,, \quad y'(0) = 0$
21. $y'' - 2y' + y = \cos t - \sin t\,; \quad y(0) = 1\,, \quad y'(0) = 3$

**22.** $y'' - 6y' + 5y = te^t$; $y(0) = 2$, $y'(0) = -1$

**23.** $y'' + 4y = g(t)$; $y(0) = -1$, $y'(0) = 0$,
where

$$g(t) = \begin{cases} t, & t < 2, \\ 5, & t > 2 \end{cases}$$

**24.** $y'' - y = g(t)$; $y(0) = 1$, $y'(0) = 2$,
where

$$g(t) = \begin{cases} 1, & t < 3, \\ t, & t > 3 \end{cases}$$

*In Problems 25–28, solve the given third-order initial value problem for $y(t)$ using the method of Laplace transforms.*

**25.** $y''' - y'' + y' - y = 0$;
$y(0) = 1$, $y'(0) = 1$, $y''(0) = 3$

**26.** $y''' + 4y'' + y' - 6y = -12$;
$y(0) = 1$, $y'(0) = 4$, $y''(0) = -2$

**27.** $y''' + 3y'' + 3y' + y = 0$;
$y(0) = -4$, $y'(0) = 4$, $y''(0) = -2$

**28.** $y''' + y'' + 3y' - 5y = 16e^{-t}$;
$y(0) = 0$, $y'(0) = 2$, $y''(0) = -4$

*In Problems 29–32, use the method of Laplace transforms to find a general solution to the given differential equation by assuming $y(0) = a$ and $y'(0) = b$, where $a$ and $b$ are arbitrary constants.*

**29.** $y'' - 4y' + 3y = 0$

**30.** $y'' + 6y' + 5y = t$

**31.** $y'' + 2y' + 2y = 5$

**32.** $y'' - 5y' + 6y = -6te^{2t}$

**33.** Use Theorem 6 in Section 7.3, page 364, to show that

$$\mathscr{L}\{t^2 y'(t)\}(s) = sY''(s) + 2Y'(s),$$

where $Y(s) = \mathscr{L}\{y\}(s)$.

**34.** Use Theorem 6 in Section 7.3, page 364, to show that

$$\mathscr{L}\{t^2 y''(t)\}(s) = s^2 Y''(s) + 4sY'(s) + 2Y(s),$$

where $Y(s) = \mathscr{L}\{y\}(s)$.

*In Problems 35–38, find solutions to the given initial value problem.*

**35.** $y'' + 3ty' - 6y = 1$; $y(0) = 0$, $y'(0) = 0$

**36.** $ty'' - ty' + y = 2$; $y(0) = 2$, $y'(0) = -1$

**37.** $ty'' - 2y' + ty = 0$; $y(0) = 1$, $y'(0) = 0$
[*Hint:* $\mathscr{L}^{-1}\{1/(s^2+1)^2\}(t) = (\sin t - t\cos t)/2$.]

**38.** $y'' + ty' - y = 0$;
$y(0) = 0$, $y'(0) = 3$

**39.** Determine the error $e(t)$ for the automatic pilot in Example 5, page 381, if the shaft is initially at rest in the zero direction and the desired direction is $g(t) = a$, where $a$ is a constant.

**40.** In Example 5 assume that in order to control oscillations, a component of torque proportional to $e'(t)$, but opposite in sign, is also fed back to the steering shaft. Show that equation (17) is now replaced by

$$Iy''(t) = -ke(t) - \mu e'(t),$$

where $\mu$ is a positive constant. Determine the error $e(t)$ for the automatic pilot with mild damping (i.e., $\mu < 2\sqrt{Ik}$) if the steering shaft is initially at rest in the zero direction and the desired direction is given by $g(t) = a$, where $a$ is a constant.

**41.** In Problem 40 determine the error $e(t)$ when the desired direction is given by $g(t) = at$, where $a$ is a constant.

# 7.6 Transforms of Discontinuous Functions

In this section we study special functions that often arise when the method of Laplace transforms is applied to physical problems. Of particular interest are methods for handling functions with jump discontinuities. As we saw in the mixing problem of Section 7.1, jump discontinuities occur naturally in any physical situation that involves switching. Finding the Laplace transforms of such functions is straightforward; however, we need some theory for inverting these transforms. To facilitate this, Oliver Heaviside introduced the following step function.

## Unit Step Function

**Definition 5.** The **unit step function** $u(t)$ is defined by

**(1)** $$u(t) := \begin{cases} 0, & t < 0, \\ 1, & 0 < t. \end{cases}$$

(Any Riemann integral, like the Laplace transform, of a function is unaffected if the integrand's value at a single point is changed by a finite amount. Therefore, we do not specify a value for $u(t)$ at $t = 0$.)

By shifting the argument of $u(t)$, the jump can be moved to a different location. That is,

**(2)** $$u(t-a) = \begin{cases} 0, & t-a < 0, \\ 1, & 0 < t-a \end{cases} = \begin{cases} 0, & t < a \\ 1, & a < t \end{cases}$$

has its jump at $t = a$. By multiplying by a constant $M$, the height of the jump can also be modified:

$$Mu(t-a) = \begin{cases} 0, & t < a, \\ M, & a < t. \end{cases}$$

See Figure 7.8.



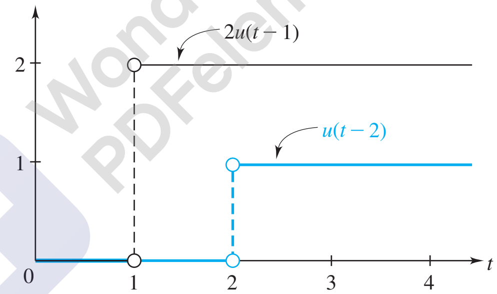



**Figure 7.8** Two-step functions expressed using the unit step function

To simplify the formulas for piecewise continuous functions, we employ the rectangular window, which turns the step function on and then turns it back off.

## Rectangular Window Function

**Definition 6.** The **rectangular window function** $\Pi_{a,b}(t)$ is defined by[†]

**(3)** $$\Pi_{a,b}(t) := u(t-a) - u(t-b) = \begin{cases} 0, & t < a, \\ 1, & a < t < b, \\ 0, & b < t. \end{cases}$$

[†]Also known as the **square pulse**, or the **boxcar function**.

The function $\Pi_{a,b}(t)$ is displayed in Figure 7.9, and Figure 7.10, illustrating multiplication of a function by $\Pi_{a,b}(t)$, justifies its name.

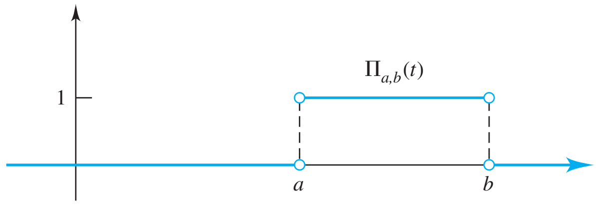



**Figure 7.9** The rectangular window

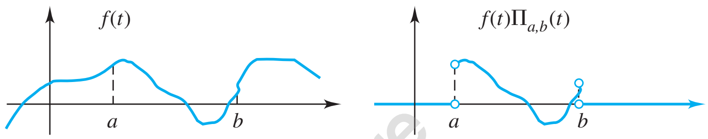



**Figure 7.10** The windowing effect of $\Pi_{a,b}(t)$



Any piecewise continuous function can be expressed in terms of window and step functions.

**Example 1** Write the function

(4) $$f(t) = \begin{cases} 3\,, & t < 2\,, \\ 1\,, & 2 < t < 5\,, \\ t\,, & 5 < t < 8\,, \\ t^2/10\,, & 8 < t \end{cases}$$

(see Figure 7.11 on page 386) in terms of window and step functions.

**Solution** Clearly, from the figure we want to window the function in the intervals (0, 2), (2, 5), and (5, 8), and to introduce a step for $t > 8$. From (5) we read off the desired representation as

(5) $$f(t) = 3\Pi_{0,2}(t) + 1\Pi_{2,5}(t) + t\Pi_{5,8}(t) + (t^2/10)u(t-8)\,. \quad ◆$$

The Laplace transform of $u(t-a)$ with $a \geq 0$ is

(6) $$\mathscr{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s}\,,$$

since, for $s > 0$,

$$\mathscr{L}\{u(t-a)\}(s) = \int_0^\infty e^{-st}u(t-a)\,dt = \int_a^\infty e^{-st}dt$$

$$= \lim_{N\to\infty} \frac{-e^{-st}}{s}\bigg|_a^N = \frac{e^{-as}}{s}\,.$$

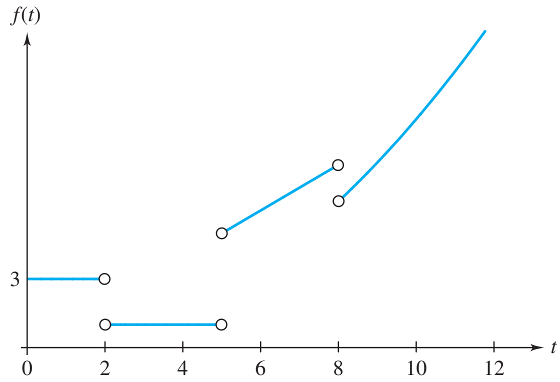



**Figure 7.11** Graph of $f(t)$ in equation (4)

Conversely, for $a > 0$, we say that the piecewise continuous function $u(t-a)$ is an inverse Laplace transform for $e^{-as}/s$ and we write

$$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\}(t) = u(t-a) .$$

For the rectangular window function, we deduce from (6) that

$$\mathscr{L}\{\Pi_{a,b}(t)\}(s) = \mathscr{L}\{u(t-a) - u(t-b)\}(s) = [e^{-sa} - e^{-sb}]/s , 0 < a < b. \tag{7}$$

The translation property of $F(s)$ discussed in Section 7.3 described the effect on the Laplace transform of multiplying a function by $e^{at}$. The next theorem illustrates an analogous effect of multiplying the Laplace transform of a function by $e^{-as}$.



**Translation in *t***

**Theorem 8.** Let $F(s) = \mathscr{L}\{f\}(s)$ exist for $s > \alpha \geq 0$. If $a$ is a positive constant, then

$$\mathscr{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s) , \tag{8}$$

and, conversely, an inverse Laplace transform[†] of $e^{-as}F(s)$ is given by

$$\mathscr{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a) . \tag{9}$$

**Proof.** By the definition of the Laplace transform,

$$\mathscr{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty e^{-st}f(t-a)u(t-a)\,dt \tag{10}$$

$$= \int_a^\infty e^{-st}f(t-a)\,dt ,$$

[†]This inverse transform is in fact a *continuous* function of $t$ if $f(0) = 0$ and $f(t)$ is continuous for $t \geq 0$; the values of $f(t)$ for $t < 0$ are of no consequence, since the factor $u(t-a)$ is zero there.

where, in the last equation, we used the fact that $u(t-a)$ is zero for $t<a$ and equals 1 for $t>a$. Now let $v = t-a$. Then we have $dv = dt$, and equation (10) becomes

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty e^{-as}e^{-sv}f(v)\,dv$$
$$= e^{-as}\int_0^\infty e^{-sv}f(v)\,dv = e^{-as}F(s)\,. \quad \blacklozenge$$

Notice that formula (8) includes as a special case the formula for $\mathcal{L}\{u(t-a)\}$; indeed, if we take $f(t) \equiv 1$, then $F(s) = 1/s$ and (8) becomes $\mathcal{L}\{u(t-a)\}(s) = e^{-as}/s$.

In practice it is more common to be faced with the problem of computing the transform of a function expressed as $g(t)u(t-a)$ rather than $f(t-a)u(t-a)$. To compute $\mathcal{L}\{g(t)u(t-a)\}$, we simply identify $g(t)$ with $f(t-a)$ so that $f(t) = g(t+a)$. Equation (8) then gives

**(11)** $$\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)\,.$$



**Example 2** Determine the Laplace transform of $t^2u(t-1)$.

**Solution** To apply equation (11), we take $g(t) = t^2$ and $a = 1$. Then

$$g(t+a) = g(t+1) = (t+1)^2 = t^2 + 2t + 1\,.$$

Now the Laplace transform of $g(t+a)$ is

$$\mathcal{L}\{g(t+a)\}(s) = \mathcal{L}\{t^2+2t+1\}(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\,.$$

So, by formula (11), we have

$$\mathcal{L}\{t^2u(t-1)\}(s) = e^{-s}\left\{\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right\}. \quad \blacklozenge$$

**Example 3** Determine $\mathcal{L}\{(\cos t)u(t-\pi)\}$.

**Solution** Here $g(t) = \cos t$ and $a = \pi$. Hence,

$$g(t+a) = g(t+\pi) = \cos(t+\pi) = -\cos t\,,$$

and so the Laplace transform of $g(t+a)$ is

$$\mathcal{L}\{g(t+a)\}(s) = -\mathcal{L}\{\cos t\}(s) = -\frac{s}{s^2+1}\,.$$

Thus, from formula (11), we get

$$\mathcal{L}\{(\cos t)u(t-\pi)\}(s) = -e^{-\pi s}\frac{s}{s^2+1}\,. \quad \blacklozenge$$

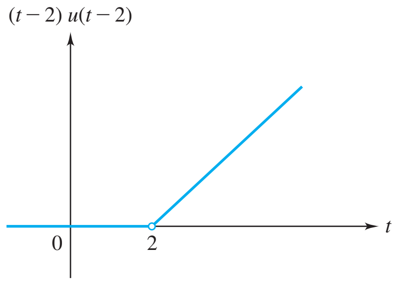



**Figure 7.12** Graph of solution to Example 4

In Examples 2 and 3, we could also have computed the Laplace transform directly from the definition. In dealing with inverse transforms, however, we do not have a simple alternative formula[†] upon which to rely, and so formula (9) is especially useful whenever the transform has $e^{-as}$ as a factor.

**Example 4** Determine $\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$ and sketch its graph.

**Solution** To use the translation property (9), we first express $e^{-2s}/s^2$ as the product $e^{-as}F(s)$. For this purpose, we put $e^{-as} = e^{-2s}$ and $F(s) = 1/s^2$. Thus, $a = 2$ and

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) = t\,.$$

It now follows from the translation property that

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}(t) = f(t-2)u(t-2) = (t-2)u(t-2)\,.$$

See Figure 7.12. ◆



As we anticipated in the beginning of this section, step functions arise in the modeling of on/off switches, changes in polarity, etc.

**Example 5** The current $I$ in an $LC$ series circuit is governed by the initial value problem

$$I''(t) + 4I(t) = g(t)\,; \qquad I(0) = 0\,, \qquad I'(0) = 0\,, \tag{12}$$

where

$$g(t) := \begin{cases} 1\,, & 0 < t < 1\,, \\ -1\,, & 1 < t < 2\,, \\ 0\,, & 2 < t\,. \end{cases}$$

Determine the current as a function of time $t$.

**Solution** Let $J(s) := \mathscr{L}\{I\}(s)$. Then we have $\mathscr{L}\{I''\}(s) = s^2J(s)$.

---

[†]Under certain conditions, the inverse transform is given by the contour integral

$$\mathscr{L}^{-1}\{F\}(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}F(s)\,ds\,.$$

See, for example, *Complex Variables and the Laplace Transform for Engineers,* by Wilbur R. LePage (Dover Publications, New York, 2010), or *Fundamentals of Complex Analysis with Applications to Engineering and Science,* 3rd ed., by E. B. Saff and A. D. Snider (Pearson Education, Boston. MA, 2003).

Writing $g(t)$ in terms of the rectangular window function $\Pi_{a,b}(t) = u(t-a) - u(t-b)$, we get

$$\begin{aligned} g(t) &= \Pi_{0,1}(t) + (-1)\Pi_{1,2}(t) = u(t) - u(t-1) - [u(t-1) - u(t-2)] \\ &= 1 - 2u(t-1) + u(t-2) , \end{aligned}$$

and so

$$\mathscr{L}\{g\}(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} .$$

Thus, when we take the Laplace transform of both sides of (12), we obtain

$$\begin{aligned} \mathscr{L}\{I''\}(s) + 4\mathscr{L}\{I\}(s) &= \mathscr{L}\{g\}(s) \\ s^2J(s) + 4J(s) &= \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} \\ J(s) &= \frac{1}{s(s^2+4)} - \frac{2e^{-s}}{s(s^2+4)} + \frac{e^{-2s}}{s(s^2+4)} . \end{aligned}$$

To find $I = \mathscr{L}^{-1}\{J\}$, we first observe that

$$J(s) = F(s) - 2e^{-s}F(s) + e^{-2s}F(s) ,$$

where

$$F(s) := \frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) .$$

Computing the inverse transform of $F(s)$ gives

$$f(t) := \mathscr{L}^{-1}\{F\}(t) = \frac{1}{4} - \frac{1}{4}\cos 2t .$$

Hence, via the translation property (9), we find

$$\begin{aligned} I(t) &= \mathscr{L}^{-1}\{F(s) - 2e^{-s}F(s) + e^{-2s}F(s)\}(t) \\ &= f(t) - 2f(t-1)u(t-1) + f(t-2)u(t-2) \\ &= \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right) - \left[\frac{1}{2} - \frac{1}{2}\cos 2(t-1)\right]u(t-1) \\ &\quad + \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-2)\right]u(t-2) . \end{aligned}$$

The current is graphed in Figure 7.13. Note that $I(t)$ is smoother than $g(t)$; the former has discontinuities in its second derivative at the points where the latter has jumps. ◆



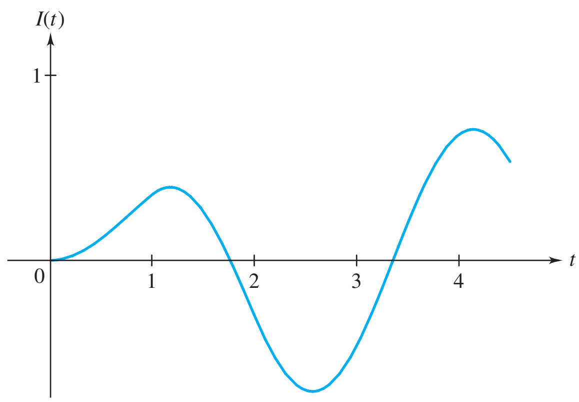



**Figure 7.13** Solution to Example 5

## 7.6 EXERCISES

*In Problems 1–4, sketch the graph of the given function and determine its Laplace transform.*

**1.** $(t-1)^2 u(t-1)$

**2.** $u(t-1) - u(t-4)$

**3.** $t^2 u(t-2)$

**4.** $tu(t-1)$

*In Problems 5–10, express the given function using window and step functions and compute its Laplace transform.*

**5.** $g(t) = \begin{cases} 0\,, & 0 < t < 1\,, \\ 2\,, & 1 < t < 2\,, \\ 1\,, & 2 < t < 3\,, \\ 3\,, & 3 < t \end{cases}$

**6.** $g(t) = \begin{cases} 0\,, & 0 < t < 2\,, \\ t+1\,, & 2 < t \end{cases}$

**7.**

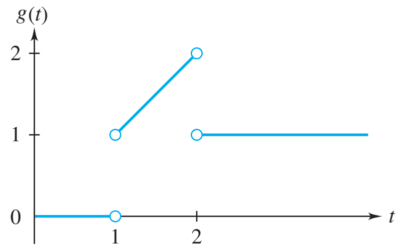



**Figure 7.14** Function in Problem 7

**8.**

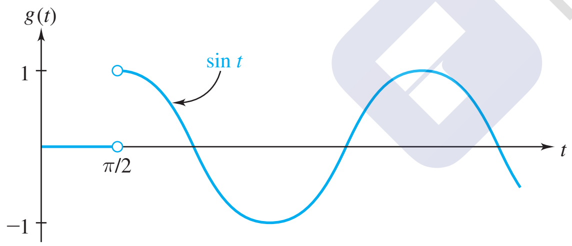



**Figure 7.15** Function in Problem 8

**9.**

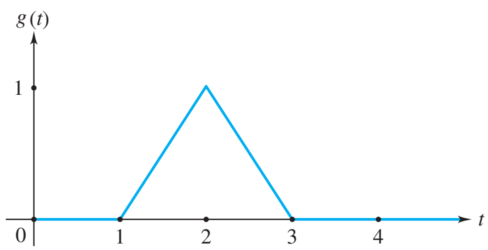



**Figure 7.16** Function in Problem 9

**10.**

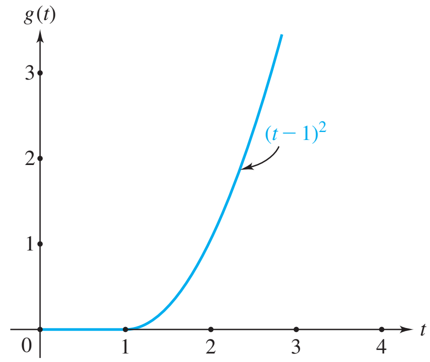



**Figure 7.17** Function in Problem 10

*In Problems 11–18, determine an inverse Laplace transform of the given function.*

**11.** $\dfrac{e^{-2s}}{s-1}$

**12.** $\dfrac{e^{-3s}}{s^2}$

**13.** $\dfrac{e^{-2s} - 3e^{-4s}}{s+2}$

**14.** $\dfrac{e^{-3s}}{s^2+9}$

**15.** $\dfrac{se^{-3s}}{s^2+4s+5}$

**16.** $\dfrac{e^{-s}}{s^2+4}$

**17.** $\dfrac{e^{-3s}(s-5)}{(s+1)(s+2)}$

**18.** $\dfrac{e^{-s}(3s^2-s+2)}{(s-1)(s^2+1)}$

**19.** The current $I(t)$ in an *RLC* series circuit is governed by the initial value problem

$$I''(t) + 2I'(t) + 2I(t) = g(t)\,;$$
$$I(0) = 10\,, \qquad I'(0) = 0\,,$$

where

$$g(t) := \begin{cases} 20\,, & 0 < t < 3\pi\,, \\ 0\,, & 3\pi < t < 4\pi\,, \\ 20\,, & 4\pi < t\,. \end{cases}$$

Determine the current as a function of time $t$. Sketch $I(t)$ for $0 < t < 8\pi$.

**20.** The current $I(t)$ in an *LC* series circuit is governed by the initial value problem

$$I''(t) + 4I(t) = g(t)\,;$$
$$I(0) = 1\,, \qquad I'(0) = 3\,,$$

where

$$g(t) := \begin{cases} 3\sin t\,, & 0 \le t \le 2\pi\,, \\ 0\,, & 2\pi < t\,. \end{cases}$$

Determine the current as a function of time $t$.

*In Problems 21–24, solve the given initial value problem using the method of Laplace transforms. Sketch the graph of the solution.*

**21.** $y'' + y = u(t-3)\,;$
$y(0) = 0\,, \quad y'(0) = 1$

**22.** $w'' + w = u(t-2) - u(t-4)\,;$
$w(0) = 1\,, \quad w'(0) = 0$

**23.** $y'' + y = t - (t-4)u(t-2)\,;$
$y(0) = 0\,, \quad y'(0) = 1$

**24.** $y'' + y = 3\sin 2t - 3(\sin 2t)u(t-2\pi)\,;$
$y(0) = 1\,, \quad y'(0) = -2$

*In Problems 25–32, solve the given initial value problem using the method of Laplace transforms.*

**25.** $y'' + 2y' + 2y = u(t-2\pi) - u(t-4\pi)\,;$
$y(0) = 1\,, \quad y'(0) = 1$

**26.** $y'' + 4y' + 4y = u(t-\pi) - u(t-2\pi)\,;$
$y(0) = 0\,, \quad y'(0) = 0$

**27.** $z'' + 3z' + 2z = e^{-3t}u(t-2)\,;$
$z(0) = 2\,, \quad z'(0) = -3$

**28.** $y'' + 5y' + 6y = tu(t-2)\,;$
$y(0) = 0\,, \quad y'(0) = 1$

**29.** $y'' + 4y = g(t)\,; \quad y(0) = 1\,, \quad y'(0) = 3\,,$
where $g(t) = \begin{cases} \sin t\,, & 0 \le t \le 2\pi\,, \\ 0\,, & 2\pi < t \end{cases}$

**30.** $y'' + 2y' + 10y = g(t)\,;$
$y(0) = -1\,, \quad y'(0) = 0\,,$
where $g(t) = \begin{cases} 10\,, & 0 \le t \le 10\,, \\ 20\,, & 10 < t < 20\,, \\ 0\,, & 20 < t \end{cases}$

**31.** $y'' + 5y' + 6y = g(t)\,;$
$y(0) = 0\,, \quad y'(0) = 2\,,$
where $g(t) = \begin{cases} 0\,, & 0 \le t < 1\,, \\ t\,, & 1 < t < 5\,, \\ 1\,, & 5 < t \end{cases}$

**32.** $y'' + 3y' + 2y = g(t)\,;$
$y(0) = 2\,, \quad y'(0) = -1\,,$
where $g(t) = \begin{cases} e^{-t}\,, & 0 \le t < 3\,, \\ 1\,, & 3 < t \end{cases}$

**33.** The mixing tank in Figure 7.18 initially holds 500 L of a brine solution with a salt concentration of 0.02 kg/L. For the first 10 min of operation, valve $A$ is open, adding 12 L/min of brine containing a 0.04 kg/L salt concentration. After 10 min, valve $B$ is switched in, adding a 0.06 kg/L concentration at 12 L/min. The exit valve $C$ removes 12 L/min, thereby keeping the volume constant. Find the concentration of salt in the tank as a function of time.

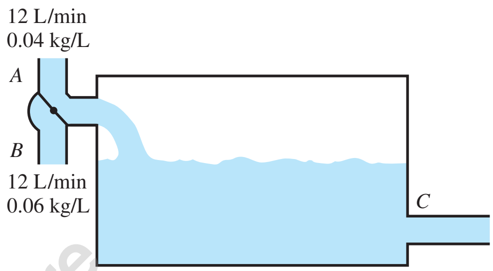



**Figure 7.18** Mixing tank

**34.** Suppose in Problem 33 valve $B$ is initially opened for 10 min and then valve $A$ is switched in for 10 min. Finally, valve $B$ is switched back in. Find the concentration of salt in the tank as a function of time.

**35.** Suppose valve $C$ removes only 6 L/min in Problem 33. Can Laplace transforms be used to solve the problem? Discuss.

**36.** The **unit triangular pulse** $\Lambda(t)$ is defined by

$$\Lambda(t) := \begin{cases} 0\,, & t < 0\,, \\ 2t\,, & 0 < t < 1/2\,, \\ 2 - 2t\,, & 1/2 < t < 1\,, \\ 0\,, & t > 1\,. \end{cases}$$

**(a)** Sketch the graph of $\Lambda(t)$. Why is it so named? Why is its symbol appropriate?

**(b)** Show that $\Lambda(t) = \int_{-\infty}^{t} 2\{\Pi_{0,1/2}(\tau) - \Pi_{1/2,1}(\tau)\}\,d\tau$.

**(c)** Find the Laplace transform of $\Lambda(t)$.

# 7.7 Transforms of Periodic and Power Functions

*Periodic* functions arise frequently in physical situations such as sinusoidal vibrations in structures, and in electromagnetic oscillations in AC machinery and microwave transmission. *Power* functions $(t^n)$ occur in more specialized applications: the square-cube law of biomechanics*, the cube rule of electoral politics,† Coulomb's inverse-square force, and, most significantly, the Taylor series of Section 3.7 and Chapter 8. The manipulation of these functions' transforms (when they exist) is facilitated by the techniques described in this section.

**Periodic Function**

**Definition 7.** A function $f(t)$ is said to be **periodic of period** $\boldsymbol{T}$ $(\neq 0)$ if

$$f(t+T) = f(t)$$

for all $t$ in the domain of $f$.

As we know, the sine and cosine functions are periodic with period $2\pi$ and the tangent function is periodic with period $\pi$.‡ To specify a periodic function, it is sufficient to give its values over one period. For example, the square wave function in Figure 7.19 can be expressed as

$$\textbf{(1)} \qquad f(t) := \begin{cases} 1\,, & 0 < t < 1\,, \\ -1\,, & 1 < t < 2\,, \end{cases} \quad \text{and } f(t) \text{ has period } 2.$$

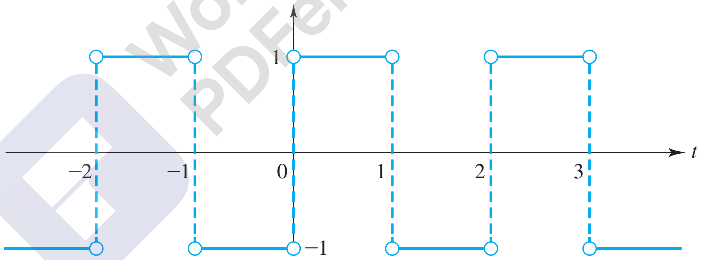



**Figure 7.19** Graph of square wave function $f(t)$

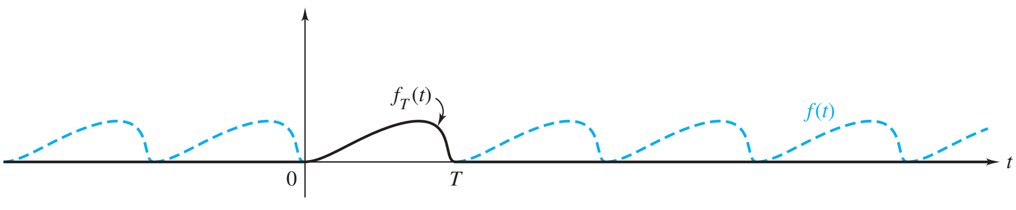



**Figure 7.20** Windowed version of periodic function

*The volume of a body increases as the cube of its length; its surface area increases as the square of the length. First formulated by Galileo in 1638 (*Discourses and Mathematical Demonstrations Relating to Two New Sciences*), this principle is useful in explaining the limitations on animal growth.

†In a two-party system, the ratio of the seats won equals the cube of the ratio of the votes cast. (G. Upton, "Blocks of voters and the cube law," *British Journal of Political Science*. Vol. 15, Issue 03 (1985): 388–398.)

‡A function that has period $T$ will also have period $2T$, $3T$, etc. For example, the sine function has periods $2\pi$, $4\pi$, $6\pi$, etc. Some authors refer to the smallest period as the **fundamental period** or just the period of the function.

It is convenient to introduce a notation for the "windowed" version of a periodic function $f(t)$, using a rectangular window whose width is the period:

(2) $$f_T(t) := f(t)\Pi_{0,T}(t) = f(t)[u(t) - u(t-T)] = \begin{cases} f(t), & 0 < t < T, \\ 0, & \text{otherwise}. \end{cases}$$

(See Figure 7.20 on page 392.) The Laplace transform of $f_T(t)$ is given by

$$F_T(s) = \int_0^\infty e^{-st} f_T(t)\,dt = \int_0^T e^{-st} f(t)\,dt\,.$$

It is related to the Laplace transform of $f(t)$ as follows.

## Transform of Periodic Function

**Theorem 9.** If $f$ has period $T$ and is piecewise continuous on $[0, T]$, then the Laplace transforms

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt \quad \text{and} \quad F_T(s) = \int_0^\infty e^{-st} f_T(t)\,dt = \int_0^T e^{-st} f(t)\,dt$$

are related by

(3) $$F_T(s) = F(s)\left[1 - e^{-sT}\right] \quad \text{or} \quad F(s) = \frac{F_T(s)}{1 - e^{-sT}}.$$



**Proof.** From (2) and the periodicity of $f$, we have

(4) $$f_T(t) = f(t)u(t) - f(t)u(t-T) = f(t)u(t) - f(t-T)u(t-T)\,,$$

so taking transforms and applying the translation-in-$t$ property (Theorem 8, page 386) yields $F_T(s) = F(s) - e^{-sT}F(s)$, which is equivalent to (3). ◆

**Example 1** Determine $\mathscr{L}\{f\}$, where $f$ is the periodic square wave function in Figure 7.19.

**Solution** Here $T = 2$. Windowing the function results in $f_T(t) = \Pi_{0,1}(t) - \Pi_{1,2}(t)$, so from the formula for the transform of the window function (equation (7) in Section 7.6, page 386) we get $F_T(s) = (1 - e^{-s})/s - (e^{-s} - e^{-2s})/s = (1 - e^{-s})^2/s$. Therefore (3) implies

$$\mathscr{L}\{f\}(s) = \frac{(1 - e^{-s})^2/s}{1 - e^{-2s}} = \frac{1 - e^{-s}}{(1 + e^{-s})s}\,. \quad ◆$$

We next turn to the problem of finding transforms of functions given by a power series. Our approach is simply to apply the formula $\mathscr{L}\{t^n\}(s) = n!/s^{n+1}$, $n = 0, 1, 2, \ldots$, to the terms of the series.

**Example 2** Determine $\mathscr{L}\{f\}$, where

$$f(t) := \begin{cases} \dfrac{\sin t}{t}, & t \neq 0\,, \\ 1\,, & t = 0\,. \end{cases}$$

**Solution** We begin by expressing $f(t)$ in a Taylor series† about $t = 0$. Since

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots ,$$

then dividing by $t$, we obtain

$$f(t) = \frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots$$

for $t > 0$. This representation also holds at $t = 0$ since

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} \frac{\sin t}{t} = 1 .$$

Observe that $f(t)$ is continuous on $[0, \infty)$ and of exponential order. Hence, its Laplace transform exists for all $s$ sufficiently large. Because of the linearity of the Laplace transform, we would expect that

$$\begin{aligned}\mathscr{L}\{f\}(s) &= \mathscr{L}\{1\}(s) - \frac{1}{3!}\mathscr{L}\{t^2\}(s) + \frac{1}{5!}\mathscr{L}\{t^4\}(s) + \cdots \\ &= \frac{1}{s} - \frac{2!}{3!s^3} + \frac{4!}{5!s^5} - \frac{6!}{7!s^7} + \cdots \\ &= \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \cdots .\end{aligned}$$

Indeed, using tools from analysis, it can be verified that this series representation is valid for all $s > 1$. Moreover, one can show that the series converges to the function $\arctan(1/s)$ (see Problem 22). Thus,

$$\mathscr{L}\left\{\frac{\sin t}{t}\right\}(s) = \arctan \frac{1}{s} . \quad (5)$$ ◆

A similar procedure involving the series expansion for $F(s)$ in powers of $1/s$ can be used to compute $f(t) = \mathscr{L}^{-1}\{F\}(t)$ (see Problems 23–25).

We have previously shown, for every nonnegative integer $n$, that $\mathscr{L}\{t^n\}(s) = n!/s^{n+1}$. But what if the power of $t$ is *not* an integer? Is this formula still valid? To answer this question, we need to extend the idea of "factorial." This is accomplished by the gamma function.‡

## Gamma Function

**Definition 8.** The **gamma function** $\Gamma(r)$ is defined by

$$\Gamma(r) := \int_0^{\infty} e^{-u} u^{r-1} du , \qquad r > 0 . \quad (6)$$

It can be shown that the integral in (6) converges for $r > 0$. A useful property of the gamma function is the recursive relation

$$\Gamma(r + 1) = r\Gamma(r) . \quad (7)$$

†For a discussion of Taylor series, see Sections 8.1 and 8.2.

‡*Historical Footnote:* The gamma function was introduced by Leonhard Euler.

This identity follows from the definition (6) after performing an integration by parts:

$$\begin{aligned}\Gamma(r+1) &= \int_0^\infty e^{-u}u^r\,du = \lim_{N\to\infty}\int_0^N e^{-u}u^r\,du \\ &= \lim_{N\to\infty}\left\{e^{-u}u^r\Big|_0^N + \int_0^N re^{-u}u^{r-1}\,du\right\} \\ &= \lim_{N\to\infty}(e^{-N}N^r) + r\lim_{N\to\infty}\int_0^N e^{-u}u^{r-1}\,du \\ &= 0 + r\Gamma(r) = r\Gamma(r)\,.\end{aligned}$$

When $r$ is a positive integer, say $r = n$, then the recursive relation (7) can be repeatedly applied to obtain

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots \\ &= n(n-1)(n-2)\cdots 2\Gamma(1)\,.\end{aligned}$$

It follows from the definition (6) that $\Gamma(1) = 1$, so we find

$$\boldsymbol{\Gamma(n+1) = n!\,.}$$

Thus, the gamma function extends the notion of factorial.

As an application of the gamma function, let's return to the problem of determining the Laplace transform of an arbitrary power of $t$. We will verify that the formula

**(8)** $$\mathcal{L}\{t^r\}(s) = \frac{\Gamma(r+1)}{s^{r+1}}$$

holds for every constant $r > -1$.

By definition,

$$\mathcal{L}\{t^r\}(s) = \int_0^\infty e^{-st}t^r\,dt\,.$$

Let's make the substitution $u = st$. Then $du = s\,dt$, and we find

$$\begin{aligned}\mathcal{L}\{t^r\}(s) &= \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^r\left(\frac{1}{s}\right)du \\ &= \frac{1}{s^{r+1}}\int_0^\infty e^{-u}u^r\,du = \frac{\Gamma(r+1)}{s^{r+1}}\,.\end{aligned}$$

Notice that when $r = n$ is a nonnegative integer, then $\Gamma(n+1) = n!$, and so formula (8) reduces to the familiar formula for $\mathcal{L}\{t^n\}$.

**Example 3** Given that $\Gamma(1/2) = \sqrt{\pi}$ (see Problem 26), find the Laplace transform of $f(t) = t^{3/2}e^{2t}$.

**Solution** We'll apply the translation-in-$s$ property (Theorem 3, page 361) to the transform for $t^{3/2}$, which from (8) is given by $\Gamma\left(\frac{3}{2}+1\right)/s^{\frac{3}{2}+1}$. Thanks to the basic gamma function property (7), we can write

$$\Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}\,.$$

Hence $\mathcal{L}\{t^{3/2}\}(s) = \dfrac{3\sqrt{\pi}}{4s^{5/2}}$, and so

$$\mathcal{L}\left\{t^{3/2}e^{2t}\right\}(s) = \frac{3\sqrt{\pi}}{4(s-2)^{5/2}}\,. \quad ◆$$

## 7.7 EXERCISES

*In Problems 1–4, determine $\mathcal{L}\{f\}$, where $f(t)$ is periodic with the given period. Also graph $f(t)$.*

**1.** $f(t) = t\,,\quad 0 < t < 2\,,$ and $f(t)$ has period 2.

**2.** $f(t) = e^t\,,\quad 0 < t < 1\,,$ and $f(t)$ has period 1.

**3.** $f(t) = \begin{cases} e^{-t}, & 0 < t < 1\,, \\ 1, & 1 < t < 2\,, \end{cases}$ and $f(t)$ has period 2.

**4.** $f(t) = \begin{cases} t, & 0 < t < 1\,, \\ 1 - t, & 1 < t < 2\,, \end{cases}$ and $f(t)$ has period 2.

*In Problems 5–8, determine $\mathcal{L}\{f\}$, where the periodic function is described by its graph.*

**5.**

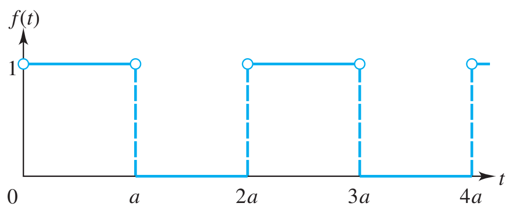



**Figure 7.21** Square wave

**6.**

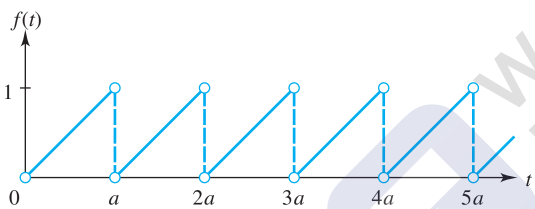



**Figure 7.22** Sawtooth wave

**7.**

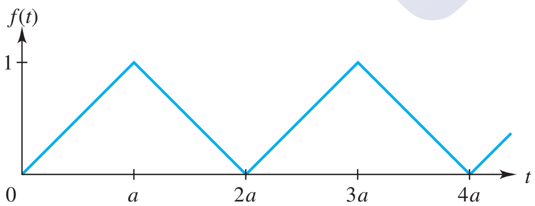



**Figure 7.23** Triangular wave

**8.**

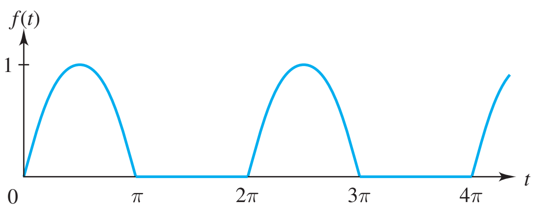



**Figure 7.24** Half-rectified sine wave

**9.** Show that if $\mathcal{L}\{g\}(s) = [(s + \alpha)(1 - e^{-Ts})]^{-1}$, where $T > 0$ is fixed, then

$$\textbf{(9)}\qquad g(t) = e^{-\alpha t} + e^{-\alpha(t-T)}u(t - T) + e^{-\alpha(t-2T)}u(t - 2T) + e^{-\alpha(t-3T)}u(t - 3T) + \cdots\,.$$

[*Hint:* Use the fact that $1 + x + x^2 + \cdots = 1/(1 - x)$.]

**10.** The function $g(t)$ in (9) can be expressed in a more convenient form as follows:

**(a)** Show that for each $n = 0, 1, 2, \ldots,$

$$g(t) = e^{-\alpha t}\left[\frac{e^{(n+1)\alpha T} - 1}{e^{\alpha T} - 1}\right] \text{ for } nT < t < (n + 1)T.$$

[*Hint:* Use the fact that $1 + x + x^2 + \cdots + x^n = (x^{n+1} - 1)/(x - 1)$.]

**(b)** Let $v = t - (n + 1)T$. Show that when $nT < t < (n + 1)T$, then $-T < v < 0$ and

$$\textbf{(10)}\qquad g(t) = \frac{e^{-\alpha v}}{e^{\alpha T} - 1} - \frac{e^{-\alpha t}}{e^{\alpha T} - 1}\,.$$

**(c)** Use the facts that the first term in (10) is periodic with period $T$ and the second term is independent of $n$ to sketch the graph of $g(t)$ in (10) for $\alpha = 1$ and $T = 2$.

**11.** Show that if $\mathcal{L}\{g\}(s) = \beta[(s^2 + \beta^2)(1 - e^{-Ts})]^{-1}$, then

$$g(t) = \sin\beta t + [\sin\beta(t - T)]u(t - T) + [\sin\beta(t - 2T)]u(t - 2T) + [\sin\beta(t - 3T)]u(t - 3T) + \cdots\,.$$

**12.** Use the result of Problem 11 to show that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + 1)(1 - e^{-\pi s})}\right\}(t) = g(t)\,,$$

where $g(t)$ is periodic with period $2\pi$ and

$$g(t) := \begin{cases} \sin t\,, & 0 \le t \le \pi\,, \\ 0\,, & \pi \le t \le 2\pi\,. \end{cases}$$

*In Problems 13 and 14, use the method of Laplace transforms and the results of Problems 9 and 10 to solve the initial value problem.*

$$y'' + 3y' + 2y = f(t)\,;$$
$$y(0) = 0\,,\qquad y'(0) = 0\,,$$

*where $f(t)$ is the periodic function defined in the stated problem.*

**13.** Problem 2

**14.** Problem 5 with $a = 1$

*In Problems 15–18, find a Taylor series for $f(t)$ about $t = 0$. Assuming the Laplace transform of $f(t)$ can be computed term by term, find an expansion for $\mathscr{L}\{f\}(s)$ in powers of $1/s$. If possible, sum the series.*

**15.** $f(t) = e^t$

**16.** $f(t) = \sin t$

**17.** $f(t) = \dfrac{1 - \cos t}{t}$

**18.** $f(t) = e^{-t^2}$

**19.** Using the recursive relation (7) and the fact that $\Gamma(1/2) = \sqrt{\pi}$, determine

**(a)** $\mathscr{L}\{t^{-1/2}\}$.

**(b)** $\mathscr{L}\{t^{7/2}\}$.

**20.** Use the recursive relation (7) and the fact that $\Gamma(1/2) = \sqrt{\pi}$ to show that

$$\mathscr{L}^{-1}\{s^{-(n+1/2)}\}(t) = \frac{2^n t^{n-1/2}}{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}},$$

where $n$ is a positive integer.

**21.** Verify (3) in Theorem 9 for the function $f(t) = \sin t$, taking the period as $2\pi$. Repeat, taking the period as $4\pi$.

**22.** By replacing $s$ by $1/s$ in the Maclaurin series expansion for arctan $s$, show that

$$\arctan \frac{1}{s} = \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \cdots .$$

**23.** Find an expansion for $e^{-1/s}$ in powers of $1/s$. Use the expansion for $e^{-1/s}$ to obtain an expansion for $s^{-1/2}e^{-1/s}$ in terms of $1/s^{n+1/2}$. Assuming the inverse Laplace transform can be computed term by term, show that

$$\mathscr{L}^{-1}\{s^{-1/2}e^{-1/s}\}(t) = \frac{1}{\sqrt{\pi t}} \cos 2\sqrt{t}.$$

[*Hint:* Use the result of Problem 20.]

**24.** Use the procedure discussed in Problem 23 to show that

$$\mathscr{L}^{-1}\{s^{-3/2}e^{-1/s}\}(t) = \frac{1}{\sqrt{\pi}} \sin 2\sqrt{t}.$$

**25.** Find an expansion for $\ln[1 + (1/s^2)]$ in powers of $1/s$. Assuming the inverse Laplace transform can be computed term by term, show that

$$\mathscr{L}^{-1}\left\{\ln\left(1 + \frac{1}{s^2}\right)\right\}(t) = \frac{2}{t}(1 - \cos t).$$

**26.** Evaluate $\Gamma(1/2)$ by setting $r = x^2$ in (6) and relating it to the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. (The latter formula is proved by using polar coordinates to evaluate its square; type "Gaussian integral" into your web browser.)

**27.** Which of these periodic functions coincides with the square wave in Figure 7.19?

**(a)** $f(t) = -1, \quad -1 < t < 0; \quad f(t) = 1, \quad 0 < t < 1;$ and $f$ has period 2.

**(b)** $f(t) = 1, \quad 2 < t < 3; \quad f(t) = -1, \quad 3 < t < 4;$ and $f$ has period 2.

**(c)** $f(t) = 1, \quad 3 < t < 4; \quad f(t) = -1, \quad 4 < t < 5;$ and $f$ has period 2.



# 7.8 Convolution

Consider the initial value problem

$$\text{(1)} \qquad y'' + y = g(t); \qquad y(0) = 0, \qquad y'(0) = 0.$$

If we let $Y(s) = \mathscr{L}\{y\}(s)$ and $G(s) = \mathscr{L}\{g\}(s)$, then taking the Laplace transform of both sides of (1) yields

$$s^2 Y(s) + Y(s) = G(s),$$

and hence

$$\text{(2)} \qquad Y(s) = \left(\frac{1}{s^2 + 1}\right) G(s).$$

That is, the Laplace transform of the solution to (1) is the product of the Laplace transform of $\sin t$ and the Laplace transform of the forcing term $g(t)$. What we would now like to have is a simple formula for $y(t)$ in terms of $\sin t$ and $g(t)$. Just as the integral of a product is not the product of the integrals, $y(t)$ is not the product of $\sin t$ and $g(t)$. However, we can express $y(t)$ as the "convolution" of $\sin t$ and $g(t)$.

## Convolution

**Definition 9.** Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$. The **convolution** of $f(t)$ and $g(t)$, denoted $f * g$, is defined by

(3) $$(f * g)(t) := \int_0^t f(t - v)g(v)\,dv\,.$$

For example, the convolution of $t$ and $t^2$ is

$$t * t^2 = \int_0^t (t - v)v^2\,dv = \int_0^t (tv^2 - v^3)\,dv$$
$$= \left(\frac{tv^3}{3} - \frac{v^4}{4}\right)\Big|_0^t = \frac{t^4}{3} - \frac{t^4}{4} = \frac{t^4}{12}\,.$$

Convolution is certainly different from ordinary multiplication. For example, $1 * 1 = t \neq 1$ and in general $1 * f \neq f$. However, convolution does satisfy some of the same properties as multiplication.



## Properties of Convolution

**Theorem 10.** Let $f(t)$, $g(t)$, and $h(t)$ be piecewise continuous on $[0, \infty)$. Then

(4) $$f * g = g * f\,,$$

(5) $$f * (g + h) = (f * g) + (f * h)\,,$$

(6) $$(f * g) * h = f * (g * h)\,,$$

(7) $$f * 0 = 0\,.$$

**Proof.** To prove equation (4), we begin with the definition

$$(f * g)(t) := \int_0^t f(t - v)g(v)\,dv\,.$$

Using the change of variables $w = t - v$, we have

$$(f * g)(t) = \int_t^0 f(w)g(t - w)(-dw) = \int_0^t g(t - w)f(w)\,dw = (g * f)(t)\,,$$

which proves (4). The proofs of equations (5) and (6) are left to the exercises (see Problems 33 and 34). Equation (7) is obvious, since $f(t - v) \cdot 0 \equiv 0$. ◆

Returning to our original goal, we now prove that if $Y(s)$ is the product of the Laplace transforms $F(s)$ and $G(s)$, then $y(t)$ is the convolution $(f * g)(t)$.

**Convolution Theorem**

**Theorem 11.** Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order $\alpha$ and set $F(s) = \mathscr{L}\{f\}(s)$ and $G(s) = \mathscr{L}\{g\}(s)$. Then

**(8)** $$\mathscr{L}\{f * g\}(s) = F(s)G(s),$$

or, equivalently,

**(9)** $$\mathscr{L}^{-1}\{F(s)G(s)\}(t) = (f * g)(t).$$

**Proof.** Starting with the left-hand side of (8), we use the definition of convolution to write for $s > \alpha$

$$\mathscr{L}\{f * g\}(s) = \int_0^\infty e^{-st}\left[\int_0^t f(t-v)g(v)\,dv\right] dt.$$

To simplify the evaluation of this iterated integral, we introduce the unit step function $u(t-v)$ and write

$$\mathscr{L}\{f * g\}(s) = \int_0^\infty e^{-st}\left[\int_0^\infty u(t-v)f(t-v)g(v)\,dv\right] dt,$$

where we have used the fact that $u(t-v) = 0$ if $v > t$. Reversing the order of integration† gives

**(10)** $$\mathscr{L}\{f * g\}(s) = \int_0^\infty g(v)\left[\int_0^\infty e^{-st}u(t-v)f(t-v)\,dt\right] dv.$$

Recall from the translation property in Section 7.6 that the integral in brackets in equation (10) equals $e^{-sv}F(s)$. Hence,

$$\mathscr{L}\{f * g\}(s) = \int_0^\infty g(v)e^{-sv}F(s)\,dv = F(s)\int_0^\infty e^{-sv}g(v)\,dv = F(s)G(s).$$

This proves formula (8). ◆

For the initial value problem (1), recall that we found

$$Y(s) = \left(\frac{1}{s^2+1}\right)G(s) = \mathscr{L}\{\sin t\}(s)\,\mathscr{L}\{g\}(s).$$

It now follows from the convolution theorem that

$$y(t) = \sin t * g(t) = \int_0^t \sin(t-v)g(v)\,dv.$$

Thus we have obtained an integral representation for the solution to the initial value problem (1) for any forcing function $g(t)$ that is piecewise continuous on $[0, \infty)$ and of exponential order.

**Example 1** Use the convolution theorem to solve the initial value problem

**(11)** $$y'' - y = g(t); \qquad y(0) = 1, \qquad y'(0) = 1,$$

where $g(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order.

†This is permitted since, for each $s > \alpha$, the absolute value of the integrand is integrable on $(0, \infty) \times (0, \infty)$.

**Solution** Let $Y(s) = \mathscr{L}\{y\}(s)$ and $G(s) = \mathscr{L}\{g\}(s)$. Taking the Laplace transform of both sides of the differential equation in (11) and using the initial conditions gives

$$s^2Y(s) - s - 1 - Y(s) = G(s) .$$

Solving for $Y(s)$, we have

$$Y(s) = \frac{s+1}{s^2-1} + \left(\frac{1}{s^2-1}\right)G(s) = \frac{1}{s-1} + \left(\frac{1}{s^2-1}\right)G(s) .$$

Hence,

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) + \mathscr{L}^{-1}\left\{\frac{1}{s^2-1}\,G(s)\right\}(t) \\ &= e^t + \mathscr{L}^{-1}\left\{\frac{1}{s^2-1}\,G(s)\right\}(t) . \end{aligned}$$

Referring to the table of Laplace transforms on the inside back cover, we find

$$\mathscr{L}\{\sinh t\}(s) = \frac{1}{s^2-1} ,$$

so we can now express

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2-1}\,G(s)\right\}(t) = \sinh t * g(t) .$$

Thus,

$$y(t) = e^t + \int_0^t \sinh(t-v)g(v)\,dv$$

is the solution to the initial value problem (11). ◆



**Example 2** Use the convolution theorem to find $\mathscr{L}^{-1}\{1/(s^2+1)^2\}$.

**Solution** Write

$$\frac{1}{(s^2+1)^2} = \left(\frac{1}{s^2+1}\right)\left(\frac{1}{s^2+1}\right).$$

Since $\mathscr{L}\{\sin t\}(s) = 1/(s^2+1)$, it follows from the convolution theorem that

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}(t) &= \sin t * \sin t = \int_0^t \sin(t-v)\sin v\,dv \\ &= \frac{1}{2}\int_0^t [\cos(2v-t) - \cos t]dv^{\dagger} \\ &= \frac{1}{2}\left[\frac{\sin(2v-t)}{2}\right]_0^t - \frac{1}{2}t\cos t \\ &= \frac{1}{2}\left[\frac{\sin t}{2} - \frac{\sin(-t)}{2}\right] - \frac{1}{2}t\cos t \\ &= \frac{\sin t - t\cos t}{2} . \end{aligned}$$ ◆

† Here we used the identity $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\beta-\alpha) - \cos(\beta+\alpha)]$.

As the preceding example attests, the convolution theorem is useful in determining the inverse transforms of rational functions of $s$. In fact, it provides an alternative to the method of partial fractions. For example,

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}(t) = \mathscr{L}^{-1}\left\{\left(\frac{1}{s-a}\right)\left(\frac{1}{s-b}\right)\right\}(t) = e^{at} * e^{bt},$$

and all that remains in finding the inverse is to compute the convolution $e^{at} * e^{bt}$.

In the early 1900s, V. Volterra introduced **integro-differential** equations in his study of population growth. These equations enabled him to take into account "hereditary influences." In certain cases, these equations involved a convolution. As the next example shows, the convolution theorem helps to solve such integro-differential equations.

**Example 3** Solve the integro-differential equation

$$\textbf{(12)} \qquad y'(t) = 1 - \int_0^t y(t-v)e^{-2v}\,dv\,, \qquad y(0) = 1\,.$$

**Solution** Equation (12) can be written as

$$\textbf{(13)} \qquad y'(t) = 1 - y(t) * e^{-2t}\,.$$

Let $Y(s) = \mathscr{L}\{y\}(s)$. Taking the Laplace transform of (13) (with the help of the convolution theorem) and solving for $Y(s)$, we obtain

$$sY(s) - 1 = \frac{1}{s} - Y(s)\left(\frac{1}{s+2}\right)$$

$$sY(s) + \left(\frac{1}{s+2}\right)Y(s) = 1 + \frac{1}{s}$$

$$\left(\frac{s^2+2s+1}{s+2}\right)Y(s) = \frac{s+1}{s}$$

$$Y(s) = \frac{(s+1)(s+2)}{s(s+1)^2} = \frac{s+2}{s(s+1)}$$

$$Y(s) = \frac{2}{s} - \frac{1}{s+1}\,.$$

Hence, $y(t) = 2 - e^{-t}$. ◆

The **transfer function** $H(s)$ of a linear system is defined as the ratio of the Laplace transform of the output function $y(t)$ to the Laplace transform of the input function $g(t)$, under the assumption that all initial conditions are zero. That is, $H(s) = Y(s)/G(s)$. If the linear system is governed by the differential equation

$$\textbf{(14)} \qquad ay'' + by' + cy = g(t)\,, \qquad t > 0\,,$$

where $a$, $b$, and $c$ are constants, we can compute the transfer function as follows. Take the Laplace transform of both sides of (14) to get

$$as^2Y(s) - asy(0) - ay'(0) + bsY(s) - by(0) + cY(s) = G(s)\,.$$

Because the initial conditions are assumed to be zero, the equation reduces to

$$(as^2 + bs + c)Y(s) = G(s)\,.$$

Thus the transfer function for equation (14) is

**(15)** $$H(s) = \frac{Y(s)}{G(s)} = \frac{1}{as^2 + bs + c}\,.$$

You may note the similarity of these calculations to those for finding the auxiliary equation for the homogeneous equation associated with (14) (recall Section 4.2, page 157). Indeed, the first step in inverting $Y(s) = G(s)/(as^2 + bs + c)$ would be to find the roots of the denominator $as^2 + bs + c$, which is identical to solving the characteristic equation for (14).

The function $h(t) \coloneqq \mathscr{L}^{-1}\{H\}(t)$ is called the **impulse response function** for the system because, physically speaking, it describes the solution when a mass–spring system is struck by a hammer (see Section 7.9). We can also characterize $h(t)$ as the unique solution to the homogeneous problem

**(16)** $$ah'' + bh' + ch = 0\,; \qquad h(0) = 0\,, \qquad h'(0) = 1/a\,.$$

Indeed, observe that taking the Laplace transform of the equation in (16) gives

**(17)** $$a[s^2H(s) - sh(0) - h'(0)] + b[sH(s) - h(0)] + cH(s) = 0\,.$$

Substituting in $h(0) = 0$ and $h'(0) = 1/a$ and solving for $H(s)$ yields

$$H(s) = \frac{1}{as^2 + bs + c}\,,$$

which is the same as the formula for the transfer function given in equation (15).

One nice feature of the impulse response function $h$ is that it can help us describe the solution to the *general* initial value problem

**(18)** $$ay'' + by' + cy = g(t)\,; \qquad y(0) = y_0\,, \qquad y'(0) = y_1\,.$$

From the discussion of equation (14), we can see that the convolution $h * g$ is the solution to (18) in the special case when the initial conditions are zero (i.e., $y_0 = y_1 = 0$). To deal with nonzero initial conditions, let $y_k$ denote the solution to the corresponding *homogeneous* initial value problem; that is, $y_k$ solves

**(19)** $$ay'' + by' + cy = 0\,; \qquad y(0) = y_0\,, \qquad y'(0) = y_1\,.$$

Then, the desired solution to the general initial value problem (18) must be $h * g + y_k$. Indeed, it follows from the superposition principle (see Theorem 3 in Section 4.5) that since $h * g$ is a solution to equation (14) and $y_k$ is a solution to the corresponding homogeneous equation, then $h * g + y_k$ is a solution to equation (14). Moreover, since $h * g$ has initial conditions zero,

$$(h * g)(0) + y_k(0) = 0 + y_0 = y_0\,,$$
$$(h * g)'(0) + y_k'(0) = 0 + y_1 = y_1\,.$$

We summarize these observations in the following theorem.

## Solution Using Impulse Response Function

**Theorem 12.** Let $I$ be an interval containing the origin. The unique solution to the initial value problem

$$ay'' + by' + cy = g\,; \qquad y(0) = y_0\,, \qquad y'(0) = y_1\,,$$

where $a$, $b$, and $c$ are constants and $g$ is continuous on $I$, is given by

**(20)** $$y(t) = (h * g)(t) + y_k(t) = \int_0^t h(t-v)g(v)dv + y_k(t)\,,$$

where $h$ is the impulse response function for the system and $y_k$ is the unique solution to (19).

Equation (20) is instructive in that it highlights how the value of $y$ at time $t$ depends on the initial conditions (through $y_k(t)$) and on the nonhomogeneity $g(t)$ (through the convolution integral). It even displays the *causal* nature of the dependence, in that the value of $g(v)$ cannot influence $y(t)$ until $t \ge v$.

A proof of Theorem 12 that does not involve Laplace transforms is outlined in Project E in Chapter 4.

In the next example, we use Theorem 12 to find a formula for the solution to an initial value problem.

**Example 4** A linear system is governed by the differential equation

**(21)** $$y'' + 2y' + 5y = g(t)\,; \qquad y(0) = 2\,, \qquad y'(0) = -2\,.$$

Find the transfer function for the system, the impulse response function, and a formula for the solution.

**Solution** According to formula (15), the transfer function for (21) is

$$H(s) = \frac{1}{as^2 + bs + c} = \frac{1}{s^2 + 2s + 5} = \frac{1}{(s+1)^2 + 2^2}\,.$$

The inverse Laplace transform of $H(s)$ is the impulse response function

$$h(t) = \mathscr{L}^{-1}\{H\}(t) = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s+1)^2 + 2^2}\right\}(t)$$
$$= \frac{1}{2}e^{-t}\sin 2t\,.$$

To solve the initial value problem, we need the solution to the corresponding homogeneous problem. The auxiliary equation for the homogeneous equation is $r^2 + 2r + 5 = 0$, which has roots $r = -1 \pm 2i$. Thus a general solution is $C_1e^{-t}\cos 2t + C_2e^{-t}\sin 2t$. Choosing $C_1$ and $C_2$ so that the initial conditions in (21) are satisfied, we obtain $y_k(t) = 2e^{-t}\cos 2t$.

Hence, a formula for the solution to the initial value problem (21) is

$$(h * g)(t) + y_k(t) = \frac{1}{2}\int_0^t e^{-(t-v)}\sin[2(t-v)]g(v)\,dv + 2e^{-t}\cos 2t\,. \ \blacklozenge$$

## 7.8 EXERCISES

*In Problems 1–4, use the convolution theorem to obtain a formula for the solution to the given initial value problem, where $g(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order.*

**1.** $y'' - 2y' + y = g(t)$; $\quad y(0) = -1$, $\quad y'(0) = 1$

**2.** $y'' + 9y = g(t)$; $\quad y(0) = 1$, $\quad y'(0) = 0$

**3.** $y'' + 4y' + 5y = g(t)$; $\quad y(0) = 1$, $\quad y'(0) = 1$

**4.** $y'' + y = g(t)$; $\quad y(0) = 0$, $\quad y'(0) = 1$

*In Problems 5–12, use the convolution theorem to find the inverse Laplace transform of the given function.*

**5.** $\dfrac{1}{s(s^2+1)}$

**6.** $\dfrac{1}{(s+1)(s+2)}$

**7.** $\dfrac{14}{(s+2)(s-5)}$

**8.** $\dfrac{1}{(s^2+4)^2}$

**9.** $\dfrac{s}{(s^2+1)^2}$

**10.** $\dfrac{1}{s^3(s^2+1)}$

**11.** $\dfrac{s}{(s-1)(s+2)}$ $\left[\textit{Hint: } \dfrac{s}{s-1} = 1 + \dfrac{1}{s-1}.\right]$

**12.** $\dfrac{s+1}{(s^2+1)^2}$

**13.** Find the Laplace transform of

$$f(t) := \int_0^t (t-v)e^{3v}\,dv .$$

**14.** Find the Laplace transform of

$$f(t) := \int_0^t e^{v}\sin(t-v)\;dv .$$

*In Problems 15–22, solve the given integral equation or integro-differential equation for $y(t)$.*

**15.** $y(t) + 3\displaystyle\int_0^t y(v)\sin(t-v)\,dv = t$

**16.** $y(t) + \displaystyle\int_0^t e^{t-v}y(v)\,dv = \sin t$

**17.** $y(t) + \displaystyle\int_0^t (t-v)y(v)\,dv = 1$

**18.** $y(t) + \displaystyle\int_0^t (t-v)y(v)\,dv = t^2$

**19.** $y(t) + \displaystyle\int_0^t (t-v)^2y(v)\,dv = t^3 + 3$

**20.** $y'(t) + \displaystyle\int_0^t (t-v)y(v)\,dv = t$, $\quad y(0) = 0$

**21.** $y'(t) + y(t) - \displaystyle\int_0^t y(v)\sin(t-v)\,dv = -\sin t$, $\quad y(0) = 1$

**22.** $y'(t) - 2\displaystyle\int_0^t e^{t-v}y(v)\,dv = t$, $\quad y(0) = 2$

*In Problems 23–28, a linear system is governed by the given initial value problem. Find the transfer function $H(s)$ for the system and the impulse response function $h(t)$ and give a formula for the solution to the initial value problem.*

**23.** $y'' + 9y = g(t)$; $\quad y(0) = 2$, $\quad y'(0) = -3$

**24.** $y'' - 9y = g(t)$; $\quad y(0) = 2$, $\quad y'(0) = 0$

**25.** $y'' - y' - 6y = g(t)$; $\quad y(0) = 1$, $\quad y'(0) = 8$

**26.** $y'' + 2y' - 15y = g(t)$; $\quad y(0) = 0$, $\quad y'(0) = 8$

**27.** $y'' - 2y' + 5y = g(t)$; $\quad y(0) = 0$, $\quad y'(0) = 2$

**28.** $y'' - 4y' + 5y = g(t)$; $\quad y(0) = 0$, $\quad y'(0) = 1$

*In Problems 29 and 30, the current $I(t)$ in an RLC circuit with voltage source $E(t)$ is governed by the initial value problem*

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = e(t),$$
$$I(0) = a, \qquad I'(0) = b,$$

*where $e(t) = E'(t)$ (see Figure 7.25). For the given constants R, L, C, a, and b, find a formula for the solution $I(t)$ in terms of $e(t)$.*

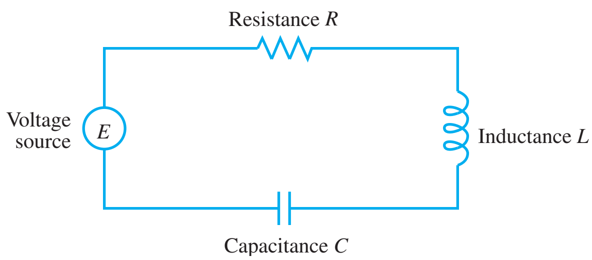



**Figure 7.25** Schematic representation of an *RLC* series circuit

**29.** $R = 20\ \Omega$, $L = 5$ H, $C = 0.005$ F, $a = -1$ A, $b = 8$ A/sec.

**30.** $R = 80\ \Omega$, $L = 10$ H, $C = 1/410$ F, $a = 2$ A, $b = -8$ A/sec.

**31.** Use the convolution theorem and Laplace transforms to compute $1 * 1 * 1$.

**32.** Use the convolution theorem and Laplace transforms to compute $1 * t * t^2$.

**33.** Prove property (5) in Theorem 10.

**34.** Prove property (6) in Theorem 10.

**35.** Use the convolution theorem to show that

$$\mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\}(t) = \int_0^t f(v)\,dv\,,$$

where $F(s) = \mathscr{L}\{f\}(s)$.

**36.** Using Theorem 5 in Section 7.3 and the convolution theorem, show that

$$\int_0^t \int_0^v f(z)dz dv = \mathscr{L}^{-1}\left\{\frac{F(s)}{s^2}\right\}(t)$$
$$= t\int_0^t f(v)dv - \int_0^t vf(v)dv\,,$$

where $F(s) = \mathscr{L}\{f\}(s)$.

**37.** Prove directly that if $h(t)$ is the impulse response function characterized by equation (16), then for any continuous $g(t)$, we have $(h * g)(0) = (h * g)'(0) = 0$. [*Hint:* Use Leibniz's rule, described in Project E of Chapter 4.]

# 7.9 Impulses and the Dirac Delta Function

In mechanical systems, electrical circuits, bending of beams, and other applications, one encounters functions that have a very large value over a very short interval. For example, the strike of a hammer exerts a relatively large force over a relatively short time, and a heavy weight concentrated at a spot on a suspended beam exerts a large force over a very small section of the beam. To deal with violent forces of short duration, physicists and engineers use the delta function introduced by Paul A. M. Dirac. Relaxing our standards of rigor for the moment, we present the following somewhat informal definition.



**Dirac Delta Function**

**Definition 10.** The **Dirac delta function** $\delta(t)$ is characterized by the following two properties:

**(1)** $$\delta(t) = \begin{cases} 0\,, & t \neq 0\,, \\ \text{“infinite,”} & t = 0\,, \end{cases}$$

and

**(2)** $$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

for any function $f(t)$ that is continuous on an open interval containing $t = 0$.

By shifting the argument of $\delta(t)$, we have $\delta(t-a) = 0$, $t \neq a$, and

**(3)** $$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

for any function $f(t)$ that is continuous on an interval containing $t = a$.

It is obvious that $\delta(t-a)$ is not a function in the usual sense; instead it is an example of what is called a **generalized function** or a **distribution.** Despite this shortcoming, the Dirac

## A TABLE OF LAPLACE TRANSFORMS

| $f(t)$ | $F(s)=\mathscr{L}\{f\}(s)$ | $f(t)$ | $F(s)=\mathscr{L}\{f\}(s)$ |
|---|---|---|---|
| **1.** $f(at)$ | $\frac{1}{a}F\left(\frac{s}{a}\right)$ | **20.** $\frac{1}{\sqrt{t}}$ | $\frac{\sqrt{\pi}}{\sqrt{s}}$ |
| **2.** $e^{at}f(t)$ | $F(s-a)$ | **21.** $\sqrt{t}$ | $\frac{\sqrt{\pi}}{2s^{3/2}}$ |
| **3.** $f'(t)$ | $sF(s)-f(0)$ | **22.** $t^{n-(1/2)}, \quad n=1,2,\ldots$ | $\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+(1/2)}}$ |
| **4.** $f^{(n)}(t)$ | $s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0) -\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$ | **23.** $t^r, \quad r>-1$ | $\frac{\Gamma(r+1)}{s^{r+1}}$ |
| | | **24.** $\sin bt$ | $\frac{b}{s^2+b^2}$ |
| **5.** $t^nf(t)$ | $(-1)^nF^{(n)}(s)$ | **25.** $\cos bt$ | $\frac{s}{s^2+b^2}$ |
| **6.** $\frac{1}{t}f(t)$ | $\int_s^\infty F(u)du$ | **26.** $e^{at}\sin bt$ | $\frac{b}{(s-a)^2+b^2}$ |
| **7.** $\int_0^t f(v)dv$ | $\frac{F(s)}{s}$ | **27.** $e^{at}\cos bt$ | $\frac{s-a}{(s-a)^2+b^2}$ |
| **8.** $(f*g)(t)$ | $F(s)G(s)$ | **28.** $\sinh bt$ | $\frac{b}{s^2-b^2}$ |
| **9.** $f(t+T)=f(t)$ | $\frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}$ | **29.** $\cosh bt$ | $\frac{s}{s^2-b^2}$ |
| **10.** $f(t-a)u(t-a),\ a\geq 0$ | $e^{-as}F(s)$ | **30.** $\sin bt - bt\cos bt$ | $\frac{2b^3}{(s^2+b^2)^2}$ |
| **11.** $g(t)u(t-a), \quad a\geq 0$ | $e^{-as}\mathscr{L}\{g(t+a)\}(s)$ | **31.** $t\sin bt$ | $\frac{2bs}{(s^2+b^2)^2}$ |
| **12.** $u(t-a), \quad a\geq 0$ | $\frac{e^{-as}}{s}$ | **32.** $\sin bt + bt\cos bt$ | $\frac{2bs^2}{(s^2+b^2)^2}$ |
| **13.** $\prod_{a,b}(t), \quad 0<a<b$ | $\frac{e^{-sa}-e^{-sb}}{s}$ | **33.** $t\cos bt$ | $\frac{s^2-b^2}{(s^2+b^2)^2}$ |
| **14.** $\delta(t-a), \quad a\geq 0$ | $e^{-as}$ | **34.** $\sin bt\cosh bt - \cos bt\sinh bt$ | $\frac{4b^3}{s^4+4b^4}$ |
| **15.** $e^{at}$ | $\frac{1}{s-a}$ | **35.** $\sin bt\sinh bt$ | $\frac{2b^2s}{s^4+4b^4}$ |
| **16.** $t^n, \quad n=1,2,\ldots$ | $\frac{n!}{s^{n+1}}$ | **36.** $\sinh bt - \sin bt$ | $\frac{2b^3}{s^4-b^4}$ |
| **17.** $e^{at}t^n, \quad n=1,2,\ldots$ | $\frac{n!}{(s-a)^{n+1}}$ | **37.** $\cosh bt - \cos bt$ | $\frac{2b^2s}{s^4-b^4}$ |
| **18.** $e^{at}-e^{bt}$ | $\frac{(a-b)}{(s-a)(s-b)}$ | **38.** $J_\nu(bt),\ \nu>-1$ | $\frac{\left(\sqrt{s^2+b^2}-s\right)^\nu}{b^\nu\sqrt{s^2+b^2}}$ |
| **19.** $ae^{at}-be^{bt}$ | $\frac{(a-b)s}{(s-a)(s-b)}$ | | |

# CHAPTER 7: Laplace Transforms

## EXERCISES 7.2: Definition of the Laplace Transform, page 359

**1.** For $s > 0$, using Definition 1 on page 351 and integration by parts, we compute

$$\begin{aligned}\mathcal{L}\{t\}(s) &= \int_0^\infty e^{-st}t\,dt = \lim_{N\to\infty}\int_0^N e^{-st}t\,dt = \lim_{N\to\infty}\int_0^N t\,d\left(-\frac{e^{-st}}{s}\right)\\ &= \lim_{N\to\infty}\left[-\frac{te^{-st}}{s}\Big|_0^N + \frac{1}{s}\int_0^N e^{-st}\,dt\right] = \lim_{N\to\infty}\left[-\frac{te^{-st}}{s}\Big|_0^N - \frac{e^{-st}}{s^2}\Big|_0^N\right]\\ &= \lim_{N\to\infty}\left[-\frac{Ne^{-sN}}{s} + 0 - \frac{e^{-sN}}{s^2} + \frac{1}{s^2}\right] = \frac{1}{s^2}\end{aligned}$$

because, for $s > 0$, $e^{-sN} \to 0$ and $Ne^{-sN} = N/e^{sN} \to 0$ as $N \to \infty$.

**3.** For $s > 6$, we have

$$\begin{aligned}\mathcal{L}\{t\}(s) &= \int_0^\infty e^{-st}e^{6t}\,dt = \int_0^\infty e^{(6-s)t}\,dt = \lim_{N\to\infty}\int_0^N e^{(6-s)t}\,dt\\ &= \lim_{N\to\infty}\left[\frac{e^{(6-s)t}}{6-s}\Big|_0^N\right] = \lim_{N\to\infty}\left[\frac{e^{(6-s)N}}{6-s} - \frac{1}{6-s}\right] = 0 - \frac{1}{6-s} = \frac{1}{s-6}\,.\end{aligned}$$

**5.** For $s > 0$,

$$\begin{aligned}\mathcal{L}\{\cos 2t\}(s) &= \int_0^\infty e^{-st}\cos 2t\,dt = \lim_{N\to\infty}\int_0^N e^{-st}\cos 2t\,dt\\ &= \lim_{N\to\infty}\left[\frac{e^{-st}(-s\cos 2t + 2\sin 2t)}{s^2+4}\Big|_0^N\right]\\ &= \lim_{N\to\infty}\left[\frac{e^{-sN}(-s\cos 2N + 2\sin 2N)}{s^2+4} - \frac{-s}{s^2+4}\right] = \frac{s}{s^2+4}\,,\end{aligned}$$

where we have used integration by parts to find an antiderivative of $e^{-st}\cos 2t$.

# Chapter 7

**7.** For $s > 2$,

$$\begin{aligned}\mathcal{L}\left\{e^{2t}\cos 3t\right\}(s) &= \int_0^\infty e^{-st}e^{2t}\cos 3t\,dt = \int_0^\infty e^{(2-s)t}\cos 3t\,dt \\ &= \lim_{N\to\infty}\left[\frac{e^{(2-s)t}\left((2-s)\cos 3t + 3\sin 3t\right)}{(2-s)^2+9}\Bigg|_0^N\right] \\ &= \lim_{N\to\infty}\frac{e^{(2-s)N}\left[(2-s)\cos 3N + 3\sin 3N\right]-(2-s)}{(2-s)^2+9} = \frac{s-2}{(s-2)^2+9}\,.\end{aligned}$$

**9.** As in Example 4 on page 353 in the text, we first break the integral into separate parts. Thus,

$$\mathcal{L}\left\{f(t)\right\}(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^2 e^{-st}\cdot 0\,dt + \int_2^\infty te^{-st}\,dt = \int_2^\infty te^{-st}\,dt\,.$$

An antiderivative of $te^{-st}$ was, in fact, obtained in Problem 1 using integration by parts. Thus, we have

$$\begin{aligned}\int_2^\infty te^{-st}\,dt &= \lim_{N\to\infty}\left[\left(-\frac{te^{-st}}{s}-\frac{e^{-st}}{s^2}\right)\Bigg|_2^N\right] = \lim_{N\to\infty}\left[-\frac{Ne^{-sN}}{s}-\frac{e^{-sN}}{s^2}+\frac{2e^{-2s}}{s}+\frac{e^{-2s}}{s^2}\right] \\ &= \frac{2e^{-2s}}{s}+\frac{e^{-2s}}{s^2} = e^{-2s}\left(\frac{2}{s}+\frac{1}{s^2}\right) = e^{-2s}\left(\frac{2s+1}{s^2}\right).\end{aligned}$$

**11.** In this problem, $f(t)$ is also a piecewise defined function. So, we split the integral and obtain

$$\begin{aligned}\mathcal{L}\left\{f(t)\right\}(s) &= \int_0^\infty e^{-st}f(t)\,dt = \int_0^\pi e^{-st}\sin t\,dt + \int_\pi^\infty e^{-st}\cdot 0\,dt = \int_0^\pi e^{-st}\sin t\,dt \\ &= \frac{e^{-st}\left(-s\sin t - \cos t\right)}{s^2+1}\Bigg|_0^\pi = \frac{e^{-\pi s}-(-1)}{s^2+1} = \frac{e^{-\pi s}+1}{s^2+1}\,,\end{aligned}$$

which is valid for all $s$.

**13.** By the linearity of the Laplace transform,

$$\mathcal{L}\left\{6e^{-3t}-t^2+2t-8\right\}(s) = 6\mathcal{L}\left\{e^{-3t}\right\}(s) - \mathcal{L}\left\{t^2\right\}(s) + 2\mathcal{L}\left\{t\right\}(s) - 8\mathcal{L}\left\{1\right\}(s).$$

From Table 7.1 on page 358 in the text, we see that

$$\mathcal{L}\left\{e^{-3t}\right\}(s) = \frac{1}{s-(-3)} = \frac{1}{s+3}\,, \qquad s > -3;$$

$$\mathcal{L}\left\{t^2\right\}(s)=\frac{2!}{s^{2+1}}=\frac{2}{s^3}\,,\quad \mathcal{L}\left\{t\right\}(s)=\frac{1!}{s^{1+1}}=\frac{1}{s^2}\,,\quad \mathcal{L}\left\{1\right\}(s)=\frac{1}{s}\,,\qquad s>0.$$

Thus the formula

$$\mathcal{L}\left\{6e^{-3t}-t^2+2t-8\right\}(s)=6\frac{1}{s+3}-\frac{2}{s^3}+2\frac{1}{s^2}-8\frac{1}{s}=\frac{6}{s+3}-\frac{2}{s^3}+\frac{2}{s^2}-\frac{8}{s}\,,$$

is valid for $s$ in the intersection of the sets $s>-3$ and $s>0$, which is $s>0$.

**15.** Using the linearity of Laplace transform and Table 7.1 on page 358 in the text, we get

$$\begin{aligned}\mathcal{L}\left\{t^3-te^t+e^{4t}\cos t\right\}(s) &= \mathcal{L}\left\{t^3\right\}(s)-\mathcal{L}\left\{te^t\right\}(s)+\mathcal{L}\left\{e^{4t}\cos t\right\}(s)\\ &= \frac{3!}{s^{3+1}}-\frac{1!}{(s-1)^{1+1}}+\frac{s-4}{(s-4)^2+1^2}\\ &= \frac{6}{s^4}-\frac{1}{(s-1)^2}+\frac{s-4}{(s-4)^2+1}\,,\end{aligned}$$

which is valid for $s>4$.

**17.** Using the linearity of Laplace transform and Table 7.1 on page 358 in the text, we get

$$\begin{aligned}\mathcal{L}\left\{e^{3t}\sin 6t-t^3+e^t\right\}(s) &= \mathcal{L}\left\{e^{3t}\sin 6t\right\}(s)-\mathcal{L}\left\{t^3\right\}(s)+\mathcal{L}\left\{e^t\right\}(s)\\ &= \frac{6}{(s-3)^2+6^2}-\frac{3!}{s^{3+1}}+\frac{1}{s-1}=\frac{6}{(s-3)^2+36}-\frac{6}{s^4}+\frac{1}{s-1}\,,\end{aligned}$$

valid for $s>3$.

**19.** For $s>5$, we have

$$\begin{aligned}\mathcal{L}\left\{t^4e^{5t}-e^t\cos\sqrt{7}t\right\}(s) &= \mathcal{L}\left\{t^4e^{5t}\right\}(s)-\mathcal{L}\left\{e^t\cos\sqrt{7}t\right\}(s)\\ &= \frac{4!}{(s-5)^{4+1}}-\frac{s-1}{(s-1)^2+(\sqrt{7})^2}=\frac{24}{(s-5)^5}-\frac{s-1}{(s-1)^2+7}\,.\end{aligned}$$

**21.** Since the function $g_1(t)\equiv 1$ is continuous on $(-\infty,\infty)$ and $f(t)=g_1(t)$ for $t$ in $[0,1]$, we conclude that $f(t)$ is continuous on $[0,1)$ and continuous from the left at $t=1$. The function $g_2(t)\equiv(t-2)^2$ is also continuous on $(-\infty,\infty)$, and so $f(t)$ (which is the same as $g_2(t)$ on $(1,10]$) is continuous on $(1,10]$. Moreover,

$$\lim_{t\to 1^+}f(t)=\lim_{t\to 1^+}g_2(t)=g_2(1)=(1-2)^2=1=f(1),$$

which implies that $f(t)$ is continuous from the right at $t = 1$. Thus $f(t)$ is continuous at $t = 1$ and, therefore, is continuous at any $t$ in $[0, 10]$.

**23.** All the functions involved in the definition of $f(t)$, that is, $g_1(t) \equiv 1$, $g_2(t) = t - 1$, and $g_3(t) = t^2 - 4$, are continuous on $(-\infty, \infty)$. So, $f(t)$, being a restriction of these functions, on $[0, 1)$, $(1, 3)$, and $(3, 10]$, respectively, is continuous on these three intervals. At points $t = 1$ and 3, $f(t)$ is not defined and so is not continuous. But one-sided limits

$$\begin{aligned}
\lim_{t\to 1^-} f(t) &= \lim_{t\to 1^-} g_1(t) = g_1(1) = 1,\\
\lim_{t\to 1^+} f(t) &= \lim_{t\to 1^+} g_2(t) = g_2(1) = 0,\\
\lim_{t\to 3^-} f(t) &= \lim_{t\to 3^-} g_2(t) = g_2(3) = 2,\\
\lim_{t\to 3^+} f(t) &= \lim_{t\to 3^+} g_3(t) = g_3(3) = 5,
\end{aligned}$$



exist and pairwise different. Therefore, $f(t)$ has jump discontinuities at $t = 1$ and $t = 3$, and hence piecewise continuous on $[0, 10]$.

**25.** Given function is a rational function and, therefore, continuous on its domain, which is all reals except zeros of the denominator. Solving $t^2 + 7t + 10 = 0$, we conclude that the points of discontinuity of $f(t)$ are $t = -2$ and $t = -5$. These points are not in $[0, 10]$. So, $f(t)$ is continuous on $[0, 10]$.

**27.** Since

$$\lim_{t\to 0^+} f(t) = \lim_{t\to 0^+} \frac{1}{t} = \infty,$$

$f(t)$ has infinite discontinuity at $t = 0$, and so neither continuous nor piecewise continuous $[0, 10]$.

**29.** **(a)** First observe that $|t^3 \sin t| \le |t^3|$ for all $t$. Next, three applications of L'Hospital's rule show that

$$\lim_{t\to\infty} \frac{t^3}{e^{\alpha t}} = \lim_{t\to\infty} \frac{3t^2}{\alpha e^{\alpha t}} = \lim_{t\to\infty} \frac{6t}{\alpha^2 e^{\alpha t}} = \lim_{t\to\infty} \frac{6}{\alpha^3 e^{\alpha t}} = 0$$

for all $\alpha > 0$. Thus, fixed $\alpha > 0$, for some $T = T(\alpha) > 0$, we have $|t^3| < e^{\alpha t}$ for all $t > T$, and so

$$\left|t^3 \sin t\right| \leq \left|t^3\right| < e^{\alpha t}, \qquad t > T.$$

Therefore, $t^3 \sin t$ is of exponential order $\alpha$, for any $\alpha > 0$.

**(b)** Clearly, for any $t$, $|f(t)| = 100e^{49t}$, and so Definition 3 is satisfied with $M = 100$, $\alpha = 49$, and any $T$. Hence, $f(t)$ is of exponential order 49.

**(c)** Since

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} = \lim_{t\to\infty} e^{t^3-\alpha t} = \lim_{t\to\infty} e^{(t^2-\alpha)t} = \infty,$$

we see that $f(t)$ grows faster than $e^{\alpha t}$ for any $\alpha$. Thus $f(t)$ is *not* of exponential order.

**(d)** Similarly to (a), for any $\alpha > 0$, we get

$$\lim_{t\to\infty} \frac{|t \ln t|}{e^{\alpha t}} = \lim_{t\to\infty} \frac{t \ln t}{e^{\alpha t}} = \lim_{t\to\infty} \frac{\ln t + 1}{\alpha e^{\alpha t}} = \lim_{t\to\infty} \frac{1/t}{\alpha^2 e^{\alpha t}} = 0\,,$$

and so $f(t)$ is of exponential order $\alpha$ for any positive $\alpha$.

**(e)** Since,

$$f(t) = \cosh\left(t^2\right) = \frac{e^{t^2} + e^{-t^2}}{2} > \frac{1}{2}\, e^{t^2}$$

and $e^{t^2}$ grows faster than $e^{\alpha t}$ for any fixed $\alpha$ (see page 357 in the text), we conclude that $\cosh\left(t^2\right)$ is *not* of exponential order.

**(f)** This function is bounded:

$$|f(t)| = \left|\frac{1}{t^2+1}\right| = fr1t^2 + 1 \leq \frac{1}{0+1} = 1,$$

and so Definition 3 is satisfied with $M = 1$ and $\alpha = 0$. Hence, $f(t)$ is of exponential order 0.

**(g)** The function $\sin\left(t^2\right)$ is bounded, namely, $|\sin\left(t^2\right)| \leq 1$. For any fixed $\beta > 0$, the limit of $t^4/e^{\beta t}$, as $t \to \infty$, is 0, which implies that $t^4 \leq e^{\beta t}$ for all $t > T = T(\beta)$. Thus,

$$\left|\sin\left(t^2\right) + t^4 e^{6t}\right| \leq 1 + e^{\beta t} e^{6t} = 2e^{\beta + 6t}\,.$$

This means that $f(t)$ is of exponential order $\alpha$ for any $\alpha > 6$.

**(h)** The function $3 + \cos 4t$ is bounded because

$$|3 + \cos 4t| \leq 3 + |\cos 4t| \leq 4.$$

Therefore, by the triangle inequality,

$$|f(t)| \geq \left|e^{t^2}\right| - |3 + \cos 4t| \geq e^{t^2} - 4,$$

and, therefore, for any fixed $\alpha$, $f(t)$ grows faster than $e^{\alpha t}$ (because $e^{t^2}$ does, and the other term is bounded). So, $f(t)$ is *not* of exponential order.

**(i)** Clearly, for any $t > 0$,

$$\frac{t^2}{t+1} = \frac{t}{t+1}t < (1)t = t.$$

Therefore,

$$e^{t^2/(t+1)} < e^t,$$

and Definition 3 holds with $M = 1$, $\alpha = 1$, and $T = 0$.

**(j)** Since, for any $x$, $-1 \leq \sin x \leq 1$, the given function is bounded. Indeed,

$$\left|\sin\left(e^{t^2}\right) + e^{\sin t}\right| \leq \left|\sin\left(e^{t^2}\right)\right| + e^{\sin t} \leq 1 + e$$

Thus it is of exponential order 0.



**31. (a)**

$$\mathcal{L}\left\{e^{(a+ib)t}\right\}(s) := \int_0^\infty e^{-st}e^{(a+ib)t}\,dt = \int_0^\infty e^{(a+ib-s)t}\,dt = \lim_{N\to\infty}\int_0^N e^{(a+ib-s)t}\,dt$$

$$= \lim_{N\to\infty}\left(\frac{e^{(a+ib-s)t}}{a+ib-s}\Big|_0^N\right) = \frac{1}{a+ib-s}\lim_{N\to\infty}\left(e^{(a-s+ib)N} - 1\right). \quad (7.1)$$

Since

$$e^{(a-s+ib)x} = e^{(a-s)x}e^{ibx},$$

where the first factor vanishes at $\infty$ if $a - s < 0$ while the second factor is a bounded ($\left|e^{ibx}\right| \equiv 1$) and periodic function, the limit in (7.1) exists if and only if $a - s < 0$. Assuming that $s > a$, we get

$$\frac{1}{a+ib-s}\lim_{N\to\infty}\left(e^{(a-s+ib)N} - 1\right) = \frac{1}{a+ib-s}(0-1) = \frac{1}{s-(a+ib)}.$$

**(b)** Note that $s-(a+ib)=(s-a)-ib$. Multiplying the result in (a) by the complex conjugate of the denominator, that is, $(s-a)+bi$, we get

$$\frac{1}{s-(a+ib)}=\frac{(s-a)+ib}{[(s-a)-ib]\cdot[(s-a)+ib]}=\frac{(s-a)+ib}{(s-a)^2+b^2},$$

where we used the fact that, for any complex number $z$, $z\overline{z}=|z|^2$.

**(c)** From (a) and (b) we klnow that

$$\mathcal{L}\left\{e^{(a+ib)t}\right\}(s)=\frac{(s-a)+ib}{(s-a)^2+b^2}.$$

Writing

$$\frac{(s-a)+ib}{(s-a)^2+b^2}=\frac{s-a}{(s-a)^2+b^2}+\frac{b}{(s-a)^2+b^2}i,$$

we see that

$$\operatorname{Re}\left[\mathcal{L}\left\{e^{(a+ib)t}\right\}(s)\right]=\operatorname{Re}\left[\frac{s-a}{(s-a)^2+b^2}+\frac{b}{(s-a)^2+b^2}i\right]=\frac{s-a}{(s-a)^2+b^2}, \quad (7.2)$$

$$\operatorname{Im}\left[\mathcal{L}\left\{e^{(a+ib)t}\right\}(s)\right]=\operatorname{Im}\left[\frac{s-a}{(s-a)^2+b^2}+\frac{b}{(s-a)^2+b^2}i\right]=\frac{b}{(s-a)^2+b^2}. \quad (7.3)$$



On the other hand, by Euler's formulas,

$$\operatorname{Re}\left[e^{-st}e^{(a+ib)t}\right]=e^{-st}\operatorname{Re}\left[e^{at}(\cos bt+i\sin bt)\right]=e^{-st}e^{at}\cos bt$$

and so

$$\begin{aligned}\operatorname{Re}\left[\mathcal{L}\left\{e^{(a+ib)t}\right\}(s)\right] &= \operatorname{Re}\left[\int_0^\infty e^{-st}e^{(a+ib)t}\,dt\right]=\operatorname{Re}\left[\int_0^\infty e^{-s}e^{(a+ib)t}\,dt\right]\\ &= \int_0^\infty \operatorname{Re}\left[e^{-s}e^{(a+ib)t}\right]dt=\int_0^\infty e^{-st}e^{at}\cos bt\,dt=\mathcal{L}\left\{e^{at}\cos bt\right\}(s),\end{aligned}$$

which together with (7.2) gives the last entry in Table 7.1. Similarly,

$$\operatorname{Im}\left[\mathcal{L}\left\{e^{(a+ib)t}\right\}(s)\right]=\mathcal{L}\left\{e^{at}\sin bt\right\}(s),$$

and so (7.3) gives the Laplace transform of $e^{at}\sin bt$.

**33.** Let $f(t)$ be a piecewise continuous function on $[a, b]$, and let a function $g(t)$ be continuous on $[a, b]$. At any point of continuity of $f(t)$, the function $(fg)(t)$ is continuous as the product of two continuous functions at this point. Suppose now that $c$ is a point of discontinuity of $f(t)$. Then one-sided limits

$$\lim_{t\to c^-} f(t) = L_- \qquad \text{and} \qquad \lim_{t\to c^+} f(t) = L_+$$

exist. At the same time, continuity of $g(t)$ yields

$$\lim_{t\to c^-} g(t) = \lim_{t\to c^+} g(t) = \lim_{t\to c} g(t) = g(c).$$

Thus, the product rule implies that one-sided limits

$$\begin{aligned}\lim_{t\to c^-}(fg)(t) &= \lim_{t\to c^-} f(t) \cdot \lim_{t\to c^-} g(t) = L_- g(c)\\ \lim_{t\to c^+}(fg)(t) &= \lim_{t\to c^+} f(t) \lim_{t\to c^+} g(t) = L_+ g(c)\end{aligned}$$

exist. So, $fg$ has a jump (even removable if $g(c) = 0$) discontinuity at $t = c$.

Therefore, the product $(fg)(t)$ is continuous at any point on $[a, b]$ except possibly a finite number of points (namely, points of discontinuity of $f(t)$).



**EXERCISES 7.3: Properties of the Laplace Transform, page 365**

**1.** Using the linearity of the Laplace transform we get

$$\mathcal{L}\left\{t^2 + e^t \sin 2t\right\}(s) = \mathcal{L}\left\{t^2\right\}(s) + \mathcal{L}\left\{e^t \sin 2t\right\}(s).$$

From Table 7.1 in Section 7.2 we know that

$$\mathcal{L}\left\{t^2\right\}(s) = \frac{2!}{s^3} = \frac{2}{s^3}\,, \quad \mathcal{L}\left\{e^t \sin 2t\right\}(s) = \frac{2}{(s-1)^2 + 2^2} = \frac{2}{(s-1)^2 + 4}\,.$$

Thus

$$\mathcal{L}\left\{t^2 + e^t \sin 2t\right\}(s) = \frac{2}{s^3} + \frac{2}{(s-1)^2 + 4}\,.$$

**3.** By the linearity of the Laplace transform,

$$\mathcal{L}\left\{e^{-t}\cos 3t + e^{6t} - 1\right\}(s) = \mathcal{L}\left\{e^{-t}\cos 3t\right\}(s) + \mathcal{L}\left\{e^{6t}\right\}(s) - \mathcal{L}\left\{1\right\}(s).$$

From Table 7.1 of the text we see that

$$\mathcal{L}\left\{e^{-t}\cos 3t\right\}(s) = \frac{s-(-1)}{[s-(-1)]^2+3^2} = \frac{s+1}{(s+1)^2+9}, \quad s > -1; \tag{7.4}$$

$$\mathcal{L}\left\{e^{6t}\right\}(s) = \frac{1}{s-6}, \quad s > 6; \tag{7.5}$$

$$\mathcal{L}\left\{1\right\}(s) = \frac{1}{s}, \quad s > 0. \tag{7.6}$$

Since (7.4), (7.5), and (7.6) all hold for $s > 6$, we see that our answer,

$$\mathcal{L}\left\{e^{-t}\cos 3t + e^{6t} - 1\right\}(s) = \frac{s+1}{(s+1)^2+9} + \frac{1}{s-6} - \frac{1}{s},$$

is valid for $s > 6$. Note that (7.4) and (7.5) could also be obtained from the Laplace transforms for $\cos 3t$ and 1, respectively, by applying the translation Theorem 3.



**5.** We use the linearity of the Laplace transform and Table 7.1 to get

$$\begin{aligned}\mathcal{L}\left\{2t^2e^{-t} - t + \cos 4t\right\}(s) &= 2\mathcal{L}\left\{t^2e^{-t}\right\}(s) - \mathcal{L}\left\{t\right\}(s) + \mathcal{L}\left\{\cos 4t\right\}(s) \\ &= 2\cdot\frac{2}{(s+1)^3} - \frac{1}{s^2} + \frac{s}{s^2+4^2} = \cdot\frac{4}{(s+1)^3} - \frac{1}{s^2} + \frac{s}{s^2+16},\end{aligned}$$

which is valid for $s > 0$.

**7.** Since $(t-1)^4 = t^4 - 4t^3 + 6t^2 - 4t + 1$, we have from the linearity of the Laplace transform that

$$\mathcal{L}\left\{(t-1)^4\right\}(s) = \mathcal{L}\left\{t^4\right\}(s) - 4\mathcal{L}\left\{t^3\right\}(s) + 6\mathcal{L}\left\{t^2\right\}(s) - 4\mathcal{L}\left\{t\right\}(s) + \mathcal{L}\left\{1\right\}(s).$$

From Table 7.1 of the text, we get that, for $s > 0$,

$$\mathcal{L}\left\{t^4\right\}(s) = \frac{4!}{s^5} = \frac{24}{s^5},$$

$$\mathcal{L}\left\{t^3\right\}(s) = \frac{3!}{s^4} = \frac{6}{s^4},$$

$$\mathcal{L}\left\{t^2\right\}(s)=\frac{2!}{s^3}=\frac{2}{s^3}\,,$$
$$\mathcal{L}\left\{t\right\}(s)=\frac{1!}{s^2}=\frac{1}{s^2}\,,$$
$$\mathcal{L}\left\{1\right\}(s)=\frac{1}{s}\,.$$

Thus

$$\mathcal{L}\left\{(t-1)^4\right\}(s)=\frac{24}{s^5}-\frac{24}{s^4}+\frac{12}{s^3}-\frac{4}{s^2}+\frac{1}{s}\,,\qquad s>0.$$

**9.** Since

$$\mathcal{L}\left\{e^{-t}\sin 2t\right\}(s)=\frac{2}{(s+1)^2+4}\,,$$

we use Theorem 6 to get

$$\begin{aligned}\mathcal{L}\left\{e^{-t}t\sin 2t\right\}(s) &= \mathcal{L}\left\{t\left(e^{-t}\sin 2t\right)\right\}(s)=-\left[\mathcal{L}\left\{e^{-t}\sin 2t\right\}(s)\right]'=-\left[\frac{2}{(s+1)^2+4}\right]' \\ &= -2(-1)\left[(s+1)^2+4\right]^{-2}\left[(s+1)^2+4\right]'=\frac{4(s+1)}{\left[(s+1)^2+4\right]^2}\,.\end{aligned}$$

**11.** We use the definition of $\cosh x$ and the linear property of the Laplace transform.

$$\begin{aligned}\mathcal{L}\left\{\cosh bt\right\}(s) &= \mathcal{L}\left\{\frac{e^{bt}+e^{-bt}}{2}\right\}(s) \\ &= \frac{1}{2}\left[\mathcal{L}\left\{e^{bt}\right\}(s)+\mathcal{L}\left\{e^{-bt}\right\}(s)\right]=\frac{1}{2}\left[\frac{1}{s-b}+\frac{1}{s+b}\right]=\frac{s}{s^2-b^2}\,.\end{aligned}$$

**13.** In this problem, we need the trigonometric identity $\sin^2 t=(1-\cos 2t)/2$ and the linearity of the Laplace transform.

$$\begin{aligned}\mathcal{L}\left\{\sin^2 t\right\}(s) &= \mathcal{L}\left\{\frac{1-\cos 2t}{2}\right\}(s) \\ &= \frac{1}{2}\left[\mathcal{L}\left\{1\right\}(s)-\mathcal{L}\left\{\cos 2t\right\}(s)\right]=\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]=\frac{2}{s(s^2+4)}\,.\end{aligned}$$

**15.** From the trigonometric identity $\cos^2 t=(1+\cos 2t)/2$, we find that

$$\cos^3 t=\cos t\cos^2 t=\frac{1}{2}\,\cos t+\frac{1}{2}\,\cos t\cos 2t\,.$$

Next we write

$$\cos t \cos 2t = \frac{1}{2}[\cos(2t+t) + \cos(2t-t)] = \frac{1}{2}\cos 3t + \frac{1}{2}\cos t.$$

Thus,

$$\cos^3 t = \frac{1}{2}\cos t + \frac{1}{4}\cos 3t + \frac{1}{4}\cos t = \frac{3}{4}\cos t + \frac{1}{4}\cos 3t.$$

We now use the linearity of the Laplace transform and Table 7.1 to find that

$$\mathcal{L}\left\{\cos^3 t\right\}(s) = \frac{3}{4}\mathcal{L}\{\cos t\}(s) + \frac{1}{4}\mathcal{L}\{\cos 3t\}(s) = \frac{3}{4}\frac{s}{s^2+1} + \frac{1}{4}\frac{s}{s^2+9},$$

which holds for $s > 0$.

**17.** Since $\sin A \sin B = [\cos(A-B) - \cos(A+B)]/2$, we get

$$\begin{aligned}\mathcal{L}\{\sin 2t \sin 5t\}(s) &= \mathcal{L}\left\{\frac{\cos 3t - \cos 7t}{2}\right\}(s) = \frac{1}{2}[\mathcal{L}\{\cos 3t\}(s) - \mathcal{L}\{\cos 7t\}(s)] \\ &= \frac{1}{2}\left[\frac{s}{s^2+9} - \frac{s}{s^2+49}\right] = \frac{20s}{(s^2+9)(s^2+49)}.\end{aligned}$$

**19.** Since $\sin A \cos B = [\sin(A+B) + \sin(A-B)]/2$, we get

$$\begin{aligned}\mathcal{L}\{\cos nt \sin mt\}(s) &= \mathcal{L}\left\{\frac{\sin[(m+n)t] + \sin[(m-n)t]}{2}\right\}(s) \\ &= \frac{1}{2}\frac{m+n}{s^2+(m+n)^2} + \frac{1}{2}\frac{m-n}{s^2+(m-n)^2}.\end{aligned}$$

**21.** By the translation property of the Laplace transform (Theorem 3),

$$\mathcal{L}\left\{e^{at}\cos bt\right\}(s) = \mathcal{L}\{\cos bt\}(s-a) = \left.\frac{u}{u^2+b^2}\right|_{u=s-a} = \frac{s-a}{(s-a)^2+b^2}.$$

**23.** Clearly,

$$(t \sin bt)' = (t)' \sin bt + t(\sin bt)' = \sin bt + bt \cos bt.$$

Therefore, using Theorem 4 and the entry 30, that is, $\mathcal{L}\{t \sin bt\}(s) = (2bs)/[(s^2+b^2)^2]$, we obtain

$$\begin{aligned}\mathcal{L}\{\sin bt + bt\cos bt\}(s) &= \mathcal{L}\{(t\sin bt)'\}(s) = s\mathcal{L}\{t \sin bt\}(s) - (t \sin bt)\big|_{t=0} \\ &= \frac{s(2bs)}{(s^2+b^2)^2} - 0 = \frac{2bs^2}{(s^2+b^2)^2}.\end{aligned}$$

**25.** **(a)** By property (6) on page 363 of the text,

$$\mathcal{L}\left\{t\cos bt\right\}(s) = -\left[\mathcal{L}\left\{\cos bt\right\}(s)\right]' = -\left[\frac{s}{s^2+b^2}\right]' = \frac{s^2-b^2}{(s^2+b^2)^2}\,, \qquad s>0.$$

**(b)** Again using the same property, we get

$$\begin{aligned}\mathcal{L}\left\{t^2\cos bt\right\}(s) &= \mathcal{L}\left\{t(t\cos bt)\right\}(s) = -\left[\mathcal{L}\left\{t\cos bt\right\}(s)\right]' \\ &= -\left[\frac{s^2-b^2}{(s^2+b^2)^2}\right]' = \frac{2s^3-6sb^2}{(s^2+b^2)^3}\,, \qquad s>0.\end{aligned}$$

**27.** First observe that since $f(t)$ is piecewise continuous on $[0,\infty)$ and $f(t)/t$ approaches a finite limit as $t\to 0^+$, we conclude that $f(t)/t$ is also piecewise continuous on $[0,\infty)$. Next, since for $t\geq 1$ we have $|f(t)/t|\leq |f(t)|$, we see that $f(t)/t$ is of exponential order $\alpha$ since $f(t)$ is. These observations and Theorem 2 on page 357 of the text show that $\mathcal{L}\{f(t)/t\}$ exists. When the results of Problem 26 are applied to $f(t)/t$, we see that

$$\lim_{N\to\infty}\mathcal{L}\left\{\frac{f(t)}{t}\right\}(N) = 0.$$

By Theorem 6, we have that

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{te^{-st}f(t)}{t}\,dt = -\frac{d}{ds}\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s)\,.$$

Thus,

$$\begin{aligned}\int_s^\infty F(u)\,du &= \int_s^\infty\left[-\frac{d}{du}\mathcal{L}\left\{\frac{f(t)}{t}\right\}(u)\right]du = \int_\infty^s \frac{d}{du}\mathcal{L}\left\{\frac{f(t)}{t}\right\}(u)\,du \\ &= \mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) - \lim_{N\to\infty}\mathcal{L}\left\{\frac{f(t)}{t}\right\}(N) = \mathcal{L}\left\{\frac{f(t)}{t}\right\}(s)\,.\end{aligned}$$

**29.** From the linearity properties (2) and (3) on page 354 of the text we have

$$\mathcal{L}\left\{g(t)\right\}(s) = \mathcal{L}\left\{y''(t)+6y'(t)+10y(t)\right\}(s) = \mathcal{L}\left\{y''(t)\right\}(s)+6\mathcal{L}\left\{y'(t)\right\}(s)+10\mathcal{L}\left\{y(t)\right\}(s).$$

Next, applying properties (2) and (4) on pages 361 and 362 yields

$$\mathcal{L}\left\{g\right\}(s) = \left[s^2\mathcal{L}\left\{y\right\}(s)-sy(0)-y'(0)\right]+6\left[s\mathcal{L}\left\{y\right\}(s)-y(0)\right]+10\mathcal{L}\left\{y\right\}(s).$$

Keeping in mind the fact that all initial conditions are zero the above becomes

$$G(s)=\left(s^2+6s+10\right)Y(s), \qquad \text{where} \qquad Y(s)=\mathcal{L}\{y\}(s).$$

Therefore, the transfer function $H(s)$ is given by

$$H(s)=\frac{Y(s)}{G(s)}=\frac{1}{s^2+6s+10}\,.$$

**31.** Using Definition 1 of the Laplace transform in Section 7.2, we obtain

$$\begin{aligned}\mathcal{L}\{g(t)\}(s) &= \int_0^\infty e^{-st}g(t)\,dt=\int_0^c(0)\,dt+\int_c^\infty e^{-st}f(t-c)\,dt=\left(\,t-c\to u,\ dt\to du\,\right)\\ &= \int_0^\infty e^{-s(u+c)}f(u)\,du=e^{-cs}\int_0^\infty e^{-su}f(u)\,du=e^{-cs}\mathcal{L}\{f(t)\}(s).\end{aligned}$$

**33.** The graphs of the function $f(t)=t$ and its translation $g(t)$ to the right by $c=1$ are shown in Figure 7-A(a).

We use the result of Problem 31 to find $\mathcal{L}\{g(t)\}$.

$$\mathcal{L}\{g(t)\}(s)=e^{-(1)s}\mathcal{L}\{t\}(s)=\frac{e^{-s}}{s^2}\,.$$



**35.** The graphs of the function $f(t)=\sin t$ and its translation $g(t)$ to the right by $c=\pi/2$ units are shown in Figure 7-A(b).

We use the formula in Problem 31 to find $\mathcal{L}\{g(t)\}$.

$$\mathcal{L}\{g(t)\}(s)=e^{-(\pi/2)s}\mathcal{L}\{\sin t\}(s)=\frac{e^{-(\pi/2)s}}{s^2+1}\,.$$

**37.** Since $f'(t)$ is of exponential order on $[0,\infty)$, for some $\alpha$, $M>0$, and $T>0$,

$$|f'(t)|\le Me^{\alpha t}, \qquad \text{for all} \quad t\ge T. \tag{7.7}$$

On the other hand, piecewise continuity of $f'(t)$ on $[0,\infty)$ implies that $f'(t)$ is bounded on any finite interval, in particular, on $[0,T]$. That is,

$$|f'(t)|\le C, \qquad \text{for all} \quad t \text{ in } [0,T]. \tag{7.8}$$

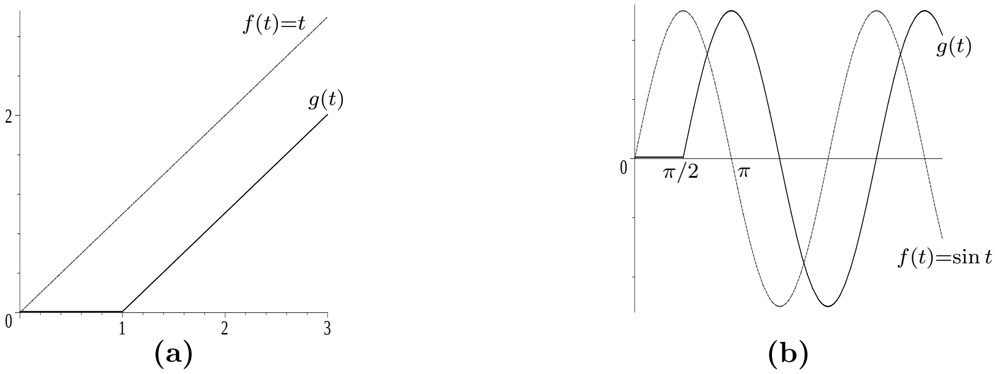



**Figure 7–A**: Graphs of functions in Problems 33 and 35.

From (7.7) and (7.8) it follows that, for $s > \alpha$,

$$\begin{aligned}\int_0^\infty e^{-st}|f'(t)|\,dt &= \int_0^T e^{-st}|f'(t)|\,dt + \int_T^\infty e^{-st}|f'(t)|\,dt \le C\int_0^T e^{-st}\,dt + M\int_T^\infty e^{-st}e^{\alpha t}\,dt \\ &= \left.\frac{Ce^{-st}}{-s}\right|_0^T + \lim_{N\to\infty}\left[\left.\frac{Me^{(\alpha-s)t}}{\alpha - s}\right|_T^N\right] = \frac{C\left[1-e^{-sT}\right]}{s} + \frac{Me^{(\alpha-s)T}}{s-\alpha} \longrightarrow 0\end{aligned}$$

as $s \to \infty$. Therefore, (7) yields

$$0 \le |s\mathcal{L}\{f\}(s) - f(0)| = \left|\int_0^\infty e^{-st}f'(t)\,dt\right| \le \int_0^\infty e^{-st}|f'(t)|\,dt \longrightarrow 0 \quad \text{as} \quad s \to \infty.$$

Hence, by the squeeze theorem,

$$\lim_{s\to\infty}|s\mathcal{L}\{f\}(s) - f(0)| = 0 \quad\Leftrightarrow\quad \lim_{s\to\infty}[s\mathcal{L}\{f\}(s) - f(0)] = 0 \quad\Leftrightarrow\quad \lim_{s\to\infty}s\mathcal{L}\{f\}(s) = f(0).$$

## EXERCISES 7.4: Inverse Laplace Transform, page 374

**1.** From Table 7.1, the function $6/(s-1)^4 = (3!)/(s-1)^4$ is the Laplace transform of $e^{\alpha t}t^n$ with $\alpha = 1$ and $n = 3$. Therefore,

$$\mathcal{L}^{-1}\left\{\frac{6}{(s-1)^4}\right\}(t) = e^t t^3\,.$$

**3.** Writing

$$\frac{s+1}{s^2+2s+10} = \frac{s+1}{(s^2+2s+1)+9} = \frac{s+1}{(s+1)^2+3^2},$$

we see that this function is the Laplace transform of $e^{-t}\cos 3t$ (the last entry in Table 7.1 with $\alpha = -1$ and $b = 3$). Hence

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+10}\right\}(t) = e^{-t}\cos 3t\,.$$

**5.** We complete the square in the denominator and use the linearity of the inverse Laplace transform to get

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+8}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+2^2}\right\}(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+2^2}\right\}(t) = \frac{1}{2}e^{-2t}\sin 2t.$$

(See the Laplace transform formula for $e^{\alpha t}\sin bt$ in Table 7.1).



**7.** By completing the square in the denominator, we can rewrite $(2s+16)/(s^2+4s+13)$ as

$$\frac{2s+16}{s^2+4s+4+9} = \frac{2s+16}{(s+2)^2+3^2} = \frac{2(s+2)}{(s+2)^2+3^2} + \frac{4(3)}{(s+2)^2+3^2}\,.$$

Thus, by the linearity of the inverse Laplace transform,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2s+16}{s^2+4s+13}\right\}(t) &= 2\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\}(t) + 4\mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\}(t)\\ &= 2e^{-2t}\cos 3t + 4e^{-2t}\sin 3t\,.\end{aligned}$$

**9.** We complete the square in the denominator, rewrite the given function as a sum of two entries in Table 7.1, and use the linearity of the inverse Laplace transform. This yields

$$\frac{3s-15}{2s^2-4s+10} = \frac{3}{2}\cdot\frac{s-5}{s^2-2s+5} = \frac{3}{2}\cdot\frac{(s-1)-4}{(s-1)^2+2^2} = \frac{(3/2)(s-1)}{(s-1)^2+2^2} - \frac{3(2)}{(s-1)^2+2^2}$$

$$\begin{aligned}\Rightarrow\quad \mathcal{L}^{-1}\left\{\frac{3s-15}{2s^2-4s+10}\right\} &= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}\\ &= \frac{3}{2}e^{t}\cos 2t - 3e^{t}\sin 2t.\end{aligned}$$

**11.** In this problem, we use the partial fractions decomposition method. Since the denominator, $(s-1)(s+2)(s+5)$, is a product of three nonrepeated linear factors, the expansion has the form

$$\begin{aligned}\frac{s^2-26s-47}{(s-1)(s+2)(s+5)} &= \frac{A}{s-1}+\frac{B}{s+2}+\frac{C}{s+5}\\ &= \frac{A(s+2)(s+5)+B(s-1)(s+5)+C(s-1)(s+2)}{(s-1)(s+2)(s+5)}.\end{aligned}$$

Therefore,

$$s^2-26s-47 = A(s+2)(s+5)+B(s-1)(s+5)+C(s-1)(s+2). \qquad (7.9)$$

Evaluating both sides of (7.9) for $s=1$, $s=-2$, and $s=-5$, we find constants $A$, $B$, and $C$.

$$\begin{aligned}&s=1: && (1)^2-26(1)-47 = A(1+2)(1+5) && \Rightarrow && A=-4,\\ &s=-2: && (-2)^2-26(-2)-47 = B(-2-1)(-2+5) && \Rightarrow && B=-1,\\ &s=-5: && (-5)^2-26(-5)-47 = C(-5-1)(-5+2) && \Rightarrow && C=6.\end{aligned}$$

Hence,

$$\frac{s^2-26s-47}{(s-1)(s+2)(s+5)} = \frac{6}{s+5}-\frac{1}{s+2}-\frac{4}{s-1}.$$



**13.** The denominator has a simple linear factor, $s$, and a double linear factor, $s+1$. Thus, we look for the decomposition of the form

$$\frac{-2s^2-3s-2}{s(s+1)^2} = \frac{A}{s}+\frac{B}{s+1}+\frac{C}{(s+1)^2} = \frac{A(s+1)^2+Bs(s+1)+Cs}{s(s+1)^2},$$

which yields

$$-2s^2-3s-2 = A(s+1)^2+Bs(s+1)+Cs. \qquad (7.10)$$

Evaluating this equality for $s=0$ and $s=-1$, we find $A$ and $C$, respectively.

$$\begin{aligned}&s=0: && -2 = A(0+1)^2 && \Rightarrow && A=-2,\\ &s=-1: && -2(-1)^2-3(-1)-2 = C(-1) && \Rightarrow && C=1.\end{aligned}$$

To find $B$, we compare the coefficients at $s^2$ in both sides of (7.10).

$$-2 = A+B \qquad \Rightarrow \qquad B=-2-A=0.$$

Hence,

$$\frac{-2s^2-3s-2}{s(s+1)^2} = \frac{1}{(s+1)^2} - \frac{2}{s}.$$

**15.** First, we complete the square in the quadratic $s^2-2s+5$ to make sure that this polynomial is irreducible and to find the form of the decomposition. Since

$$s^2-2s+5 = (s^2-2s+1)+4 = (s-1)^2+2^2,$$

we have

$$\frac{-8s-2s^2-14}{(s+1)(s^2-2s+5)} = \frac{A}{s+1} + \frac{B(s-1)+C(2)}{(s-1)^2+2^2} = \frac{A\left[(s-1)^2+4\right]+\left[B(s-1)+2C\right](s+1)}{(s+1)\left[(s-1)^2+4\right]}$$

which implies that

$$-8s-2s^2-14 = A\left[(s-1)^2+4\right] + \left[B(s-1)+2C\right](s+1).$$

Taking $s=-1$, $s=1$, and $s=0$, we find $A$, $B$, and $C$, respectively.

$$\begin{array}{lll} s=-1: & 8(-1)-2(-1)^2-14 = A\left[(-1-1)^2+4\right] & \Rightarrow \quad A=-3, \\ s=1: & 8(1)-2(1)^2-14 = A\left[(1-1)^2+4\right]+2C(1+1) & \Rightarrow \quad C=1, \\ s=0: & 8(0)-2(0)^2-14 = A\left[(0-1)^2+4\right]+\left[B(0-1)+2C\right](0+1) & \Rightarrow \quad B=1, \end{array}$$

and so

$$\frac{-8s-2s^2-14}{(s+1)(s^2-2s+5)} = -\frac{3}{s+1} + \frac{(s-1)+2}{(s-1)^2+4}$$



**17.** First we need to completely factor the denominator. Since $s^2+s-6=(s-2)(s+3)$, we have

$$\frac{3s+5}{s(s^2+s-6)} = \frac{3s+5}{s(s-2)(s+3)}.$$

Since the denominator has only nonrepeated linear factors, we can write

$$\frac{3s+5}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

for some choice of $A$, $B$ and $C$. Clearing fractions gives us

$$3s+5 = A(s-2)(s+3) + Bs(s+3) + Cs(s-2).$$

With $s = 0$, this yields $5 = A(-2)(3)$ so that $A = -5/6$. With $s = 2$, we get $11 = B(2)(5)$ so that $B = 11/10$. Finally, $s = -3$ yields $-4 = C(-3)(-5)$ so that $C = -4/15$. Thus,

$$\frac{3s+5}{s(s^2+s-6)} = -\frac{5}{6s} + \frac{11}{10(s-2)} - \frac{4}{15(s+3)}.$$

**19.** First observe that the quadratic polynomial $s^2+2s+2$ is irreducible because the discriminant $2^2 - 4(1)(2) = -4$ is negative. Since the denominator has one nonrepeated linear factor and one nonrepeated quadratic factor, we can write

$$\frac{1}{(s-3)(s^2+2s+2)} = \frac{1}{(s-3)[(s+1)^2+1]} = \frac{A}{s-3} + \frac{B(s+1)+C}{(s+1)^2+1},$$

where we have chosen a form which is more convenient for taking the inverse Laplace transform. Clearing fractions gives us

$$1 = A\left[(s+1)^2+1\right] + [B(s+1)+C](s-3). \qquad (7.11)$$

With $s = 3$, this yields $1 = 17A$ so that $A = 1/17$. Substituting $s = -1$, we see that $1 = A(1)+C(-4)$, or $C = (A-1)/4 = -4/17$. Finally, the coefficient $A+B$ at $s^2$ in the right-hand side of (7.11) must be the same as in the left-hand side, that is, 0. So $B = -A = -1/17$ and

$$\frac{1}{(s-3)(s^2+2s+2)} = \frac{1}{17}\left[\frac{1}{s-3} - \frac{s+1}{(s+1)^2+1} - \frac{4}{(s+1)^2+1}\right].$$

**21.** Since the denominator contains only nonrepeated linear factors, the partial fractions decomposition has the form

$$\frac{6s^2-13s+2}{s(s-1)(s-6)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-6} = \frac{A(s-1)(s-6)+Bs(s-6)+Cs(s-1)}{s(s-1)(s-6)}.$$

Therefore,

$$6s^2-13s+2 = A(s-1)(s-6)+Bs(s-6)+Cs(s-1).$$

Evaluating both sides of this equation for $s = 0$, $s = 1$, and $s = 6$, we find constants $A$, $B$, and $C$.

$$\begin{aligned} s=0: &\quad 2 = 6A &\Rightarrow&\quad A = 1/3,\\ s=1: &\quad -5 = -5B &\Rightarrow&\quad B = 1,\\ s=6: &\quad 140 = 30C &\Rightarrow&\quad C = 14/3. \end{aligned}$$

Hence,

$$\frac{6s^2-13s+2}{s(s-1)(s-6)}=\frac{1/3}{s}+\frac{1}{s-1}+\frac{14/3}{s-6}$$

and the linear property of the inverse Laplace transform yields

$$\mathcal{L}^{-1}\left\{\frac{6s^2-13s+2}{s(s-1)(s-6)}\right\}=\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{14}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}=\frac{1}{3}+e^{t}+\frac{14}{3}e^{6t}.$$

**23.** In this problem, the denominator of $F(s)$ has a simple linear factor, $s+1$, and a double linear factor, $s+3$. Thus, the decomposition is the form

$$\frac{5s^2+34s+53}{(s+3)^2(s+1)}=\frac{A}{(s+3)^2}+\frac{B}{s+3}+\frac{C}{s+1}=\frac{A(s+1)+B(s+1)(s+3)+C(s+3)^2}{(s+3)^2(s+1)}.$$

Therefore, we must have

$$5s^2+34s+53=A(s+1)+B(s+1)(s+3)+C(s+3)^2.$$



Substitutions $s=-3$ and $s=-1$ yield values of $A$ and $C$, respectively.

$$\begin{aligned} s=-3:&\quad -4=-2A \quad\Rightarrow\quad A=2,\\ s=-1:&\quad 24=4C \quad\Rightarrow\quad C=6.\end{aligned}$$

To find $B$, we take, say, $s=0$ and get

$$53=A+3B+9C \qquad\Rightarrow\qquad B=\frac{53-A-9C}{3}=-1.$$

Hence,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{5s^2+34s+53}{(s+3)^2(s+1)}\right\}(t) &= 2\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\}(t)-\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}(t)+6\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)\\ &= 2te^{-3t}-e^{-3t}+6e^{-t}.\end{aligned}$$

**25.** Observing that the quadratic $s^2+2s+5=(s+1)^2+2^2$ is irreducible, the partial fractions decomposition for $F(s)$ has the form

$$\frac{7s^2+23s+30}{(s-2)(s^2+2s+5)}=\frac{A}{s-2}+\frac{B(s+1)+C(2)}{(s+1)^2+2^2}.$$

Clearing fractions gives us

$$7s^2 + 23s + 30 = A\left[(s+1)^2 + 4\right] + \left[B(s+1) + C(2)\right](s-2).$$

With $s = 2$, this yields $104 = 13A$ so that $A = 8$; $s = -1$ gives $14 = A(4) + C(-6)$, or $C = 3$. Finally, the coefficient $A + B$ at $s^2$ in the right-hand side must match the one in the left-hand side, which is 7. So $B = 7 - A = -1$. Therefore,

$$\frac{7s^2 + 23s + 30}{(s-2)(s^2 + 2s + 5)} = \frac{8}{s-2} + \frac{-(s+1) + 3(2)}{(s+1)^2 + 2^2},$$

which yields

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{7s^2 + 23s + 30}{(s-2)(s^2 + 2s + 5)}\right\} &= 8\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2 + 2^2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2 + 2^2}\right\} \\ &= 8e^{2t} - e^{-t}\cos 2t + 3e^{-t}\sin 2t\,.\end{aligned}$$

**27.** First, we find $F(s)$.

$$F(s)\left(s^2 - 4\right) = \frac{5}{s+1} \qquad \Rightarrow \qquad F(s) = \frac{5}{(s+1)(s^2 - 4)} = \frac{5}{(s+1)(s-2)(s+2)}\,.$$

The partial fractions expansion yields

$$\frac{5}{(s+1)(s-2)(s+2)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s+2}\,.$$

Clearing fractions gives us

$$5 = A(s-2)(s+2) + B(s+1)(s+2) + C(s+1)(s-2).$$

With $s = -1$, $s = 2$, and $s = -2$ this yields $A = -5/3$, $B = 5/12$, and $C = 5/4$. So,

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\}(t) &= -\frac{5}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) + \frac{5}{12}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t) + \frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) \\ &= -\frac{5}{3}e^{-t} + \frac{5}{12}e^{2t} + \frac{5}{4}e^{-2t}\,.\end{aligned}$$

**29.** Solving for $F(s)$ yields

$$F(s) = \frac{10s^2 + 12s + 14}{(s+2)(s^2 - 2s + 2)} = \frac{10s^2 + 12s + 14}{(s+2)[(s-1)^2 + 1]}\,.$$

Since, in the denominator, we have nonrepeated linear and quadratic factors, we seek for the decomposition

$$\frac{10s^2+12s+14}{(s+2)[(s-1)^2+1]} = \frac{A}{s+2} + \frac{B(s-1)+C(1)}{(s-1)^2+1}.$$

Clearing fractions, we conclude that

$$10s^2+12s+14 = A[(s-1)^2+1] + [B(s-1)+C](s+2).$$

Substitution $s=-2$ into this equation yields $30 = 10A$ or $A=3$. With $s=1$, we get $36 = A+3C$ and so $C=(36-A)/3 = 11$. Finally, substitution $s=0$ results $14 = 2A+2(C-B)$ or $B = A+C-7 = 7$. Now we apply the linearity of the inverse Laplace transform and obtain

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\}(t) &= 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) + 7\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\}(t) + 11\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}(t)\\ &= 3e^{-2t} + 7e^t\cos t + 11e^t\sin t.\end{aligned}$$

**31.** Functions $f_1(t)$, $f_2(t)$, and $f_3(t)$ coincide for all $t$ in $[0,\infty)$ except a finite number of points. Since the Laplace transform a function is a definite integral, it does not depend on values of the function at finite number of points. Therefore, in (a), (b), and (c) we have one and the same answer, that is

$$\mathcal{L}\{f_1(t)\}(s) = \mathcal{L}\{f_2(t)\}(s) = \mathcal{L}\{f_3(t)\}(s) = \mathcal{L}\{t\}(s) = \frac{1}{s^2}.$$

By Definition 4, the inverse Laplace transform is a continuous function on $[0,\infty)$. $f_3(t)=t$ clearly satisfies this condition while $f_1(t)$ and $f_2(t)$ have removable discontinuities at $t=2$ and $t=1,6$, respectively. Therefore,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) = f_3(t) = t.$$

**33.** We are looking for $\mathcal{L}^{-1}\{F(s)\}(t) = f(t)$. According to the formula given just before this problem,

$$f(t) = \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{dF}{ds}\right\}(t)$$

(take $n = 1$ in the formula). Since

$$F(s) = \ln\left(\frac{s+2}{s-5}\right) = \ln(s+2) - \ln(s-5),$$

we have

$$\begin{aligned} \frac{dF(s)}{ds} &= \frac{d}{ds}\left(\ln(s+2) - \ln(s-5)\right) = \frac{1}{s+2} - \frac{1}{s-5} \\ \Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{dF}{ds}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s-5}\right\}(t) = e^{-2t} - e^{5t} \\ \Rightarrow \quad \mathcal{L}^{-1}\{F(s)\}(t) &= \frac{-1}{t}\left(e^{-2t} - e^{5t}\right) = \frac{e^{5t} - e^{-2t}}{t}. \end{aligned}$$

**35.** Taking the derivative of $F(s)$, we get

$$\frac{dF(s)}{ds} = \frac{d}{ds}\ln\frac{s^2+9}{s^2+1} = \frac{d}{ds}\left[\ln(s^2+9) - \ln(s^2+1)\right] = \frac{2s}{s^2+9} - \frac{2s}{s^2+1}.$$

So, using the linear property of the inverse Laplace transform, we obtain

$$\mathcal{L}^{-1}\left\{\frac{dF(s)}{ds}\right\}(t) = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}(t) - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) = 2(\cos 3t - \cos t).$$

Thus

$$\mathcal{L}^{-1}\{F(s)\}(t) = \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{dF(s)}{ds}\right\}(t) = \frac{2(\cos t - \cos 3t)}{t}.$$

**37.** By the definition, both, $\mathcal{L}^{-1}\{F_1\}(t)$ and $\mathcal{L}^{-1}\{F_2\}(t)$, are continuous functions on $[0,\infty)$. Therefore, their sum, $\left(\mathcal{L}^{-1}\{F_1\} + \mathcal{L}^{-1}\{F_2\}\right)(t)$, is also continuous on $[0,\infty)$. Furthermore, the linearity of the Laplace transform yields

$$\mathcal{L}\left\{\left(\mathcal{L}^{-1}\{F_1\} + \mathcal{L}^{-1}\{F_2\}\right)\right\}(s) = \mathcal{L}\left\{\mathcal{L}^{-1}\{F_1\}\right\}(s) + \mathcal{L}\left\{\mathcal{L}^{-1}\{F_2\}\right\}(s) = F_1(s) + F_2(s).$$

Therefore, $\mathcal{L}^{-1}\{F_1\} + \mathcal{L}^{-1}\{F_2\}$ is a continuous function on $[0,\infty)$ whose Laplace transform is $F_1 + F_2$. By the definition of the inverse Laplace transform, this function is the inverse Laplace transform of $F_1 + F_2$, that is,

$$\mathcal{L}^{-1}\{F_1\}(t) + \mathcal{L}^{-1}\{F_2\}(t) = \mathcal{L}^{-1}\{F_1 + F_2\}(t),$$

and (3) in Theorem 7 is proved.

To show (4), we use the continuity of $\mathcal{L}^{-1}\{F\}$ to conclude that $c\mathcal{L}^{-1}\{F\}$ is a continuous function. Since the linearity of the Laplace transform yields

$$\mathcal{L}\left\{c\mathcal{L}^{-1}\{F\}\right\}(s) = c\mathcal{L}\left\{\mathcal{L}^{-1}\{F\}\right\}(s) = cF(s),$$

we have $c\mathcal{L}^{-1}\{F\}(t) = \mathcal{L}^{-1}\{cF\}(t)$.

**39.** In this problem, the denominator $Q(s) := s(s-1)(s+2)$ has only nonrepeated linear factors, and so the partial fractions decomposition has the form

$$F(s) := \frac{2s+1}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}.$$

To find $A$, $B$, and $C$, we use the residue formula in Problem 38. This yields

$$A = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{2s+1}{(s-1)(s+2)} = \frac{2(0)+1}{(0-1)(0+2)} = -\frac{1}{2},$$
$$B = \lim_{s\to 1}(s-1)F(s) = \lim_{s\to 1} \frac{2s+1}{s(s+2)} = \frac{2(1)+1}{(1)(1+2)} = 1,$$
$$C = \lim_{s\to -2}(s+2)F(s) = \lim_{s\to 2} \frac{2s+1}{s(s-1)} = \frac{2(-2)+1}{(-2)(-2-1)} = -\frac{1}{2}.$$

Therefore,

$$\frac{2s+1}{s(s-1)(s+2)} = -\frac{1/2}{s} + \frac{1}{s-1} - \frac{1/2}{s+2}.$$

**41.** In notation of Problem 40,

$$P(s) = 3s^2 - 16s + 5, \qquad Q(s) = (s+1)(s-3)(s-2).$$

We can apply the Heaviside's expansion formula because $Q(s)$ has only nonrepeated linear factors. We need the values of $P(s)$ and $Q'(s)$ at the points $r_1 = -1$, $r_2 = 3$, and $r_3 = 2$. Using the product rule, we find that

$$Q'(s) = (s-3)(s-2) + (s+1)(s-2) + (s+1)(s-3),$$

and so

$$Q'(-1) = (-1-3)(-1-2) = 12, \quad Q'(3) = (3+1)(3-2) = 4, \quad Q'(2) = (2+1)(2-3) = -3.$$

Also, we compute

$$P(-1) = 24, \quad P(3) = -16, \quad P(2) = -15.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{3s^2 - 16s + 5}{(s+1)(s-3)(s-2)}\right\}(t) = \frac{P(-1)}{Q'(-1)}\,e^{(-1)t} + \frac{P(3)}{Q'(3)}\,e^{(3)t} + \frac{P(2)}{Q'(2)}\,e^{(2)t} = 2e^{-t} - 4e^{3t} + 5e^{2t}\,.$$

**43.** Since $s^2 - 2s + 5 = (s-1)^2 + 2^2$, we see that the denominator of $F(s)$ has nonrepeated linear factor $s + 2$ and nonrepeated irreducible quadratic factor $s^2 - 2s + 5$ with $\alpha = 1$ and $\beta = 2$ (in notation of Problem 40). Thus the partial fractions decomposition has the form

$$F(s) = \frac{6s^2 + 28}{(s^2 - 2s + 5)(s+2)} = \frac{A(s-1) + 2B}{(s-1)^2 + 2^2} + \frac{C}{s+2}\,.$$

We find $C$ by applying the real residue formula derived in Problem 38.

$$C = \lim_{s \to -2} \frac{(s+2)(6s^2 + 28)}{(s^2 - 2s + 5)(s+2)} = \lim_{s \to -2} \frac{6s^2 + 28}{s^2 - 2s + 5} = \frac{52}{13} = 4.$$

Next, we use the complex residue formula given in Problem 42, to find $A$ and $B$. Since $\alpha = 1$ and $\beta = 2$, the formula becomes

$$2B + i2A = \lim_{s \to 1+2i} \frac{(s^2 - 2s + 5)(6s^2 + 28)}{(s^2 - 2s + 5)(s+2)} = \lim_{s \to 1+2i} \frac{6s^2 + 28}{s+2} = \frac{6(1+2i)^2 + 28}{(1+2i) + 2} = \frac{10 + 24i}{3 + 2i}\,.$$

Dividing we get

$$2B + i2A = \frac{(10 + 24i)(3 - 2i)}{(3 + 2i)(3 - 2i)} = \frac{78 + 52i}{13} = 6 + 4i.$$

Taking the real and imaginary parts yields

$$\begin{aligned} 2B &= 6, & & & B &= 3, \\ 2A &= 4 & &\Rightarrow & A &= 2. \end{aligned}$$

Therefore,

$$\frac{6s^2 + 28}{(s^2 - 2s + 5)(s+2)} = \frac{2(s-1) + 2(3)}{(s-1)^2 + 2^2} + \frac{4}{s+2}\,.$$

**EXERCISES 7.5: Solving Initial Value Problems, page 383**

**1.** Let $Y(s) := \mathcal{L}\{y\}(s)$. Taking the Laplace transform of both sides of the given differential equation and using its linearity, we obtain

$$\mathcal{L}\{y''\}(s) - 2\mathcal{L}\{y'\}(s) + 5Y(s) = \mathcal{L}\{0\}(s) = 0. \tag{7.12}$$

We can express $\mathcal{L}\{y''\}(s)$ and $\mathcal{L}\{y'\}(s)$ in terms of $Y(s)$ using the initial conditions and Theorem 5 in Section 7.3.

$$\begin{aligned} \mathcal{L}\{y'\}(s) &= sY(s) - y(0) = sY(s) - 2, \\ \mathcal{L}\{y''\}(s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 4. \end{aligned}$$

Substituting back into (7.12) and solving for $Y(s)$ yield

$$\begin{aligned} &\left[s^2Y(s) - 2s - 4\right] - 2\left[sY(s) - 2\right] + 5Y(s) = 0 \\ \Rightarrow \quad & Y(s)\left(s^2 - 2s + 5\right) = 2s \\ \Rightarrow \quad & Y(s) = \frac{2s}{s^2 - 2s + 5} = \frac{2s}{(s-1)^2 + 2^2} = \frac{2(s-1)}{(s-1)^2 + 2^2} + \frac{2}{(s-1)^2 + 2^2}. \end{aligned}$$

Applying now the inverse Laplace transform to both sides, we obtain

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2 + 2^2}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2 + 2^2}\right\}(t) = 2e^t\cos 2t + e^t \sin 2t.$$

**3.** Let $Y(s) := \mathcal{L}\{y\}(s)$. Taking the Laplace transform of both sides of the given differential equation, $y'' + 6y' + 9y = 0$, and using the linearity of the Laplace transform, we obtain

$$\mathcal{L}\{y''\}(s) + 6\mathcal{L}\{y'\}(s) + 9Y(s) = 0.$$

We use formula (4), page 362, to express $\mathcal{L}\{y''\}(s)$ and $\mathcal{L}\{y'\}(s)$ in terms of $Y(s)$.

$$\begin{aligned} \mathcal{L}\{y'\}(s) &= sY(s) - y(0) = sY(s) + 1, \\ \mathcal{L}\{y''\}(s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) + s - 6. \end{aligned}$$

Therefore,

$$\left[s^2Y(s) + s - 6\right] + 6\left[sY(s) + 1\right] + 9Y(s) = 0$$

$$\Rightarrow \qquad Y(s)\left(s^2+6s+9\right) = -s$$
$$\Rightarrow \qquad Y(s) = \frac{-s}{s^2+6s+9} = \frac{-s}{(s+3)^2} = \frac{3}{(s+3)^2} - \frac{1}{s+3},$$

where the last equality comes from the partial fraction expansion of $-s/(s+3^2)$. We apply the inverse Laplace transform to both sides and use Table 7.1 to obtain

$$y(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}(t) = 3te^{-3t} - e^{-3t}.$$

**5.** Let $W(s) = \mathcal{L}\{w\}(s)$. Then taking the Laplace transform of the equation and using linearity yield

$$\mathcal{L}\{w''\}(s) + W(s) = \mathcal{L}\left\{t^2+2\right\}(s) = \mathcal{L}\left\{t^2\right\}(s) + 2\mathcal{L}\{1\}(s) = \frac{2}{s^3} + \frac{2}{s}.$$

Since $\mathcal{L}\{w''\}(s) = s^2W(s) - sw(0) - w'(0) = s^2W(s) - s + 1$, we have

$$\left[s^2W(s) - s + 1\right] + W(s) = \frac{2}{s^3} + \frac{2}{s}$$
$$\Rightarrow \qquad \left(s^2+1\right)W(s) = s - 1 + \frac{2(s^2+1)}{s^3} \qquad \Rightarrow \qquad W(s) = \frac{s}{s^2+1} - \frac{1}{s^2+1} + \frac{2}{s^3}.$$

Now, taking the inverse Laplace transform, we obtain

$$w = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \cos t - \sin t + t^2.$$

**7.** Let $Y(s) := \mathcal{L}\{y\}(s)$. Using the initial conditions and Theorem 5 in Section 7.3 we can express $\mathcal{L}\{y''\}(s)$ and $\mathcal{L}\{y'\}(s)$ in terms of $Y(s)$, namely,

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 5,$$
$$\mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 5s + 4.$$

Taking the Laplace transform of both sides of the given differential equation and using its linearity, we obtain

$$\mathcal{L}\{y'' - 7y' + 10y\}(s) = \mathcal{L}\{9\cos t + 7\sin t\}(s)$$
$$\Rightarrow \qquad \left[s^2Y(s) - 5s + 4\right] - 7\left[sY(s) - 5\right] + 10Y(s) = \frac{9s}{s^2+1} + \frac{7}{s^2+1}$$

$$\Rightarrow \qquad \left(s^2-7s+10\right)Y(s)=\frac{9s+7}{s^2+1}+5s-39=\frac{5s^3-39s^2+14s-32}{s^2+1}$$

$$\Rightarrow \qquad Y(s)=\frac{9s+7}{s^2+1}+5s-39=\frac{5s^3-39s^2+14s-32}{(s^2+1)(s^2-7s+10)}=\frac{5s^3-39s^2+14s-32}{(s^2+1)(s-5)(s-2)}\,.$$

The partial fractions decomposition of $Y(s)$ has the form

$$\frac{5s^3-39s^2+14s-32}{(s^2+1)(s-5)(s-2)}=\frac{As+B}{s^2+1}+\frac{C}{s-5}+\frac{D}{s-2}\,.$$

Clearing fractions yields

$$5s^3-39s^2+14s-32=(As+B)(s-5)(s-2)+C(s^2+1)(s-2)+D(s^2+1)(s-5).$$

We substitute $s=5$ and $s=2$ to find $C$ and $D$, resprectively, and then $s=0$ to find $B$.

$$\begin{aligned}
s=5: &\quad -312=78C && \Rightarrow && C=-4,\\
s=2: &\quad -120=-15D && \Rightarrow && D=8,\\
s=0: &\quad -32=10B-2C-5D && \Rightarrow && B=0.
\end{aligned}$$

Equating the coefficients at $s^3$, we also get $A+C+D=5$, which implies that $A=1$. Thus

$$Y(s)=\frac{s}{s^2+1}-\frac{4}{s-5}+\frac{8}{s-2} \qquad \Rightarrow \qquad y(t)=\mathcal{L}^{-1}\left\{Y(s)\right\}(t)=\cos t-4e^{5t}+8e^{2t}\,.$$

**9.** First, note that the initial conditions are given at $t=1$. Thus, to use the method of Laplace transform, we make a shift in $t$ and move the initial conditions to $t=0$.

$$\begin{aligned}
&z''(t)+5z'(t)-6z(t)=21e^{t-1}\\
&\Rightarrow \qquad z''(t+1)+5z'(t+1)-6z(t+1)=21e^{(t+1)-1}=21e^{t}. \qquad (7.13)
\end{aligned}$$

Now, let $y(t):=z(t+1)$. Then the chain rule yields

$$\begin{aligned}
y'(t)&=z'(t+1)(t+1)'=z'(t+1),\\
y''(t)&=[y'(t)]'=z''(t+1)(t+1)'=z''(t+1),
\end{aligned}$$

and (7.13) becomes

$$y''(t)+5y'(t)-6y(t)=21e^{t} \qquad (7.14)$$

with initial conditions

$$y(0) = z(0+1) = z(1) = -1, \qquad y'(0) = z'(0+1) = z'(1) = 9.$$

With $Y(s) := \mathcal{L}\{y(t)\}(s)$, we apply the Laplace transform to both sides of (7.14) and obtain

$$\mathcal{L}\{y''\}(s) + 5\mathcal{L}\{y'\}(s) - 6Y(s) = \mathcal{L}\left\{21e^t\right\}(s) = \frac{21}{s-1}. \tag{7.15}$$

By Theorem 5, Section 7.3,

$$\begin{aligned}\mathcal{L}\{y'\}(s) &= sY(s) - y(0) = sY(s) + 1,\\ \mathcal{L}\{y''\}(s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) + s - 9.\end{aligned}$$

Substituting these expressions back into (7.15) and solving for $Y(s)$ yield

$$\begin{aligned}&\left[s^2Y(s) + s - 9\right] + 5\left[sY(s) + 1\right] - 6Y(s) = \frac{21}{s-1}\\ \Rightarrow \qquad &\left(s^2 + 5s - 6\right)Y(s) = \frac{21}{s-1} - s + 4 = \frac{-s^2 + 5s + 17}{s-1}\\ \Rightarrow \qquad &Y(s) = \frac{-s^2+5s+17}{(s-1)(s^2+5s-6)} = \frac{-s^2+5s+17}{(s-1)(s-1)(s+6)} = \frac{-s^2+5s+17}{(s-1)^2(s+6)}.\end{aligned}$$

The partial fractions decomposition for $Y(s)$ has the form

$$\frac{-s^2+5s+17}{(s-1)^2(s+6)} = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s+6}.$$

Clearing fractions yields

$$-s^2 + 5s + 17 = A(s+6) + B(s-1)(s+6) + C(s-1)^2.$$

Substitutions $s = 1$ and $s = -6$ give $A = 3$ and $C = -1$. Also, with $s = 0$, we have $17 = 6A - 6B + C$ or $B = 0$. Therefore,

$$Y(s) = \frac{3}{(s-1)^2} - \frac{1}{s+6} \qquad \Rightarrow \qquad y(t) = \mathcal{L}^{-1}\left\{\frac{3}{(s-1)^2} - \frac{1}{s+6}\right\}(t) = 3te^t - e^{-6t}.$$

Finally, shifting the argument back, we obtain

$$z(t) = y(t-1) = 3(t-1)e^{t-1} - e^{-6(t-1)}.$$

**11.** As in the previous problem (and in Example 3 in the text), we first need to shift the initial conditions to 0. If we set $v(t) = y(t+2)$, the initial value problem for $v(t)$ becomes

$$v''(t) - v(t) = (t+2) - 2 = t, \qquad v(0) = y(2) = 3,\ v'(0) = y'(2) = 0.$$

Taking the Laplace transform of both sides of this new differential equation gives us

$$\mathcal{L}\{v''\}(s) - \mathcal{L}\{v\}(s) = \mathcal{L}\{t\}(s) = \frac{1}{s^2}.$$

If we denote $V(s) := \mathcal{L}\{v\}(s)$ and express $\mathcal{L}\{v''\}(s)$ in terms of $V(s)$ using (4) in Section 4.3 (with $n = 2$), that is, $\mathcal{L}\{v''\}(s) = s^2V(s) - 3s$, we obtain

$$\begin{aligned} &\left[s^2V(s) - 3s\right] - V(s) = \frac{1}{s^2} \\ \Rightarrow \qquad & V(s) = \frac{3s^3+1}{s^2(s^2-1)} = \frac{3s^3+1}{s^2(s+1)(s-1)} = -\frac{1}{s^2} + \frac{1}{s+1} + \frac{2}{s-1}. \end{aligned}$$

Hence,

$$v(t) = \mathcal{L}^{-1}\{V(s)\}(t) = \mathcal{L}^{-1}\left\{-\frac{1}{s^2} + \frac{1}{s+1} + \frac{2}{s-1}\right\}(t) = -t + e^{-t} + 2e^{t}.$$

Since $v(t) = y(t+2)$, we have $y(t) = v(t-2)$ and so

$$y(t) = -(t-2) + e^{-(t-2)} + 2e^{t-2} = 2 - t + e^{2-t} + 2e^{t-2}.$$

**13.** To shift the initial conditions to $t = 0$, we make the substitution $x(t) := y(t+\pi/2)$ in the original equation and use the fact that

$$x'(t) := y'(t+\pi/2), \qquad x''(t) := y''(t+\pi/2).$$

This yields

$$\begin{aligned} & y''(t) - y'(t) - 2y(t) = -8\cos t - 2\sin t \\ \Rightarrow \quad & -8\cos\left(t+\frac{\pi}{2}\right) - 2\sin\left(t+\frac{\pi}{2}\right) = -8\cos\left(t+\frac{\pi}{2}\right) - 2\sin\left(t+\frac{\pi}{2}\right) = 8\sin t - 2\cos t \\ \Rightarrow \quad & x''(t) - x'(t) - 2x(t) = 8\sin t - 2\cos t, \qquad x(0) = 1,\ x'(0) = 0. \end{aligned}$$

Taking the Laplace transform of both sides in this last differential equation and using the fact that, with $X(s) := \mathcal{L}\{x\}(s)$,

$$\mathcal{L}\{x'\}(s) = sX(s) - 1 \quad \text{and} \quad \mathcal{L}\{x''\}(s) = s^2X(s) - s$$

(which comes from the initial conditions and (4) in Section 7.3), we obtain

$$\begin{aligned}
&\left[s^2X(s) - s\right] - \left[sX(s) - 1\right] - 2X(s) = \mathcal{L}\{8\sin t - 2\cos t\}(s) = \frac{8}{s^2+1} - \frac{2s}{s^2+1} \\
\Rightarrow \quad & \left(s^2 - s - 2\right)X(s) = \frac{8-2s}{s^2+1} + s - 1 = \frac{s^3 - s^2 - s + 7}{s^2+1} \\
\Rightarrow \quad & X(s) = \frac{s^3 - s^2 - s + 7}{(s^2+1)(s^2-s-2)} = \frac{s^3 - s^2 - s + 7}{(s^2+1)(s-2)(s+1)}\,.
\end{aligned}$$

We seek for the partial fractions decomposition of $X(s)$ in the form

$$\frac{s^3 - s^2 - s + 7}{(s^2+1)(s-2)(s+1)} = \frac{As+B}{s^2+1} + \frac{C}{(s-2)} + \frac{D}{s+1}\,.$$

Solving yields

$$A = \frac{7}{5}\,, \quad B = -\frac{11}{5}\,, \quad C = \frac{3}{5}\,, \quad D = -1.$$

Therefore,

$$\begin{aligned}
X(s) &= \frac{(7/5)s}{s^2+1} + \frac{(-11/5)}{s^2+1} + \frac{(3/5)}{(s-2)} - \frac{1}{s+1} \\
\Rightarrow \quad x(t) &= \mathcal{L}^{-1}\{X(s)\}(t) = \frac{7}{5}\cos t - \frac{11}{5}\sin t + \frac{3}{5}e^{2t} - e^{-t}\,.
\end{aligned}$$

Finally, since $y(t) = x(t - \pi/2)$, we obtain the solution

$$\begin{aligned}
y(t) &= \frac{7}{5}\cos\left(t - \frac{\pi}{2}\right) - \frac{11}{5}\sin\left(t - \frac{\pi}{2}\right) + \frac{3}{5}e^{2(t-\pi/2)} - e^{-(t-\pi/2)} \\
&= \frac{7}{5}\sin t + \frac{11}{5}\cos t + \frac{3}{5}e^{2t-\pi} - e^{(\pi/2)-t}
\end{aligned}$$

**15.** Taking the Laplace transform of $y'' - 3y' + 2y = \cos t$ and applying the linearity of the Laplace transform yields

$$\mathcal{L}\{y''\}(s) - 3\mathcal{L}\{y'\}(s) + 2\mathcal{L}\{y\}(s) = \mathcal{L}\{\cos t\}(s) = \frac{s}{s^2+1}\,. \tag{7.16}$$

If we put $Y(s) = \mathcal{L}\{y\}(s)$ and apply the property (4), page 362 of the text, we get

$$\mathcal{L}\{y'\}(s) = sY(s), \qquad \mathcal{L}\{y''\}(s) = s^2Y(s) + 1.$$

Substitution back into (7.16) yields

$$\begin{aligned} \left[s^2Y(s)+1\right] - 3\left[sY(s)\right] + 2Y(s) &= \frac{s}{s^2+1} \\ \Rightarrow \qquad \left(s^2-3s+2\right)Y(s) &= \frac{s}{s^2+1} - 1 = \frac{-s^2+s-1}{s^2+1} \\ \Rightarrow \qquad Y(s) &= \frac{-s^2+s-1}{(s^2+1)(s^2-3s+2)} = \frac{-s^2+s-1}{(s^2+1)(s-1)(s-2)}. \end{aligned}$$

**17.** With $Y(s) := \mathcal{L}\{y\}(s)$, we find that

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 1, \quad \mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s,$$

and so the Laplace transform of both sides of the original equation yields

$$\begin{aligned} \mathcal{L}\{y''+y'-y\}(s) &= \mathcal{L}\left\{t^3\right\}(s) \\ \Rightarrow \qquad \left[s^2Y(s)-s\right] + \left[sY(s)-1\right] - Y(s) &= \frac{6}{s^4} \\ \Rightarrow \qquad Y(s) &= \frac{1}{s^2+s-1}\left(\frac{6}{s^4}+s+1\right) = \frac{s^5+s^4+6}{s^4(s^2+s-1)}. \end{aligned}$$

**19.** Let us denote $Y(s) := \mathcal{L}\{y\}(s)$. From the initial conditions and formula (4) on page 362 of the text we get

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 1, \quad \mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s - 1.$$

The Laplace transform, applied to both sides of the given equation, yields

$$\begin{aligned} \left[s^2Y(s)-s-1\right] + 5\left[sY(s)-1\right] - Y(s) &= \mathcal{L}\left\{e^t\right\}(s) - \mathcal{L}\{1\}(s) = \frac{1}{s-1} - \frac{1}{s} = \frac{1}{s(s-1)} \\ \Rightarrow \qquad \left(s^2+5s-1\right)Y(s) &= \frac{1}{s(s-1)} + s + 6 = \frac{s^3+5s^2-6s+1}{s(s-1)} \\ \Rightarrow \qquad Y(s) &= \frac{s^3+5s^2-6s+1}{s(s-1)(s^2+5s-1)}. \end{aligned}$$

# Chapter 7

**21.** Applying the Laplace transform to both sides of the given equation yields

$$\mathcal{L}\{y''\}(s) - 2\mathcal{L}\{y'\}(s) + \mathcal{L}\{t\}(s) = \mathcal{L}\{\cos t\}(s) - \mathcal{L}\{\sin t\}(s) = \frac{s-1}{s^2+1}.$$

If $\mathcal{L}\{y\}(s) =: Y(s)$, then it follows from the initial conditions and (4) on page 362 of the text that

$$\mathcal{L}\{y'\}(s) = sY(s) - 1, \quad \mathcal{L}\{y''\}(s) = s^2Y(s) - s - 3.$$

Therefore, $Y(s)$ satisfies

$$\left[s^2Y(s) - s - 3\right] - 2\left[sY(s) - 1\right] + Y(s) = \frac{s-1}{s^2+1}.$$

Solving for $Y(s)$ gives us

$$\left(s^2 - 2s + 1\right)Y(s) = \frac{s-1}{s^2+1} + s + 1 = \frac{s^3+s^2+2s}{s^2+1}$$

$$\Rightarrow \qquad Y(s) = \frac{s^3+s^2+2s}{(s^2+1)(s^2-2s+1)} = \frac{s^3+s^2+2s}{(s^2+1)(s-1)^2}.$$

**23.** In this equation, the right-hand side is a piecewise defined function. Let us find its Laplace transform first.

$$\begin{aligned}
\mathcal{L}\{g(t)\}(s) &= \int_0^\infty e^{-st}g(t)\,dt = \int_0^2 e^{-st}t\,dt + \int_2^\infty e^{-st}5\,dt \\
&= \left.\frac{te^{-st}}{-s}\right|_0^2 - \int_0^2 \frac{e^{-st}}{-s}\,dt + \lim_{N\to\infty}\left.\frac{5e^{-st}}{-s}\right|_2^N \\
&= -\left[\frac{2e^{-2s}}{s}\right] - \left[\frac{e^{-2s}}{s^2} + \frac{1}{s^2}\right] + \frac{5e^{-2s}}{s} = \frac{1+3se^{-2s}-e^{-2s}}{s^2},
\end{aligned}$$

where we used integration by parts integrating $e^{-st}t$.

Using this formula and applying the Laplace transform to the given equation yields

$$\begin{aligned}
&\mathcal{L}\{y''\}(s) + 4\mathcal{L}\{y\}(s) = \mathcal{L}\{g(t)\}(s) \\
\Rightarrow \qquad & s^2\mathcal{L}\{y\}(s) + s + 4\mathcal{L}\{y\}(s) = \mathcal{L}\{g(t)\}(s) \\
\Rightarrow \qquad & \left(s^2+4\right)\mathcal{L}\{y\}(s) = \mathcal{L}\{g(t)\}(s) - s = \frac{-s^3+1+3se^{-2s}-e^{-2s}}{s^2}
\end{aligned}$$

$$\Rightarrow \qquad \mathcal{L}\{y\}(s) = \frac{-s^3 + 1 + 3se^{-2s} - e^{-2s}}{s^2(s^2+4)}.$$

**25.** Taking the Laplace transform of $y''' - y'' + y' - y = 0$ and applying the linearity of the Laplace transform yields

$$\mathcal{L}\{y'''\}(s) - \mathcal{L}\{y''\}(s) + \mathcal{L}\{y'\}(s) - \mathcal{L}\{y\}(s) = \mathcal{L}\{0\}(s) = 0. \tag{7.17}$$

If we denote $Y(s) := \mathcal{L}\{y\}(s)$ and and apply property (4) on page 362 of the text, we get

$$\mathcal{L}\{y'\}(s) = sY(s) - 1, \quad \mathcal{L}\{y''\}(s) = s^2Y(s) - s - 1, \; LTy''' = s^3Y(s) - s^2 - s - 3.$$

Combining these equations with (7.17) gives us

$$\begin{aligned} \left[s^3Y(s) - s^2 - s - 3\right] - \left[s^2Y(s) - s - 1\right] + [sY(s) - 1] - Y(s) = 0 \\ \Rightarrow \qquad \left(s^3 - s^2 + s - 1\right)Y(s) = s^2 + 3 \\ \Rightarrow \qquad Y(s) = \frac{s^2+3}{s^3 - s^2 + s - 1} = \frac{s^2+3}{(s-1)(s^2+1)}. \end{aligned}$$

Expanding $Y(s)$ by partial fractions results

$$Y(s) = \frac{2}{s-1} - \frac{s+1}{s^2+1} = \frac{2}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}.$$

From Table 7.1 on page 358 of the text, we see that

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = 2e^t - \cos t - \sin t.$$



**27.** Let $Y(s) := \mathcal{L}\{y\}(s)$. Then, by Theorem 5 in Section 7.3,

$$\begin{aligned} \mathcal{L}\{y'\}(s) &= sY(s) - y(0) = sY(s) + 4, \\ \mathcal{L}\{y''\}(s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) + 4s - 4, \\ \mathcal{L}\{y'''\}(s) &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) = s^3Y(s) + 4s^2 - 4s + 2. \end{aligned}$$

Using these equations and applying the Laplace transform to both sides of the given differential equation, we get

$$\left[s^3Y(s) + 4s^2 - 4s + 2\right] + 3\left[s^2Y(s) + 4s - 4\right] + 3\left[sY(s) + 4\right] + Y(s) = 0$$

$$\Rightarrow \qquad \left(s^3+3s^2+3s+1\right)Y(s)+\left(4s^2+8s+2\right)=0$$
$$\Rightarrow \qquad Y(s)=-\frac{4s^2+8s+2}{s^3+3s^2+3s+1}=-\frac{4s^2+8s+2}{(s+1)^3}\,.$$

Therefore, the partial fractions decomposition of $Y(s)$ has the form

$$-\frac{4s^2+8s+2}{(s+1)^3}=\frac{A}{(s+1)^3}+\frac{B}{(s+1)^2}+\frac{C}{s+1}=\frac{A+B(s+1)+C(s+1)^2}{(s+1)^3}$$
$$\Rightarrow \qquad -(4s^2+8s+2)=A+B(s+1)+C(s+1)^2\,.$$

Substitution $s=-1$ yields $A=2$. Equating coefficients at $s^2$, we get $C=-4$. At last, substituting $s=0$ we obtain

$$-2=A+B+C \qquad \Rightarrow \qquad B=-2-A-C=0.$$

Therefore,

$$Y(s)=\frac{2}{(s+1)^3}+\frac{-4}{s+1} \qquad \Rightarrow \qquad y(t)=\mathcal{L}^{-1}\{Y\}(t)=t^2e^{-t}-4e^{-t}=\left(t^2-4\right)e^{-t}\,.$$



**29.** Using the initial conditions, $y(0)=a$ and $y'(0)=b$, and formula (4) on page 362 of the text, we conclude that

$$\mathcal{L}\{y'\}(s)=sY(s)-y(0)=sY(s)-a,$$
$$\mathcal{L}\{y''\}(s)=s^2Y(s)-sy(0)-y'(0)=s^2Y(s)-as-b,$$

where $Y(s)=\mathcal{L}\{y\}(s)$. Applying the Laplace transform to the original equation yields

$$\left[s^2Y(s)-as-b\right]-4\left[sY(s)-a\right]+3Y(s)=\mathcal{L}\{0\}(s)=0$$
$$\Rightarrow \qquad \left(s^2-4s+3\right)Y(s)=as+b-4a$$
$$\Rightarrow \qquad Y(s)=\frac{as+b-4a}{s^2-4s+3}=\frac{as+b-4a}{(s-1)(s-3)}=\frac{A}{s-1}+\frac{B}{s-3}\,.$$

Solving for $A$ and $B$, we find that $A=(3a-b)/2$, $B=(b-a)/2$. Hence

$$Y(s)=\frac{(3a-b)/2}{s-1}+\frac{(b-a)/2}{s-3}$$
$$\Rightarrow \qquad y(t)=\mathcal{L}^{-1}\{Y\}(t)=\frac{3a-b}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t)+\frac{b-a}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}(t)$$
$$=\frac{3a-b}{2}e^{t}+\frac{b-a}{2}e^{3t}\,.$$

**31.** Similarly to Problem 29, we have

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - a, \quad \mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - as - b,$$

with $Y(s) := \mathcal{L}\{y\}(s)$. Thus the Laplace transform of both sides of the the given equation yields

$$\begin{aligned} &\mathcal{L}\{y'' + 2y' + 2y\}(s) = \mathcal{L}\{5\}(s) \\ \Rightarrow \quad & \left[s^2Y(s) - as - b\right] + 2\left[sY(s) - a\right] + 2Y(s) = \frac{5}{s} \\ \Rightarrow \quad & \left(s^2 + 2s + 2\right)Y(s) = \frac{5}{s} + as + 2a + b = \frac{as^2 + (2a+b)s + 5}{s} \\ \Rightarrow \quad & Y(s) = \frac{as^2 + (2a+b)s + 5}{s(s^2 + 2s + 2)} = \frac{as^2 + (2a+b)s + 5}{s[(s+1)^2 + 1]}. \end{aligned}$$

We seek for an expansion of $Y(s)$ of the form

$$\frac{as^2 + (2a+b)s + 5}{s[(s+1)^2 + 1]} = \frac{A}{s} + \frac{B(s+1) + C}{(s+1)^2 + 1}.$$

Clearing fractions, we obtain

$$as^2 + (2a+b)s + 5 = A\left[(s+1)^2 + 1\right] + \left[B(s+1) + C\right]s.$$

Substitutions $s = 0$ and $s = -1$ give us

$$\begin{aligned} s = 0: \quad & 5 = 2A & \Rightarrow \quad & A = 5/2, \\ s = -1: \quad & 5 - a - b = A - C & \Rightarrow \quad & C = A + a + b - 5 = a + b - 5/2. \end{aligned}$$

To find $B$, we can compare coefficients at $s^2$:

$$a = A + B \qquad \Rightarrow \qquad B = a - A = a - 5/2.$$

So,

$$\begin{aligned} & Y(s) = \frac{5/2}{s} + \frac{(a - 5/2)(s+1)}{(s+1)^2 + 1} + \frac{a + b - 5/2}{(s+1)^2 + 1} \\ \Rightarrow \quad & y(t) = \mathcal{L}^{-1}\{Y\}(t) = \frac{5}{2} + \left(a - \frac{5}{2}\right)e^{-t}\cos t + \left(a + b - \frac{5}{2}\right)e^{-t}\sin t. \end{aligned}$$

**33.** By Theorem 6 in Section 7.3,

$$\mathcal{L}\left\{t^2 y'(t)\right\}(s) = (-1)^2 \frac{d^2}{ds^2}\left[\mathcal{L}\left\{y'(t)\right\}(s)\right] = \frac{d^2}{ds^2}\left[\mathcal{L}\left\{y'(t)\right\}(s)\right]. \tag{7.18}$$

On the other hand, equation (4) on page 362 says that

$$\mathcal{L}\left\{y'(t)\right\}(s) = sY(s) - y(0), \qquad Y(s) := \mathcal{L}\left\{y\right\}(s).$$

Substitution back into (7.18) yields

$$\begin{aligned}\mathcal{L}\left\{t^2 y'(t)\right\}(s) &= \frac{d^2}{ds^2}\left[sY(s) - y(0)\right] = \frac{d}{ds}\left\{\frac{d}{ds}\left[sY(s) - y(0)\right]\right\} \\ &= \frac{d}{ds}\left[sY'(s) + Y(s)\right] = \left(sY''(s) + Y'(s)\right) + Y'(s) = sY''(s) + 2Y'(s).\end{aligned}$$

**35.** Taking the Laplace transform of $y'' + 3ty' - 6y = 1$ and applying the linearity of the Laplace transform yields

$$\mathcal{L}\left\{y''\right\}(s) + 3\mathcal{L}\left\{ty'\right\}(s) - 6\mathcal{L}\left\{y\right\}(s) = \mathcal{L}\left\{1\right\}(s) = \frac{1}{s}. \tag{7.19}$$

If we put $Y(s) = \mathcal{L}\left\{y\right\}(s)$ and apply property (4) on page 362 of the text with $n = 2$, we get

$$\mathcal{L}\left\{y''\right\}(s) = s^2 Y(s) - sy(0) - y'(0) = s^2 Y(s). \tag{7.20}$$

Furthermore, as it was shown in Example 4, Section 4.5,

$$\mathcal{L}\left\{ty'\right\}(s) = -sY'(s) - Y(s). \tag{7.21}$$

Substitution (7.20) and (7.21) back into (7.19) yields

$$\begin{aligned} & s^2 Y(s) + 3\left[-sY'(s) - Y(s)\right] - 6Y(s) = \frac{1}{s} \\ \Rightarrow \quad & -3sY'(s) + \left(s^2 - 9\right)Y(s) = \frac{1}{s} \\ \Rightarrow \quad & Y'(s) + \left(\frac{3}{s} - \frac{s}{3}\right)Y(s) = -\frac{1}{3s^2}. \end{aligned}$$

This is a first order linear differential equation in $Y(s)$, which can be solved by the techniques of Section 2.3. Namely, it has the integrating factor

$$\mu(s) = \exp\left[\int\left(\frac{3}{s} - \frac{s}{3}\right) ds\right] = \exp\left[3\ln s - \frac{s^2}{6}\right] = s^3 e^{-s^2/6}.$$

Thus

$$\begin{aligned} Y(s) &= \frac{1}{\mu(s)}\int \mu(s)\left(-\frac{1}{3s^2}\right)ds = \frac{1}{s^3e^{-s^2/6}}\int \frac{-s}{3}e^{-s^2/6}\,ds \\ &= \frac{1}{s^3e^{-s^2/6}}\left(e^{-s^2/6}+C\right) = \frac{1}{s^3}\left(1+Ce^{s^2/6}\right). \end{aligned}$$

Just as in Example 4 on page 380 of the text, $C$ must be zero in order to ensure that $Y(s) \to 0$ as $s \to \infty$. Thus $Y(s) = 1/s^3$, and from Table 7.1 on page 358 of the text we get

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}(t) = \frac{t^2}{2}.$$

**37.** We apply the Laplace transform to the given equation and obtain

$$\mathcal{L}\{ty''\}(s) - 2\mathcal{L}\{y'\}(s) + \mathcal{L}\{ty\}(s) = 0. \tag{7.22}$$



Using Theorem 5 (Section 7.3) and the initial conditions, we express $\mathcal{L}\{y''\}(s)$ and $\mathcal{L}\{y'\}(s)$ in terms of $Y(s) := \mathcal{L}\{y\}(s)$.

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 1, \tag{7.23}$$

$$\mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s. \tag{7.24}$$

We now involve Theorem 6 in Section 7.3 to get

$$\mathcal{L}\{ty\}(s) = -\frac{d}{ds}\left[\mathcal{L}\{y\}(s)\right] = -Y'(s). \tag{7.25}$$

Also, Theorem 6 and equation (7.24) yield

$$\mathcal{L}\{ty''\}(s) = -\frac{d}{ds}\left[\mathcal{L}\{y''\}(s)\right] = -\frac{d}{ds}\left[s^2Y(s) - s\right] = 1 - 2sY(s) - s^2Y'(s). \tag{7.26}$$

Substituting (7.23), (7.25), and (7.26) into (7.22), we obtain

$$\begin{aligned} &\left[1 - 2sY(s) - s^2Y'(s)\right] - 2\left[sY(s) - 1\right] + \left[-Y'(s)\right] = 0 \\ \Rightarrow\quad & -\left(s^2+1\right)Y'(s) - 4sY(s) + 3 = 0 \\ \Rightarrow\quad & Y'(s) + \frac{4s}{s^2+1}Y(s) = \frac{3}{s^2+1}. \end{aligned}$$

The integrating factor of this first order linear differential equation is

$$\mu(s) = \exp\left[\int \frac{4s}{s^2+1}\,ds\right] = \exp\left[2\ln\left(s^2+1\right)\right] = \left(s^2+1\right)^2 .$$

Hence

$$\begin{aligned} Y(s) &= \frac{1}{\mu(s)}\int \mu(s)\left(\frac{3}{s^2+1}\right)ds = \frac{1}{(s^2+1)^2}\int 3\left(s^2+1\right)ds \\ &= \frac{1}{(s^2+1)^2}\left(s^3+3s+C\right) = \frac{(s^3+s)+(2s+C)}{(s^2+1)^2} = \frac{s}{s^2+1}+\frac{2s}{(s^2+1)^2}+\frac{C}{(s^2+1)^2}, \end{aligned}$$

where $C$ is an arbitrary constant. Therefore,

$$y(t) = \mathcal{L}^{-1}\{Y\}(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\}(t) + \frac{C}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s^2+1)^2}\right\}(t).$$

Using formulas (24), (29) and (30) on the inside back cover of the text, we finally get

$$y(t) = \cos t + t\sin t + c(\sin t - t\cos t),$$

where $c := C/2$ is an arbitrary constant.

**39.** Similarly to Example 5, we have the initial value problem (18), namely,

$$Iy''(t) = -ke(t), \qquad y(0)=0, \quad y'(0)=0,$$

for the model of the mechanism. This equation leads to equation (19) for the Laplace transforms $Y(s) := \mathcal{L}\{y(t)\}(s)$ and $E(s) := \mathcal{L}\{e(t)\}(s)$:

$$s^2IY(s) = -kE(s). \tag{7.27}$$

But, this time, $e(t) = y(t) - a$ and so

$$E(s) = \mathcal{L}\{y(t)-a\}(s) = Y(s) - \frac{a}{s} \qquad \Rightarrow \qquad Y(s) = E(s) + \frac{a}{s}.$$

Substituting this relation into (7.27) yields

$$s^2IE(s) + aIs = -kE(s) \qquad \Rightarrow \qquad E(s) = -\frac{-aIs}{s^2I+k} = -\frac{as}{s^2+(k/I)}.$$

Taking the inverse Laplace transform, we obtain

$$e(t) = \mathcal{L}^{-1}\{E(s)\}(t) = -a\mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{k/I})^2}\right\}(t) = -a\cos\left(\sqrt{k/I}t\right).$$

**41.** As in Problem 40, the differential equation modeling the automatic pilot is

$$Iy''(t) = -ke(t) - \mu e'(t)\,, \tag{7.28}$$

but now the error $e(t)$ is given by $e(t) = y(t) - at$.

Let $Y(s) := \mathcal{L}\{y(t)\}(s)$, $E(s) := \mathcal{L}\{e(t)\}(s)$. Notice that, as in Example 5 on page 382, we have $y(0) = y'(0) = 0$, and so $e(0) = 0$. Using these initial conditions and Theorem 5 in Section 7.3, we obtain

$$\mathcal{L}\{y''(t)\}(s) = s^2Y(s) \qquad \text{and} \qquad \mathcal{L}\{e'(t)\}(s) = sE(s).$$

Applying the Laplace transform to both sides of (7.28) we then conclude that

$$\begin{aligned} &I\mathcal{L}\{y''(t)\}(s) = -k\mathcal{L}\{e(t)\}(s) - \mu\mathcal{L}\{e'(t)\}(s) \\ \Rightarrow \qquad &Is^2Y(s) = -kE(s) - \mu sE(s) = -(k+\mu s)E(s). \end{aligned} \tag{7.29}$$

Since $e(t) = y(t) - at$,

$$E(s) = \mathcal{L}\{e(t)\}(s) = \mathcal{L}\{y(t) - at\}(s) = Y(s) - a\mathcal{L}\{t\}(s) = Y(s) - \frac{a}{s^2}$$

or $Y(s) = E(s) + a/s^2$. Substitution back into (7.29) yields

$$\begin{aligned} &Is^2\left(E(s) + \frac{a}{s^2}\right) = -(k+\mu s)E(s) \\ \Rightarrow \qquad &\left(Is^2 + \mu s + k\right)E(s) = -aI \\ \Rightarrow \qquad &E(s) = \frac{-aI}{Is^2 + \mu s + k} = \frac{-a}{s^2 + (\mu/I)s + (k/I)}\,. \end{aligned}$$

Completing the square in the denominator, we write $E(s)$ in the form suitable for inverse Laplace transform.

$$\begin{aligned} E(s) &= \frac{-a}{[s+\mu/(2I)]^2 + (k/I) - \mu^2/(4I^2)} \\ &= \frac{-a}{[s+\mu/(2I)]^2 + (4kI-\mu^2)/(4I^2)} = \frac{-2Ia}{\sqrt{4kI-\mu^2}} \frac{\sqrt{4kI-\mu^2}/(2I)}{[s+\mu/(2I)]^2 + (4kI-\mu^2)/(4I^2)}\,. \end{aligned}$$

Thus, using Table 7.1 on page 358 of the text, we find that

$$e(t) = \mathcal{L}^{-1}\left\{E(s)\right\}(t) = \frac{-2Ia}{\sqrt{4kI - \mu^2}}\, e^{-\mu t/(2I)} \sin\left[\frac{\sqrt{4kI - \mu^2}t}{2I}\right].$$

Compare this with Example 5 of the text and observe, how for moderate damping with $\mu < 2\sqrt{kI}$, the oscillations of Example 5 die out exponentially.

**EXERCISES 7.6: Transforms of Discontinuous and Periodic Functions, page 395**

**1.** To find the Laplace transform of $g(t) = (t-1)^2 u(t-1)$ we apply formula (5) on page 387 of the text with $a = 1$ and $f(t) = t^2$. This yields

$$\mathcal{L}\left\{(t-1)^2 u(t-1)\right\}(s) = e^{-s}\mathcal{L}\left\{t^2\right\}(s) = \frac{2e^{-s}}{s^3}.$$

The graph of $g(t) = (t-1)^2 u(t-1)$ is shown in Figure 7-B(a).



**3.** The graph of the function $y = t^2 u(t-2)$ is shown in Figure 7-B(b). For this function, formula (8) on page 387 is more convenient. To apply the shifting property, we observe that $g(t) = t^2$ and $a = 2$. Hence

$$g(t+a) = g(t+2) = (t+2)^2 = t^2 + 4t + 4.$$

Now the Laplace transform of $g(t+2)$ is

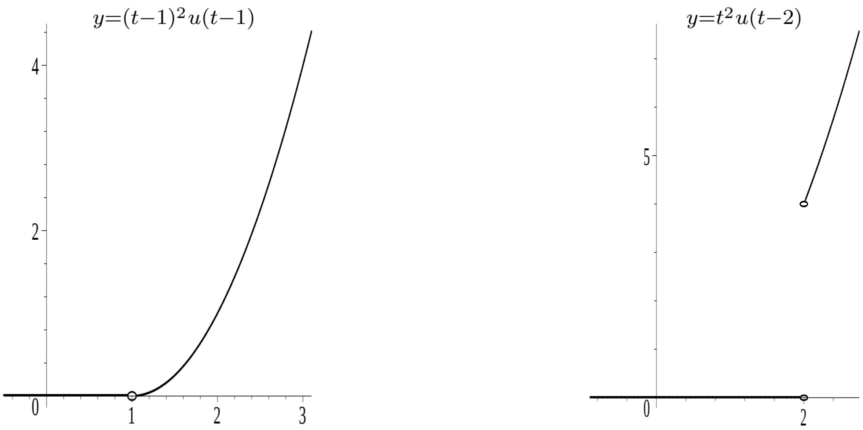
$$\mathcal{L}\left\{t^2 + 4t + 4\right\}(s) = \mathcal{L}\left\{t^2\right\}(s) + 4\mathcal{L}\left\{t\right\}(s) + 4\mathcal{L}\left\{1\right\}(s) = \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}.$$

Hence, by formula (8), we have

$$\mathcal{L}\left\{t^2 u(t-2)\right\}(s) = e^{-2s}\mathcal{L}\left\{g(t+2)\right\}(s) = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right) = \frac{e^{-2s}(4s^2 + 4s + 2)}{s^3}.$$

**5.** The function $g(t)$ equals zero until $t$ reaches 1, at which point $g(t)$ jumps to 2. We can express this jump by $(2-0)u(t-1)$. At $t = 2$ the function $g(t)$ jumps from the value 2 to the value 1. This can be expressed by adding the term $(1-2)u(t-2)$. Finally, the jump at $t = 3$ from 1 to 3 can be accomplished by the function $(3-1)u(t-3)$. Hence

$$g(t) = 0 + (2-0)u(t-1) + (1-2)u(t-2) + (3-1)u(t-3) = 2u(t-1) - u(t-2) + 2u(t-3)$$




**(a)** **(b)**

**Figure 7–B**: Graphs of functions in Problems 1 and 3.



and, by the linearity of the Laplace transform,

$$\begin{aligned}\mathcal{L}\{g(t)\}(s) &= 2\mathcal{L}\{u(t-1)\}(s) - \mathcal{L}\{u(t-2)\}(s) + 2\mathcal{L}\{u(t-3)\}(s)\\ &= 2\frac{e^{-s}}{s} - \frac{e^{-2s}}{s} + 2\frac{e^{-3s}}{s}\\ &= \frac{e^{-s} - e^{-2s} + 2e^{-3s}}{s}\,.\end{aligned}$$

**7.** Observe from the graph that $g(t)$ is given by

$$\begin{cases} 0, & t < 1,\\ t, & 1 < t < 2,\\ 1, & 2 < t.\end{cases}$$

The function $g(t)$ equals zero until $t$ reaches 1, at which point $g(t)$ jumps to the function $t$. We can express this jump by $tu(t-1)$. At $t = 2$ the function $g(t)$ jumps from the function $t$ to the value 1. This can be expressed by adding the term $(1-t)u(t-2)$. Hence

$$g(t) = 0 + tu(t-1) + (1-t)u(t-2) = tu(t-1) - (t-1)u(t-2).$$

Taking the Laplace transform of both sides and using formula (8) on page 387, we find that the Laplace transform of the function $g(t)$ is given by

$$\begin{aligned}\mathcal{L}\{g(t)\}(s) &= \mathcal{L}\{tu(t-1)\}(s) - \mathcal{L}\{(t-1)u(t-2)\}(s)\\ &= e^{-s}\mathcal{L}\{(t+1)\}(s) - e^{-2s}\mathcal{L}\{(t-1)+2\}(s)\\ &= \left(e^{-s}-e^{-2s}\right)\mathcal{L}\{t+1\}(s) = \left(e^{-s}-e^{-2s}\right)\left(\frac{1}{s^2}+\frac{1}{s}\right) = \frac{(e^{-s}-e^{-2s})(s+1)}{s^2}.\end{aligned}$$

**9.** First, we find the formula for $g(t)$ from the picture given.

$$\begin{cases} 0, & t<1,\\ t-1, & 1<t<2,\\ 3-t, & 2<t<3,\\ 0, & 3<t. \end{cases}$$

Thus, this function jumps from 0 to $t-1$ at $t=1$, from $t-1$ to $3-t$ at $t=2$, and from $3-t$ to 0 at $t=3$. Since the function $u(t-a)$ has the unit jump from 0 to 1 at $t=a$, we can express $g(t)$ as

$$\begin{aligned}g(t) &= [(t-1)-0]u(t-1) + [(3-t)-(t-1)]u(t-2) + [0-(3-t)]u(t-3)\\ &= (t-1)u(t-1) + (4-2t)u(t-2) + (t-3)u(t-3).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}\{g(t)\}(s) &= \mathcal{L}\{(t-1)u(t-1)\}(s) + \mathcal{L}\{(4-2t)u(t-2)\}(s) + \mathcal{L}\{(t-3)u(t-3)\}(s)\\ &= e^{-s}\mathcal{L}\{(t+1)-1\}(s) + e^{-2s}\mathcal{L}\{4-2(t+2)\}(s) + e^{-3s}\mathcal{L}\{(t+3)-3\}(s)\\ &= e^{-s}\mathcal{L}\{t\}(s) - 2e^{-2s}\mathcal{L}\{t\}(s) + e^{-3s}\mathcal{L}\{t\}(s) = \frac{e^{-s}-2e^{-2s}+e^{-3s}}{s^2}.\end{aligned}$$

**11.** We use formula (6) on page 387 of the text with $a=2$ and $F(s)=1/(s-1)$. Since

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) = e^t \qquad \Rightarrow \qquad f(t-2) = e^{t-2},$$

we get

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-1}\right\}(t) = f(t-2)u(t-2) = e^{t-2}u(t-2).$$

**13.** Using the linear property of the inverse Laplace transform, we obtain

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}-3e^{-4s}}{s+2}\right\}(t)=\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+2}\right\}(t)-3\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s+2}\right\}(t).$$

To each term in the above equation, we can apply now formula (6), page 387 of the text with $F(s)=1/(s+2)$ and $a=2$ and $a=4$, respectively. Since

$$f(t):=\mathcal{L}^{-1}\{F(s)\}(t)=\mathcal{L}^{-1}\{1/(s+2)\}(t)=e^{-2t},$$

we get

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+2}\right\}(t)-3\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s+2}\right\}(t) &= f(t-2)u(t-2)-3f(t-4)u(t-4)\\ &= e^{-2(t-2)}u(t-2)-3e^{-2(t-4)}u(t-4).\end{aligned}$$



**15.** Since

$$\begin{aligned}F(s):=\frac{s}{s^2+4s+5}&=\frac{s}{(s+2)^2+1^2}=\frac{s+2}{(s+2)^2+1^2}-2\frac{1}{(s+2)^2+1^2}\\ \Rightarrow\qquad f(t):=\mathcal{L}^{-1}\{F(s)\}(t)&=e^{-2t}(\cos t-2\sin t),\end{aligned}$$

applying Theorem 8 we get

$$\mathcal{L}^{-1}\left\{\frac{se^{-3s}}{s^2+4s+5}\right\}(t)=f(t-3)u(t-3)=e^{-2(t-3)}\left[\cos(t-3)-2\sin(t-3)\right]u(t-3).$$

**17.** By partial fractions,

$$\frac{s-5}{(s+1)(s+2)}=-\frac{6}{s+1}+\frac{7}{s+2}$$

so that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{e^{-3s}(s-5)}{(s+1)(s+2)}\right\}(t) &= -6\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s+1}\right\}(t)+7\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s+2}\right\}(t)\\ &= -6\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t-3)u(t-3)+7\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t-3)u(t-3)\\ &= \left[-6e^{-(t-3)}+7e^{-2(t-3)}\right]u(t-3)=\left[7e^{6-2t}-6e^{3-t}\right]u(t-3).\end{aligned}$$

**19.** In this problem, we apply methods of Section 7.5 of solving initial value problems using the Laplace transform. Taking the Laplace transform of both sides of the given equation and using the linear property of the Laplace transform, we get

$$\mathcal{L}\{I''\}(s) + 2\mathcal{L}\{I'\}(s) + 2\mathcal{L}\{I\}(s) = \mathcal{L}\{g(t)\}(s). \tag{7.30}$$

Let us denote $\mathbf{I}(s) := \mathcal{L}\{I\}(s)$. By Theorem 5, Section 7.3,

$$\begin{aligned} \mathcal{L}\{I'\}(s) &= s\mathbf{I}(s) - I(0) = s\mathbf{I}(s) - 10, \\ \mathcal{L}\{I''\}(s) &= s^2\mathbf{I}(s) - sI(0) - I'(0) = s^2\mathbf{I}(s) - 10s. \end{aligned} \tag{7.31}$$

To find the Laplace transform of $g(t)$, we express this function using the unit step function $u(t)$. Since $g(t)$ identically equals to 20 for $0 < t < 3\pi$, jumps from 20 to 0 at $t = 3\pi$ and then jumps from 0 to 20 at $t = 4\pi$, we can write

$$g(t) = 20 + (0-20)u(t-3\pi) + (20-0)u(t-4\pi) = 20 - 20u(t-3\pi) + 20u(t-4\pi).$$

Therefore,

$$\begin{aligned} \mathcal{L}\{g(t)\}(s) &= \mathcal{L}\{20 - 20u(t-3\pi) + 20u(t-4\pi)\}(s) \\ &= 20\mathcal{L}\{1 - u(t-3\pi) + u(t-4\pi)\}(s) = 20\left(\frac{1}{s} - e^{-3\pi s} + e^{-4\pi s}\right). \end{aligned}$$

Substituting this equation and (7.31) into (7.30) yields

$$\begin{aligned} \left[s^2\mathbf{I}(s) - 10s\right] + 2\left[s\mathbf{I}(s) - 10\right] + 2\mathbf{I}(s) &= 20\left(\frac{1}{s} - \frac{e^{-3\pi s}}{s} + \frac{e^{-4\pi s}}{s}\right) \\ \Rightarrow \qquad \mathbf{I}(s) &= 10\frac{1}{s} + 20\frac{-e^{-3\pi s} + e^{-4\pi s}}{s[(s+1)^2+1]}. \end{aligned} \tag{7.32}$$

Since $\mathcal{L}^{-1}\{1/s\}(t) = 1$ and

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s[(s+1)^2+1]}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{2}\left[1s - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right]\right\}(t) \\ &= \frac{1}{2}\left[1 - e^{-t}(\cos t + \sin t)\right], \end{aligned}$$

applying the inverse Laplace transform to both sides of (7.32) yields

$$I(t) = \mathcal{L}^{-1}\left\{10\frac{1}{s} + 20\frac{-e^{-3\pi s}}{s[(s+1)^2+1]} + 20\frac{e^{-4\pi s}}{s[(s+1)^2+1]}\right\}(t)$$

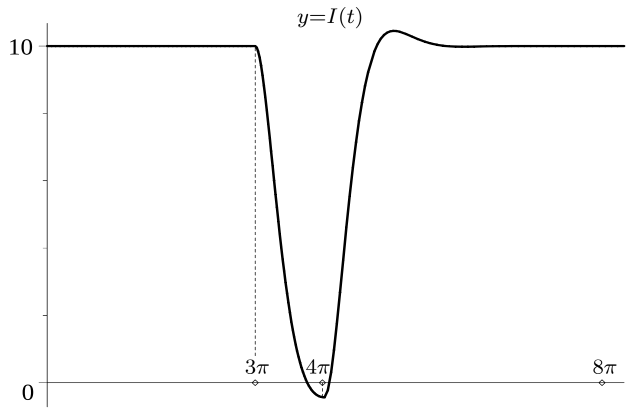



**Figure 7–C**: The graph of the function $y = I(t)$ in Problem 19.

$$
\begin{aligned}
&= 10 - 10u(t-3\pi)\left[1 - e^{-(t-3\pi)}\left(\cos(t-3\pi) + \sin(t-3\pi)\right)\right] \\
&\qquad + 10u(t-4\pi)\left[1 - e^{-(t-4\pi)}\left(\cos(t-4\pi) + \sin(t-4\pi)\right)\right] \\
&= 10 - 10u(t-3\pi)\left[1 + e^{-(t-3\pi)}\left(\cos t + \sin t\right)\right] \\
&\qquad + 10u(t-4\pi)\left[1 - e^{-(t-4\pi)}\left(\cos t + \sin t\right)\right].
\end{aligned}
$$



The graph of the solution, $y = I(t)$, $0 < t < 8\pi$, is depicted in Figure 7-C.

**21.** In the windowed version (11) of $f(t)$, $f_T(t) = t$ and $T = 2$. Thus

$$
\begin{aligned}
F_T(s) &:= \int_0^\infty e^{-st} f_T(t)\,dt = \int_0^2 e^{-st} t\,dt = -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\bigg|_0^2 \\
&= -\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2} = \frac{1 - 2se^{-2s} - e^{-2s}}{s^2}.
\end{aligned}
$$

From Theorem 9 on page 391 of the text, we obtain

$$
\mathcal{L}\{f(t)\}(s) = \frac{F_T(s)}{1 - e^{-2s}} = \frac{1 - 2se^{-2s} - e^{-2s}}{s^2(1 - e^{-2s})}.
$$

The graph of the function $y = f(t)$ is given in Figure B.45 in the answers of the text.

**23.** We use formula (12) on page 391 of the text. With the period $T = 2$, the windowed version

$f_T(t)$ of $f(t)$ is

$$f_T(t) = \begin{cases} f(t), & 0 < t < 2, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} e^{-t}, & 0 < t < 1, \\ 1, & 1 < t < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} F_T(s) &= \int_0^{\infty} e^{-st} f_T(t)\,dt = \int_0^1 e^{-st} e^{-t}\,dt + \int_1^2 e^{-st}\,dt \\ &= \left.\frac{e^{-(s+1)t}}{-(s+1)}\right|_0^1 + \left.\frac{e^{-st}}{-s}\right|_1^2 = \frac{1 - e^{-(s+1)}}{s+1} + \frac{e^{-s} - e^{-2s}}{s} \end{aligned}$$

and, by (12),

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{1 - e^{-2s}} \left[ \frac{1 - e^{-(s+1)}}{s+1} + \frac{e^{-s} - e^{-2s}}{s} \right].$$

The graph of $f(t)$ is shown in Figure B.46 in the answers of the text.



**25.** Similarly to Example 6 on page 392 of the text, $f(t)$ is a periodic function with period $T = 2a$, whose windowed version has the form

$$f_{2a}(t) = 1 - u(t-a), \qquad 0 < t < 2a.$$

Thus, using the linearity of the Laplace transform and formula (4) on page 386 for the Laplace transform of the unit step function, we have

$$F_{2a}(s) = \mathcal{L}\{f_{2a}(t)\}(s) = \mathcal{L}\{1\}(s) - \mathcal{L}\{u(t-a)\}(s) = \frac{1}{s} - \frac{e^{-as}}{s} = \frac{1 - e^{-as}}{s}.$$

Applying now Theorem 9 yields

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{1 - e^{-2as}} \frac{1 - e^{-as}}{s} = \frac{1}{(1 - e^{-as})(1 + e^{-as})} \frac{1 - e^{-as}}{s} = \frac{1}{s(1 + e^{-as})}.$$

**27.** Observe that if we let

$$f_{2a}(t) = \begin{cases} f(t), & 0 < t < 2a, \\ 0, & \text{otherwise,} \end{cases}$$

denote the windowed version of $f(t)$, then from formula (12) on page 391 of the text we have

$$\mathcal{L}\{f(t)\}(s) = \frac{\mathcal{L}\{f_{2a}(t)\}(s)}{1-e^{-2as}} = \frac{\mathcal{L}\{f_{2a}(t)\}(s)}{(1-e^{-as})(1+e^{-as})}.$$

Now

$$\begin{aligned} f_{2a}(t) &= \frac{t}{a} + \left[\left(2-\frac{t}{a}\right) - \frac{t}{a}\right] u(t-a) + \left[0 - \left(2-\frac{t}{a}\right)\right] u(t-2a) \\ &= \frac{t}{a} - \frac{2(t-a)u(t-a)}{a} + \frac{(t-2a)u(t-2a)}{a}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{L}\{f_{2a}(t)\}(s) &= \frac{1}{a}\mathcal{L}\{t\}(s) - \frac{2}{a}\mathcal{L}\{(t-a)u(t-a)\}(s) + \frac{1}{a}\mathcal{L}\{(t-2a)u(t-2a)\}(s) \\ &= \frac{1}{a}\frac{1}{s^2} - \frac{2}{a}\frac{e^{-as}}{s^2} + \frac{1}{a}\frac{e^{-as}}{s^2} = \frac{1}{as^2}\left(1-2e^{-as}+e^{-2as}\right) = \frac{(1-e^{-as})^2}{as^2} \end{aligned}$$

and

$$\mathcal{L}\{f(t)\}(s) = \frac{(1-e^{-as})^2/(as^2)}{(1-e^{-as})(1+e^{-as})} = \frac{1-e^{-as}}{as^2(1+e^{-as})}.$$

**29.** Applying the Laplace transform to both sides of the given differential equation, we obtain

$$\mathcal{L}\{y''\}(s) + \mathcal{L}\{y\}(s) = \mathcal{L}\{u(t-3)\}(s) = \frac{e^{-3s}}{s}.$$

Since

$$\mathcal{L}\{y''\}(s) = s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0) = s^2\mathcal{L}\{y\}(s) - 1,$$

substitution yields

$$\begin{aligned} &\left[s^2\mathcal{L}\{y\}(s) - 1\right] + \mathcal{L}\{y\}(s) = \frac{e^{-3s}}{s} \\ \Rightarrow\quad &\mathcal{L}\{y\}(s) = \frac{1}{s^2+1} + \frac{e^{-3s}}{s(s^2+1)} = \frac{1}{s^2+1} + e^{-3s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]. \end{aligned}$$

By formula (6) on page 387 of the text,

$$\mathcal{L}^{-1}\left\{e^{-3s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\}(t-3)u(t-3) = [1-\cos(t-3)]u(t-3).$$

Hence

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} + e^{-3s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]\right\}(t) = \sin t + [1-\cos(t-3)]u(t-3)$$

The graph of the solution is shown in Figure B.47 in the answers of the text.

**31.** We apply the Laplace transform to both sides of the differential equation and get

$$\mathcal{L}\left\{y''\right\}(s)+\mathcal{L}\left\{y\right\}(s)=\mathcal{L}\left\{t-(t-4)u(t-2)\right\}(s)=\frac{1}{s^2}-\mathcal{L}\left\{(t-4)u(t-2)\right\}(s). \qquad (7.33)$$

Since $(t-4)u(t-2)=[(t-2)-2]u(t-2)$, we can use formula (5) from Theorem 8 to find its Laplace transform. With $f(t)=t-2$ and $a=2$, this formula yields

$$\mathcal{L}\left\{(t-4)u(t-2)\right\}(s)=e^{-2s}\mathcal{L}\left\{t-2\right\}(s)=e^{-2s}\left[\frac{1}{s^2}-\frac{2}{s}\right].$$

Also,

$$\mathcal{L}\left\{y''\right\}(s)=s^2\mathcal{L}\left\{y\right\}(s)-sy(0)-y'(0)=s^2\mathcal{L}\left\{y\right\}(s)-1.$$

Substitution back into (7.33) yields

$$\left[s^2\mathcal{L}\left\{y\right\}(s)-1\right]+\mathcal{L}\left\{y\right\}(s)=\frac{1}{s^2}-e^{-2s}\left[\frac{1}{s^2}-\frac{2}{s}\right]$$

$$\Rightarrow \qquad \mathcal{L}\left\{y\right\}(s)=\frac{1}{s^2}-e^{-2s}\frac{1-2s}{s^2(s^2+1)}=\frac{1}{s^2}-e^{-2s}\left[\frac{1}{s^2}-\frac{2}{s}+\frac{2s}{s^2+1}-\frac{1}{s^2+1}\right].$$

Applying now the inverse Laplace transform and using formula (6) on page 387 of the text, we obtain

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}-e^{-2s}\left[\frac{1}{s^2}-\frac{2}{s}+\frac{2s}{s^2+1}-\frac{1}{s^2+1}\right]\right\}(t) \\ &= t-\mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{2}{s}+\frac{2s}{s^2+1}-\frac{1}{s^2+1}\right\}(t-2)u(t-2) \\ &= t-\left[(t-2)-2+2\cos(t-2)-\sin(t-2)\right]u(t-2) \\ &= t+\left[4-t+\sin(t-2)-2\cos(t-2)\right]u(t-2). \end{aligned}$$

See Figure B.48 in the answers of the text.

**33.** By formula (4) on page 386 of the text,

$$\mathcal{L}\left\{u(t-2\pi)-u(t-4\pi)\right\}(s)=\frac{e^{-2\pi s}}{s}-\frac{e^{-4\pi s}}{s}.$$

Thus, taking the Laplace transform of $y''+2y'+2y=u(t-2\pi)-u(t-4\pi)$ and applying the initial conditions $y(0)=y'(0)$ gives us

$$\left[s^2Y(s)-s-1\right]+2\left[sY(s)-1\right]+2Y(s)=\frac{e^{-2\pi s}-e^{-4\pi s}}{s},$$

where $Y(s)$ is the Laplace transform of $y(t)$. Solving for $Y(s)$ yields

$$\begin{aligned} Y(s) &= \frac{s+3}{s^2+2s+2} + \frac{e^{-2\pi s} - e^{-4\pi s}}{s(s^2+2s+2)} \\ &= \frac{s+1}{(s+1)^2+1^2} + \frac{2(1)}{(s+1)^2+1^2} + \frac{e^{-2\pi s}}{s[(s+1)^2+1^2]} - \frac{e^{-4\pi s}}{s[(s+1)^2+1^2]}. \end{aligned} \quad (7.34)$$

Since

$$\frac{1}{s[(s+1)^2+1^2]} = \frac{1}{2}\frac{(s^2+2s+2)-(s^2+2s)}{s[(s+1)^2+1^2]} = \frac{1}{2}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+1^2} - \frac{1}{(s+1)^2+1^2}\right],$$

we have

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s[(s+1)^2+1^2]}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{2}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+1^2} - \frac{1}{(s+1)^2+1^2}\right]\right\}(t) \\ &= \frac{1}{2}\left[1 - e^{-t}\cos t - e^{-t}\sin t\right] \end{aligned}$$

and, by formula (6) on page 387 of the text,

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s[(s+1)^2+1^2]}\right\}(t) &= \frac{1}{2}\left[1 - e^{-(t-2\pi)}\cos(t-2\pi) - e^{-(t-2\pi)}\sin(t-2\pi)\right]u(t-2\pi) \\ &= \frac{1}{2}\left[1 - e^{2\pi-t}(\cos t + \sin t)\right]u(t-2\pi) \\ \mathcal{L}^{-1}\left\{\frac{e^{-4\pi s}}{s[(s+1)^2+1^2]}\right\}(t) &= \frac{1}{2}\left[1 - e^{-(t-4\pi)}\cos(t-4\pi) - e^{-(t-4\pi)}\sin(t-4\pi)\right]u(t-4\pi) \\ &= \frac{1}{2}\left[1 - e^{4\pi-t}(\cos t + \sin t)\right]u(t-4\pi). \end{aligned}$$

Finally, taking the inverse Laplace transform in (7.34) yields

$$\begin{aligned} y(t) = e^{-t}\cos t + 2e^{-t}\sin t &+ \frac{1}{2}\left[1 - e^{2\pi-t}(\cos t + \sin t)\right]u(t-2\pi) \\ &- \frac{1}{2}\left[1 - e^{4\pi-t}(\cos t + \sin t)\right]u(t-4\pi). \end{aligned}$$

**35.** We take the Laplace transform of the both sides of the given equation and obtain

$$\mathcal{L}\{z''\}(s) + 3\mathcal{L}\{z'\}(s) + 2\mathcal{L}\{z\}(s) = \mathcal{L}\left\{e^{-3t}u(t-2)\right\}(s). \quad (7.35)$$

We use the initial conditions, $z(0) = 2$ and $z'(0) = -3$, and formula (4) from Section 7.3 to express $\mathcal{L}\{z'\}(s)$ and $\mathcal{L}\{z''\}(s)$ in terms of $Z(s) := \mathcal{L}\{z\}(s)$. That is,

$$\mathcal{L}\{z'\}(s) = sZ(s) - z(0) = sZ(s) - 2, \quad \mathcal{L}\{z''\}(s) = s^2Z(s) - sz(0) - z'(0) = s^2Z(s) - 2s + 3.$$

In the right-hand side of (7.35), we can use, say, the translation property of the Laplace transform (Theorem 3, Section 7.3) and the Laplace transform of the unit step function (formula (4), Section 7.6).

$$\mathcal{L}\left\{e^{-3t}u(t-2)\right\}(s) = \mathcal{L}\{u(t-2)\}(s+3) = \frac{e^{-2(s+3)}}{s+3}.$$

Therefore, (7.35) becomes

$$\begin{aligned}
&\left[s^2Z(s) - 2s + 3\right] + 3\left[sZ(s) - 2\right] + 2Z(s) = \frac{e^{-2(s+3)}}{s+3} \\
\Rightarrow \quad &\left(s^2 + 3s + 2\right)Z(s) = 2s + 3 + \frac{e^{-2(s+3)}}{s+3} \\
\Rightarrow \quad &Z(s) = \frac{2s+3}{s^2+3s+2} + e^{-2s-6}\frac{1}{(s+3)(s^2+3s+2)} \\
&= \frac{1}{s+1} + \frac{1}{s+2} + e^{-2s-6}\left[\frac{1/2}{s+3} - \frac{1}{s+2} + \frac{1/2}{s+1}\right].
\end{aligned}$$



Hence,

$$\begin{aligned}
z(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{1}{s+2} + e^{-6}e^{-2s}\left[\frac{1/2}{s+3} - \frac{1}{s+2} + \frac{1/2}{s+1}\right]\right\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) \\
&\quad + \frac{e^{-6}}{2}\left[\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}\right](t-2)u(t-2) \\
&= e^{-t} + e^{-2t} + \frac{e^{-6}}{2}\left[e^{-3(t-2)} - 2e^{-2(t-2)} + e^{-(t-2)}\right]u(t-2) \\
&= e^{-t} + e^{-2t} + \frac{1}{2}\left[e^{-3t} - 2e^{-2(t+1)} + e^{-(t+4)}\right]u(t-2)
\end{aligned}$$

**37.** Since

$$\mathcal{L}\{g(t)\}(s) = \int_0^\infty e^{-st}g(t)\,dt = \int_0^{2\pi} e^{-st}\sin t\,dt = \frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\Big|_0^{2\pi} = \frac{1-e^{-2\pi s}}{s^2+1},$$

applying the Laplace transform to the original equation yields

$$\begin{aligned}
&\mathcal{L}\{y''\}(s) + 4\mathcal{L}\{y\}(s) = \mathcal{L}\{g(t)\}(s) \\
\Rightarrow \quad & \left[s^2\mathcal{L}\{y\}(s) - s - 3\right] + 4\mathcal{L}\{y\}(s) = \frac{1-e^{-2\pi s}}{s^2+1} \\
\Rightarrow \quad & \mathcal{L}\{y\}(s) = \frac{s+3}{s^2+4} + \frac{1}{(s^2+1)(s^2+4)} - \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}.
\end{aligned}$$

Using the partial fractions decomposition

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{(s^2+4)-(s^2+1)}{(s^2+1)(s^2+4)} = \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right],$$

we conclude that

$$\mathcal{L}\{y\}(s) = \frac{s}{s^2+4} + \frac{4}{3}\frac{2}{s^2+4} + \frac{1}{3}\frac{1}{s^2+1} - e^{-2\pi s}\left[\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right]$$

and so

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}(t) + \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s^2+1}\right\}(t) \\
&\quad - \mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right\}(t-2\pi)u(t-2\pi) \\
&= \cos 2t + \frac{4}{3}\sin 2t + \frac{1}{3}\sin t - \left[\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin 2(t-2\pi)\right]u(t-2\pi) \\
&= \cos 2t + \frac{4}{3}\sin 2t + \frac{1}{3}\sin t - \left[\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right]u(t-2\pi) \\
&= \cos 2t + \frac{1}{3}\left[1 - u(t-2\pi)\right]\sin t + \frac{1}{6}\left[8 + u(t-2\pi)\right]\sin 2t.
\end{aligned}$$



**39.** We can express $g(t)$ using the unit step function as

$$g(t) = tu(t-1) + (1-t)u(t-5) = [(t-1)+1]u(t-1) - [(t-5)+4]u(t-5).$$

Thus, formula (5) on page 387 of the text yields

$$\mathcal{L}\{g(t)\}(s) = e^{-s}\mathcal{L}\{t+1\}(s) - e^{-5s}\mathcal{L}\{t+4\}(s) = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) - e^{-5s}\left(\frac{1}{s^2} + \frac{4}{s}\right).$$

Let $Y(s) = \mathcal{L}\{y\}(s)$. Applying the Laplace transform to the given equation and using the initial conditions, we obtain

$$\begin{aligned}
&\mathcal{L}\{y''\}(s) + 5\mathcal{L}\{y'\}(s) + 6Y(s) = \mathcal{L}\{g(t)\}(s)\\
\Rightarrow\quad & \left[s^2Y(s) - 2\right] + 5\left[sY(s)\right] + 6Y(s) = \mathcal{L}\{g(t)\}(s)\\
\Rightarrow\quad & \left(s^2+5s+6\right)Y(s) = 2 + e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right) - e^{-5s}\left(\frac{1}{s^2}+\frac{4}{s}\right)\\
\Rightarrow\quad & Y(s) = \frac{2}{s^2+5s+6} + e^{-s}\frac{s+1}{s^2(s^2+5s+6)} - e^{-5s}\frac{4s+1}{s^2(s^2+5s+6)}\,. \qquad (7.36)
\end{aligned}$$

Using partial fractions decomposition, we can write

$$\begin{aligned}
\frac{2}{s^2+5s+6} &= \frac{2}{s+2} - \frac{2}{s+3}\,,\\
\frac{s+1}{s^2(s^2+5s+6)} &= \frac{1/36}{s} + \frac{1/6}{s^2} - \frac{1/4}{s+2} + \frac{2/9}{s+3}\,,\\
\frac{4s+1}{s^2(s^2+5s+6)} &= \frac{1/6}{s^2} + \frac{19/36}{s} - \frac{7/4}{s+2} + \frac{11/9}{s+3}\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{2}{s^2+5s+6}\right\}(t) &= 2e^{-2t} - 2e^{-3t}\,,\\
\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s^2+5s+6)}\right\}(t) &= \frac{1}{36} + \frac{t}{6} - \frac{e^{-2t}}{4} + \frac{2e^{-3t}}{9}\,,\\
\mathcal{L}^{-1}\left\{\frac{4s+1}{s^2(s^2+5s+6)}\right\}(t) &= \frac{19}{36} + \frac{t}{6} - \frac{7e^{-2t}}{4} + \frac{11e^{-3t}}{9}\,.
\end{aligned}$$

Using these equations and taking the inverse Laplace transform in (7.36), we finally obtain

$$\begin{aligned}
y(t) = 2e^{-2t} - 2e^{-3t} &+ \left[\frac{1}{36} + \frac{t-1}{6} - \frac{e^{-2(t-1)}}{4} + \frac{2e^{-3(t-1)}}{9}\right]u(t-1)\\
&+ \left[\frac{19}{36} + \frac{t-5}{6} - \frac{7e^{-2(t-5)}}{4} + \frac{11e^{-3(t-5)}}{9}\right]u(t-5).
\end{aligned}$$

**41.** First observe that for $s > 0$, $T > 0$, we have $0 < e^{-Ts} < 1$ so that

$$\frac{1}{1-e^{-Ts}} = 1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \cdots \qquad (7.37)$$

and the series converges for all $s > 0$. Thus,

$$\begin{aligned}\frac{1}{(s+\alpha)(1-e^{-Ts})} = \frac{1}{s+\alpha}\,\frac{1}{1-e^{-Ts}} &= \frac{1}{s+\alpha}\left(1+e^{-Ts}+e^{-2Ts}+e^{-3Ts}+\cdots\right)\\ &= \frac{1}{s+\alpha}+\frac{e^{-Ts}}{s+\alpha}+\frac{e^{-2Ts}}{s+\alpha}+\cdots,\end{aligned}$$

and so

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+\alpha)(1-e^{-Ts})}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+\alpha}+\frac{e^{-Ts}}{s+\alpha}+\frac{e^{-2Ts}}{s+\alpha}+\cdots\right\}(t). \tag{7.38}$$

Taking for granted that the linearity of the inverse Laplace transform extends to the infinite sum in (7.38) and ignoring convergence questions yields

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s+\alpha)(1-e^{-Ts})}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s+\alpha}\right\}+\mathcal{L}^{-1}\left\{\frac{e^{-Ts}}{s+\alpha}\right\}+\mathcal{L}^{-1}\left\{\frac{e^{-2Ts}}{s+\alpha}\right\}+\cdots\\ &= e^{-\alpha t}+e^{-\alpha(t-T)}u(t-T)+e^{-\alpha(t-2T)}u(t-2T)+\cdots\end{aligned}$$

as claimed.



**43.** Using the expansion (7.37) obtained in Problem 41, we can represent $\mathcal{L}\{g\}(s)$ as

$$\begin{aligned}\mathcal{L}\{g\}(s) = \frac{\beta}{s^2+\beta^2}\,\frac{1}{1-e^{-Ts}} &= \frac{\beta}{s^2+\beta^2}\left(1+e^{-Ts}+e^{-2Ts}+e^{-3Ts}+\cdots\right)\\ &= \frac{\beta}{s^2+\beta^2}+e^{-Ts}\frac{\beta}{s^2+\beta^2}+e^{-2Ts}\frac{\beta}{s^2+\beta^2}+\cdots.\end{aligned}$$

Since $\mathcal{L}^{-1}\{\beta/(s^2+\beta^2)\}(t) = \sin\beta t$, using the linearity of the inverse Laplace transform (extended to infinite series) and formula (6) in Theorem 8, we obtain

$$\begin{aligned}g(t) &= \mathcal{L}^{-1}\left\{\frac{\beta}{s^2+\beta^2}\right\}(t)+\mathcal{L}^{-1}\left\{\frac{\beta}{s^2+\beta^2}\right\}(t-T)u(t-T)\\ &\qquad +\mathcal{L}^{-1}\left\{\frac{\beta}{s^2+\beta^2}\right\}(t-2T)u(t-2T)+\cdots\\ &= \sin\beta t+[\sin\beta(t-T)]u(t-T)+[\sin\beta(t-2T)]u(t-2T)+\cdots\end{aligned}$$

as stated.

**45.** In order to apply the method of Laplace transform to given initial value problem, let us find $\mathcal{L}\{f\}(s)$ first. Since the period of $f(t)$ is $T = 1$ and $f(t) = e^t$ on $(0, 1)$, the windowed version of $f(t)$ is

$$f_1(t) = \begin{cases} e^t, & 0 < t < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and so

$$F_1(s) = \int_0^\infty e^{-st} f_1(t)\, dt = \int_0^1 e^{-st} e^t\, dt = \left.\frac{e^{(1-s)t}}{1-s}\right|_0^1 = \frac{1-e^{1-s}}{s-1}\,.$$

Hence, Theorem 9 yields the following formula for $\mathcal{L}\{f\}(s)$:

$$\mathcal{L}\{f\}(s) = \frac{1-e^{1-s}}{(s-1)(1-e^{-s})}\,.$$

We can now apply the Laplace transform to the given differential equation and obtain

$$\mathcal{L}\{y''\}(s) + 3\mathcal{L}\{y'\}(s) + 2\mathcal{L}\{y\}(s) = \frac{1-e^{1-s}}{(s-1)(1-e^{-s})}$$

$$\Rightarrow \quad \left[s^2\mathcal{L}\{y\}(s)\right] + 3\left[s\mathcal{L}\{y\}(s)\right] + 2\mathcal{L}\{y\}(s) = \frac{1-e^{1-s}}{(s-1)(1-e^{-s})}$$

$$\Rightarrow \quad \mathcal{L}\{y\}(s) = \frac{1-e^{1-s}}{(s-1)(s^2+3s+2)(1-e^{-s})} = \frac{1-e^{1-s}}{(s-1)(s+1)(s+2)(1-e^{-s})}$$

$$\Rightarrow \quad \mathcal{L}\{y\}(s) = \frac{e}{(s-1)(s+1)(s+2)} + \frac{1-e}{1-e^{-s}}\,\frac{1}{(s-1)(s+1)(s+2)}\,.$$

Using the partial fractions decomposition

$$\frac{1}{(s-1)(s+1)(s+2)} = \frac{1/6}{s-1} - \frac{1/2}{s+1} + \frac{1/3}{s+2}$$

we find that

$$\mathcal{L}\{y\}(s) = \frac{e/6}{s-1} - \frac{e/2}{s+1} + \frac{e/3}{s+2} + \frac{1-e}{6}\,\frac{1}{(s-1)(1-e^{-s})} - \frac{1-e}{2}\,\frac{1}{(s+1)(1-e^{-s})} + \frac{1-e}{3}\,\frac{1}{(s+2)(1-e^{-s})}$$

$$\Rightarrow \quad y(t) = \frac{e}{6}e^t - \frac{e}{2}e^{-t} + \frac{e}{3}e^{-2t} + \frac{1-e}{6}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(1-e^{-s})}\right\}(t) - \frac{1-e}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(1-e^{-s})}\right\}(t) + \frac{1-e}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(1-e^{-s})}\right\}(t). \quad (7.39)$$

To each of the three inverse Laplace transforms in the above formula we can apply results of Problem 42(a) with $T = 1$ and $\alpha = -1$, 1, and 2, respectively. Thus, for $n < t < n+1$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(1-e^{-s})}\right\}(t) = e^{t}\left[\frac{e^{-(n+1)}-1}{e^{-1}-1}\right],$$
$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(1-e^{-s})}\right\}(t) = e^{-t}\left[\frac{e^{n+1}-1}{e-1}\right],$$
$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(1-e^{-s})}\right\}(t) = e^{-2t}\left[\frac{e^{2(n+1)}-1}{e^{2}-1}\right].$$

Finally, substitution back into (7.39) yields

$$\begin{aligned} y(t) &= \frac{e}{6}e^{t} - \frac{e}{2}e^{-t} + \frac{e}{3}e^{-2t} + \frac{1-e}{6}e^{t}\left[\frac{e^{-(n+1)}-1}{e^{-1}-1}\right] \\ &\qquad - \frac{1-e}{2}e^{-t}\left[\frac{e^{n+1}-1}{e-1}\right] + \frac{1-e}{3}e^{-2t}\left[\frac{e^{2(n+1)}-1}{e^{2}-1}\right] \\ &= \frac{e^{t-n}}{6} - \frac{e^{-t}\left(1+e-e^{n+1}\right)}{2} + \frac{e^{-2t}\left(1+e+e^{2}-e^{2n+2}\right)}{3(e+1)}. \end{aligned}$$

**47.** Since

$$e^{t} = \sum_{k=0}^{\infty}\frac{t^{k}}{k!}$$

and

$$\mathcal{L}\left\{t^{k}\right\}(s) = \frac{k!}{s^{k+1}},$$

using the linearity of the Laplace transform we have

$$\mathcal{L}\left\{e^{t}\right\}(s) = \mathcal{L}\left\{\sum_{k=0}^{\infty}\frac{t^{k}}{k!}\right\}(s) = \sum_{k=0}^{\infty}\frac{\mathcal{L}\left\{t^{k}\right\}(s)}{k!} = \sum_{k=0}^{\infty}\frac{k!/s^{k+1}}{k!} = \frac{1}{s}\sum_{k=0}^{\infty}\left(\frac{1}{s}\right)^{k}. \qquad (7.40)$$

We can apply now the summation formula for geometric series, that is,

$$1 + x + x^{2} + \cdots = \frac{1}{1-x},$$

which is valid for $|x| < 1$. With $x = 1/s$, $s > 1$, (7.40) yields

$$\mathcal{L}\left\{e^{t}\right\}(s) = \frac{1}{s}\frac{1}{1-(1/s)} = \frac{1}{s-1}.$$

**49.** Recall that the Taylor's series for $\cos t$ about $t = 0$ is

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots + (-1)^n \frac{t^{2n}}{(2n)!} + \cdots$$

so that

$$\frac{1 - \cos t}{t} = \frac{t}{2!} - \frac{t^3}{4!} + \frac{t^5}{6!} + \cdots + (-1)^{n+1} \frac{t^{2n-1}}{(2n)!} + \cdots .$$

Thus

$$\begin{aligned} \mathcal{L}\left\{\frac{1 - \cos t}{t}\right\}(s) &= \frac{1}{2!}\mathcal{L}\{t\}(s) - \frac{1}{4!}\mathcal{L}\{t^3\}(s) + \cdots + \frac{(-1)^{n+1}}{(2n)!}\mathcal{L}\{t^{2n-1}\}(s) + \cdots \\ &= \frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s^4} + \cdots + \frac{(-1)^{n+1}}{2n}\frac{1}{s^{2n}} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}\frac{1}{s^{2n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2ns^{2n}} . \end{aligned}$$

To sum this series, recall that

$$\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} .$$

Hence,

$$\ln\left(1 + \frac{1}{s^2}\right) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{ns^{2n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{ns^{2n}} .$$

Thus, we have

$$\frac{1}{2}\ln\left(1 + \frac{1}{s^2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2ns^{2n}} = \mathcal{L}\left\{\frac{1 - \cos t}{t}\right\}(s) .$$

This formula can also be obtained by using the result of Problem 27 in Section 7.3 of the text.

**51.** We use formula (17) on page 394 of the text.

**(a)** With $r = -1/2$, (17) yields

$$\mathcal{L}\left\{t^{-1/2}\right\}(s) = \frac{\Gamma[(-1/2) + 1]}{s^{(-1/2)+1}} = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}} .$$

**(b)** This time, $r = 7/2$, and (17) becomes

$$\mathcal{L}\left\{t^{7/2}\right\}(s) = \frac{\Gamma[(7/2) + 1]}{s^{(7/2)+1}} = \frac{\Gamma(9/2)}{s^{9/2}} .$$

From the recursive formula (16) we find that

$$\Gamma\left(\frac{9}{2}\right)=\Gamma\left(\frac{7}{2}+1\right)=\frac{7}{2}\Gamma\left(\frac{7}{2}\right)=\frac{7}{2}\frac{5}{2}\Gamma\left(\frac{5}{2}\right)=\frac{7}{2}\frac{5}{2}\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{105\sqrt{\pi}}{16}.$$

Therefore,

$$\mathcal{L}\left\{t^{7/2}\right\}(s)=\frac{105\sqrt{\pi}}{16s^{9/2}}.$$

**53.** According to the definition (11) of the function $f_T(t)$, $f_T(t-kT)=0$ if the point $t-kT$ does not belong to $(0,T)$. Therefore, fixed $t$, in the series (13) all the terms containing $f_T(t-kT)$ with $k$'s such that $t-kT\le 0$ or $t-kT\ge T$ vanish. In the remaining terms, $k$ satisfies

$$0<t-kT<T \qquad \Leftrightarrow \qquad \frac{t}{T}-1<k<\frac{t}{T}.$$

But, for any fixed $t$, there is at most one $k$ satisfying this condition.



**55.** Recall that

$$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots.$$

Substituting $-1/s$ for $x$ above yields

$$e^{-1/s}=1-\frac{1}{s}+\frac{1}{2!s^2}-\frac{1}{3!s^3}+\cdots+\frac{(-1)^n}{n!s^n}+\cdots.$$

Thus, we have

$$s^{-1/2}e^{-1/s}=\frac{1}{s^{1/2}}-\frac{1}{s^{3/2}}+\frac{1}{2!s^{5/2}}+\cdots+\frac{(-1)^n}{n!s^{n+1/2}}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!s^{n+1/2}}.$$

By Problem 52 of this section,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{n+(1/2)}}\right\}(t)=\frac{2^n t^{n-(1/2)}}{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}},$$

so that

$$\begin{aligned}\mathcal{L}^{-1}\left\{s^{-1/2}e^{-1/s}\right\}(t) &= \mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{n!s^{n+1/2}}\right\}(t)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\mathcal{L}^{-1}\left\{\frac{1}{s^{n+(1/2)}}\right\}(t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{2^n t^{n-(1/2)}}{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}.\end{aligned}$$

Multiplying the $n$th term by $[2 \cdot 4 \cdots (2n)]/[2 \cdot 4 \cdots (2n)]$, we obtain

$$\begin{aligned}\mathcal{L}^{-1}\left\{s^{-1/2}e^{-1/s}\right\}(t) &= \sum_{n=0}^{\infty}\frac{(-1)^n(2^n)^2t^{n-(1/2)}}{(2n)!\sqrt{\pi}} = \sum_{n=0}^{\infty}\frac{(-1)^n(2^n)^2t^n}{(2n)!\sqrt{\pi t}} \\ &= \left(\frac{1}{\sqrt{\pi t}}\right)\sum_{n=0}^{\infty}\frac{(-1)^n(2\sqrt{t})^{2n}}{(2n)!} = \left(\frac{1}{\sqrt{\pi t}}\right)\cos\left(2\sqrt{t}\right).\end{aligned}$$

**57.** Recall that the Maclaurin expansion of $\ln(1-x)$ is

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n},$$

which converges for $|x| < 1$. Hence, substitution $-1/s^2$ for $x$ yields

$$\ln\left(1+\frac{1}{s^2}\right) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{ns^{2n}} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{ns^{2n}}.$$

Assuming that the inverse Laplace transform can be computed termwise, we obtain

$$\mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s^2}\right)\right\} = \mathcal{L}^{-1}\left\{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{ns^{2n}}\right\} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\mathcal{L}^{-1}\left\{\frac{1}{s^{2n}}\right\}.$$

From Table 7.1 in Section 7.2, $\mathcal{L}\left\{t^k\right\} = k!/s^{k+1}$, $k = 1, 2, \ldots$. Thus $\mathcal{L}^{-1}\left\{1/s^{k+1}\right\} = t^k/k!$. With $k = 2n-1$, this yields

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{2n}}\right\}(t) = \frac{t^{2n-1}}{(2n-1)!}, \qquad n = 1, 2, \ldots$$

and, therefore,

$$\mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s^2}\right)\right\}(t) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\,\frac{t^{2n-1}}{(2n-1)!} = -\frac{2}{t}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}t^{2n}. \tag{7.41}$$

Since

$$\cos t = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}t^{2n} = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}t^{2n},$$

(7.41) implies that

$$\mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s^2}\right)\right\}(t) = -\frac{2}{t}\left(\cos t - 1\right) = \frac{2(1-\cos t)}{t}.$$

**59.** Applying the Laplace transform to both sides of the original equation and using its linearity, we obtain

$$\mathcal{L}\{y''\}(s) - \mathcal{L}\{y\}(s) = \mathcal{L}\{G_3(t-1)\}(s). \tag{7.42}$$

Initial conditions, $y(0) = 0$ and $y'(0)$=2, and Theorem 5 in Section 7.3 imply that

$$\mathcal{L}\{y''\}(s) = s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0) = s^2\mathcal{L}\{y\}(s) - 2.$$

In the right-hand side of (7.42), we can apply the result of Problem 58(c) with $a = 3$ and $b = 1$ to get

$$\mathcal{L}\{G_3(t-1)\}(s) = \frac{e^{-s} - e^{-4s}}{s}.$$

Thus (7.42) becomes

$$\begin{aligned} \left[s^2\mathcal{L}\{y\}(s) - 2\right] - \mathcal{L}\{y\}(s) &= \frac{e^{-s} - e^{-4s}}{s} \\ \Rightarrow \qquad \mathcal{L}\{y\}(s) &= \frac{2}{s^2-1} + \frac{e^{-s} - e^{-4s}}{s(s^2-1)}. \end{aligned}$$



Substituting partial fraction decompositions

$$\frac{2}{s^2-1} = \frac{1}{s-1} - \frac{1}{s+1}, \qquad \frac{1}{s(s^2-1)} = \frac{1/2}{s-1} + \frac{1/2}{s+1} - \frac{1}{s}$$

yields

$$\begin{aligned} \mathcal{L}\{y\}(s) &= \frac{1}{s-1} - \frac{1}{s+1} + \left(e^{-s} - e^{-4s}\right)\left[\frac{1/2}{s-1} + \frac{1/2}{s+1} - \frac{1}{s}\right] \\ &= \frac{1}{s-1} - \frac{1}{s+1} + e^{-s}\left[\frac{1/2}{s-1} + \frac{1/2}{s+1} - \frac{1}{s}\right] - e^{-4s}\left[\frac{1/2}{s-1} + \frac{1/2}{s+1} - \frac{1}{s}\right]. \end{aligned} \tag{7.43}$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1/2}{s-1} + \frac{1/2}{s+1} - \frac{1}{s}\right\}(t) = \frac{e^t + e^{-t} - 2}{2},$$

formula (6) on page 387 of the text gives us

$$\begin{aligned} \mathcal{L}^{-1}\left\{e^{-s}\left[\frac{1/2}{s-1} + \frac{1/2}{s+1} - \frac{1}{s}\right]\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{1/2}{s-1} + \frac{1/2}{s+1} - \frac{1}{s}\right\}(t-1)u(t-1) \\ &= \frac{e^{t-1} + e^{1-t} - 2}{2}u(t-1), \end{aligned}$$

$$\mathcal{L}^{-1}\left\{e^{-4s}\left[\frac{1/2}{s-1}+\frac{1/2}{s+1}-\frac{1}{s}\right]\right\}(t)=\mathcal{L}^{-1}\left\{\frac{1/2}{s-1}+\frac{1/2}{s+1}-\frac{1}{s}\right\}(t-4)u(t-4)$$
$$=\frac{e^{t-4}+e^{4-t}-2}{2}u(t-4).$$

Taking the inverse Laplace transform in (7.43) yields

$$y(t)=e^t-e^{-t}+\frac{e^{t-1}+e^{1-t}-2}{2}u(t-1)-\frac{e^{t-4}+e^{4-t}-2}{2}u(t-4).$$

**61.** In this problem, we use the method of solving "mixing problems" discussed in Section 3.2. So, let $x(t)$ denote the mass of salt in the tank at time $t$ with $t=0$ denoting the moment when the process started. Thus, using the formula

$$\text{mass} = \text{volume} \times \text{concentration}\,,$$

we have the initial condition

$$x(0)=500\,(\text{L})\times 0.2\,(\text{kg/L})=100\,(\text{kg}).$$

For the rate of change of $x(t)$, that is, $x'(t)$, we use then relation

$$x'(t)=\text{input rate}-\text{output rate}\,. \tag{7.44}$$



While the output rate (through the exit valve $C$) can be computed as

$$\text{output rate}=\frac{x(t)}{500}\,(\text{kg/L})\times 12\,(\text{L/min})=\frac{3x(t)}{125}\,(\text{kg/min})$$

for all $t$, the input rate has different formulas for the first 10 minute and after that. Namely,

$$0<t<10\ (\text{valve } A):\qquad \text{input rate}=12\,(\text{L/min})\times 0.4\,(\text{kg/L})=4.8\,(\text{kg/min});$$
$$10<t\ (\text{valve } B):\qquad \text{input rate}=12\,(\text{L/min})\times 0.6\,(\text{kg/L})=7.2\,(\text{kg/min}).$$

In other words, the input rate is a function of $t$, which can be written as

$$\text{input rate}=g(t)=\begin{cases}4.8, & 0<t<10,\\ 7.2, & 10<t.\end{cases}$$

Using the unit step function, we can express $g(t) = 4.8 + 2.4u(t-10)$ (kg/min). Therefore (7.44) becomes

$$x'(t) = g(t) - \frac{3x(t)}{125} \qquad \Rightarrow \qquad x'(t) + \frac{3}{125}x(t) = 4.8 + 2.4u(t-10) \tag{7.45}$$

with the initial condition $x(0) = 100$. Taking the Laplace transform of both sides yields

$$\begin{aligned} &\mathcal{L}\{x'\}(s) + \frac{3}{125}\mathcal{L}\{x\}(s) = \mathcal{L}\{4.8 + 2.4u(t-10)\}(s) = \frac{4.8}{s} + \frac{2.4e^{-10s}}{s} \\ \Rightarrow \quad & [s\mathcal{L}\{x\}(s) - 100] + \frac{3}{125}\mathcal{L}\{x\}(s) = \frac{4.8}{s} + \frac{2.4e^{-10s}}{s} \\ \Rightarrow \quad & \mathcal{L}\{x\}(s) = \frac{100s + 4.8}{s[s + (3/125)]} + \frac{2.4}{s[s + (3/125)]}e^{-10s}. \end{aligned} \tag{7.46}$$

Since

$$\begin{aligned} \frac{2.4}{s[s + (3/125)]} &= 100\left(\frac{1}{s} - \frac{1}{s + (3/125)}\right), \\ \frac{100s + 4.8}{s[s + (3/125)]} &= 100\left(\frac{2}{s} - \frac{1}{s + (3/125)}\right), \end{aligned}$$

applying the inverse Laplace transform in (7.46), we get

$$x(t) = 100\left(2 - e^{-3t/125}\right) + 100\left(1 - e^{-3(t-10)/125}\right)u(t-10).$$

Finally, dividing by the volume of the solution in the tank, which constantly equals to 500 L, we conclude that

$$\text{concentration } = 0.4 - 0.2e^{-3t/125} + 0.2\left(1 - e^{-3(t-10)/125}\right)u(t-10).$$

**63.** In this problem, the solution still enters the tank at the rate 12 L/min, but leaves the tank at the rate only 6 L/min. Thus, every minute, the volume of the solution in the tank increases by $12 - 6 = 6$ L. Therefore, the volume, as a function of $t$, is given by $500 + 6t$ and so

$$\text{output rate} = \frac{x(t)}{500 + 6t}\,(\text{kg/L}) \times 6\,(\text{L/min}) = \frac{3x(t)}{250 + 3t}\,(\text{kg/min}).$$

Instead of equation (7.45) in Problem 61, we now have

$$x'(t) = g(t) - \frac{3x(t)}{250 + 3t} \qquad \Rightarrow \qquad (250 + 3t)x'(t) + 3x(t) = (250 + 3t)[48 + 24u(t-10)].$$

This equation has polynomial coefficients and can also be solved using the Laplace transform method. (See the discussion in Section 7.5, page 380, and Example 4.) But, as an intermediate step, one will obtain a first order linear differential equation for $\mathcal{L}\{x\}(s)$.

## EXERCISES 7.7: Convolution, page 405

**1.** Let $Y(s) := \mathcal{L}\{y\}(s)$, $G(s) := \mathcal{L}\{g\}(s)$. Taking the Laplace transform of both sides of the given differential equation and using the linear property of the Laplace transform, we obtain

$$\mathcal{L}\{y''\}(s) - 2\mathcal{L}\{y'\}(s) + Y(s) = G(s).$$

The initial conditions and Theorem 5, Section 7.3, imply that

$$\begin{aligned}\mathcal{L}\{y'\}(s) &= sY(s) + 1,\\ \mathcal{L}\{y''\}(s) &= s^2Y(s) + s - 1.\end{aligned}$$

Thus, substitution yields

$$\begin{aligned}&\left[s^2Y(s) + s - 1\right] - 2\left[sY(s) + 1\right] + Y(s) = G(s)\\ \Rightarrow \qquad &\left(s^2 - 2s + 1\right)Y(s) = 3 - s + G(s)\\ \Rightarrow \qquad &Y(s) = \frac{3-s}{s^2-2s+1} + \frac{G(s)}{s^2-2s+1} = \frac{2}{(s-1)^2} - \frac{1}{s-1} + \frac{G(s)}{(s-1)^2}.\end{aligned}$$

Taking now the inverse Laplace transform, we obtain

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{G(s)}{(s-1)^2}\right\}(t).$$

Using Table 7.1, we find that

$$\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) = e^t, \qquad \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}(t) = te^t,$$

and, by the convolution theorem,

$$\mathcal{L}^{-1}\left\{\frac{G(s)}{(s-1)^2}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}G(s)\right\}(t) = \left(te^t\right) * g(t) = \int\limits_0^t (t-v)e^{t-v}g(v)\,dv.$$

Thus

$$y(t) = 2te^t - e^t + \int_0^t (t-v)e^{t-v}g(v)\,dv.$$

**3.** Taking the Laplace transform of $y'' + 4y' + 5y = g(t)$ and applying the initial conditions $y(0) = y'(0) = 1$ gives us

$$\left[s^2Y(s) - s - 1\right] + 4\left[sY(s) - 1\right] + 5Y(s) = G(s),$$

where $Y(s) := \mathcal{L}\{y\}(s)$, $G(s) := \mathcal{L}\{g\}(s)$. Thus

$$Y(s) = \frac{s+5}{s^2+4s+5} + \frac{G(s)}{s^2+4s+5} = \frac{s+2}{(s+2)^2+1} + \frac{3}{(s+2)^2+1} + \frac{G(s)}{(s+2)^2+1}.$$

Taking the inverse Laplace transform of $Y(s)$ with the help of the convolution theorem yields

$$y(t) = e^{-2t}\cos t + 3e^{-2t}\sin t + \int_0^t e^{-2(t-v)}\sin(t-v)g(v)\,dv..$$



**5.** Since $\mathcal{L}^{-1}\{1/s\}(t) = 1$ and $\mathcal{L}^{-1}\{1/(s^2+1)\}(t) = \sin t$, writing

$$\frac{1}{s(s^2+1)} = \frac{1}{s}\cdot\frac{1}{s^2+1}$$

and using the convolution theorem, we obtain

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}(t) = 1 * \sin t = \int_0^t \sin v\,dv = -\cos v\,\Big|_0^t = 1 - \cos t.$$

**7.** From Table 7.1, $\mathcal{L}^{-1}\{1/(s-a)\}(t) = e^{at}$. Therefore, using the linearity of the inverse Laplace transform and the convolution theorem, we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{14}{(s+2)(s-5)}\right\}(t) &= 14\mathcal{L}^{-1}\left\{\frac{1}{s+2}\cdot\frac{1}{s-5}\right\}(t) = 14e^{-2t} * e^{5t} = 14\int_0^t e^{-2(t-v)}e^{5v}\,dv \\ &= 14e^{-2t}\int_0^t e^{7v}\,dv = 2e^{-2t}\left(e^{7t}-1\right) = 2\left(e^{5t}-e^{-2t}\right).\end{aligned}$$

# Chapter 7

**9.** Since $s/(s^2+1)^2 = [s/(s^2+1)]\cdot[1/(s^2+1)]$ the convolution theorem tells us that

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\cdot\frac{s}{s^2+1}\right\}(t) = \cos t * \sin t = \int_0^t \cos(t-v)\sin v\,dv.$$

Using the identity $\sin\alpha\cos\beta = [\sin(\alpha+\beta)+\sin(\alpha-\beta)]/2$, we get

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}(t) &= \frac{1}{2}\int_0^t [\sin t + \sin(t-2v)]\,dv \\ &= \frac{1}{2}\left(v\sin t + \frac{\cos(t-2v)}{2}\right)\Bigg|_0^t = \frac{t\sin t}{2}.\end{aligned}$$

**11.** Using the hint, we can write

$$\frac{s}{(s-1)(s+2)} = \frac{1}{s+2} + \frac{1}{(s-1)(s+2)},$$

so that by the convolution theorem, Theorem 11 on page 400 of the text,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{(s-1)(s+2)}\right\}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\}(t) \\ &= e^{-2t} + e^t * e^{-2t} = e^{-2t} + \int_0^t e^{t-v}e^{-2v}\,dv \\ &= e^{-2t} + e^t\int_0^t e^{-3v}\,dv = e^{-2t} - \frac{e^t}{3}\left(e^{-3t}-1\right) = \frac{2e^{-2t}}{3} + \frac{e^t}{3}.\end{aligned}$$

**13.** Note that $f(t) = t * e^{3t}$. Hence, by (8) on page 400 of the text,

$$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{t\}(s)\mathcal{L}\left\{e^{3t}\right\}(s) = \frac{1}{s^2}\cdot\frac{1}{s-3} = \frac{1}{s^2(s-3)}.$$

**15.** Note that

$$\int_0^t y(v)\sin(t-v)\,dv = \sin t * y(t).$$

Let $Y(s) := \mathcal{L}\{y\}(s)$. Taking the Laplace transform of the original equation, we obtain

$$Y(s) + 3\mathcal{L}\{\sin t * y(t)\}(s) = \mathcal{L}\{t\}(s)$$

$$Y(s) + 3\mathcal{L}\{\sin t\}(s)Y(s) = \frac{1}{s^2} \quad \Rightarrow \quad Y(s) + \frac{3}{s^2+1}Y(s) = \frac{1}{s^2}$$

$$\Rightarrow \quad Y(s) = \frac{s^2+1}{s^2(s^2+4)} = \frac{(1/4)}{s^2} + \frac{(3/8)2}{s^2+2^2}$$

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\left\{\frac{(1/4)}{s^2} + \frac{(3/8)2}{s^2+2^2}\right\}(t) = \frac{t}{4} + \frac{3\sin 2t}{8}.$$

17. We use the convolution Theorem 11 to find the Laplace transform of the integral term.

$$\mathcal{L}\left\{\int_0^t (t-v)y(v)\,dv\right\}(s) = \mathcal{L}\{t * y(t)\}(s) = \mathcal{L}\{t\}(s)\mathcal{L}\{y(t)\}(s) = \frac{Y(s)}{s^2},$$

where $Y(s)$ denotes the Laplace transform of $y(t)$. Thus taking the Laplace transform of both sides of the given equation yields

$$Y(s) + \frac{Y(s)}{s^2} = \frac{1}{s} \quad \Rightarrow \quad Y(s) = \frac{s}{s^2+1} \quad \Rightarrow \quad y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t) = \cos t.$$

19. By the convolution theorem,

$$\mathcal{L}\left\{\int_0^t (t-v)^2 y(v)\,dv\right\}(s) = \mathcal{L}\left\{t^2 * y(t)\right\}(s) = \mathcal{L}\left\{t^2\right\}(s)\mathcal{L}\{y(t)\}(s) = \frac{2Y(s)}{s^3}.$$

Hence, applying the Laplace transform to the original equation yields

$$Y(s) + \frac{2Y(s)}{s^3} = \mathcal{L}\left\{t^3+3\right\}(s) = \frac{6}{s^4} + \frac{3}{s}$$

$$\Rightarrow \quad Y(s) = \frac{s^3}{s^3+2} \cdot \frac{6+3s^3}{s^4} = \frac{3}{s}$$

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\left\{\frac{3}{s}\right\}(t) = 3.$$

21. As in Example 3 on page 402 of the text, we first rewrite the integro-differential equation as

$$y'(t) + y(t) - y(t) * \sin t = -\sin t, \qquad y(0) = 1. \tag{7.47}$$

We now take the Laplace transform of (7.47) to obtain

$$[sY(s) - 1] + Y(s) - \frac{1}{s^2+1}Y(s) = -\frac{1}{s^2+1},$$

where $Y(s) = \mathcal{L}\{y\}(s)$. Thus,

$$\begin{aligned} Y(s) &= \frac{s^2}{s^3 + s^2 + s} = \frac{s}{s^2 + s + 1} = \frac{s}{(s + 1/2)^2 + 3/4} \\ &= \frac{s + 1/2}{(s + 1/2)^2 + 3/4} - \frac{(1/\sqrt{3})(\sqrt{3}/2)}{(s + 1/2)^2 + 3/4}. \end{aligned}$$

Taking the inverse Laplace transform yields

$$y(t) = e^{-t/2} \cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{1}{\sqrt{3}} e^{-t/2} \sin\left(\frac{\sqrt{3}t}{2}\right).$$

23. Taking the Laplace transform of the differential equation, and assuming zero initial conditions, we obtain

$$s^2 Y(s) + 9Y(s) = G(s),$$

where $Y = \mathcal{L}\{y\}$, $G = \mathcal{L}\{g\}$. Thus,

$$H(s) = \frac{Y(s)}{G(s)} = \frac{1}{s^2 + 9}.$$



The impulse response function is then

$$h(t) = \mathcal{L}^{-1}\{H(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 9}\right\}(t) = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 3^2}\right\}(t) = \frac{\sin 3t}{3}.$$

To solve the initial value problem, we need the solution to the corresponding homogeneous problem. The auxiliary equation, $r^2 + 9 = 0$, has roots, $r = \pm 3i$. Thus, a general solution to the homogeneous equation is

$$y_h(t) = C_1 \cos 3t + C_2 \sin 3t.$$

Applying the initial conditions $y(0) = 2$ and $y'(0) = -3$, we obtain

$$\begin{aligned} 2 = y(0) &= \left(C_1 \cos 3t + C_2 \sin 3t\right)\Big|_{t=0} = C_1, & \Rightarrow \quad C_1 &= 2, \\ -3 = y'(0) &= \left(-3C_1 \sin 3t + 3C_2 \cos 3t\right)\Big|_{t=0} = 3C_2 & \Rightarrow \quad C_2 &= -1. \end{aligned}$$

So

$$y_k(t) = 2\cos 3t - \sin 3t,$$

and the formula for the solution to the original initial value problem is

$$y = (h * g)(t) + y_k(t) = \frac{1}{3}\int_0^t g(v)\sin 3(t-v)\,dv + 2\cos 3t - \sin 3t.$$

**25.** Taking the Laplace transform of both sides of the given equation and assuming zero initial conditions, we get

$$\mathcal{L}\{y'' - y' - 6y\}(s) = \mathcal{L}\{g(t)\}(s) \qquad \Rightarrow \qquad s^2Y(s) - sY(s) - 6Y(s) = G(s).$$

Thus,

$$H(s) = \frac{Y(s)}{G(s)} = \frac{1}{s^2 - s - 6} = \frac{1}{(s-3)(s+2)}$$

is the transfer function. The impulse response function $h(t)$ is then given by

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s+2)}\right\}(t) = e^{3t} * e^{-2t} = \int_0^t e^{3(t-v)}e^{-2v}\,dv = e^{3t}\left.\frac{e^{-5v}}{-5}\right|_0^t = \frac{e^{3t} - e^{-2t}}{5}.$$



To solve the given initial value problem, we use Theorem 12. To this end, we need the solution $y_k(t)$ to the corresponding initial value problem for the homogeneous equation. That is,

$$y'' - y' - 6y = 0, \qquad y(0) = 1, \quad y'(0) = 8$$

(see (19) in the text). Applying the Laplace transform yields

$$\begin{aligned}
&\left[s^2Y_k(s) - s - 8\right] - \left[sY_k(s) - 1\right] - 6Y_k(s) = 0\\
\Rightarrow \qquad & Y_k(s) = \frac{s+7}{s^2 - s - 6} = \frac{s+7}{(s-3)(s+2)} = \frac{2}{s-3} - \frac{1}{s+2}\\
\Rightarrow \qquad & y_k(t) = \mathcal{L}^{-1}\{Y_k(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{2}{s-3} - \frac{1}{s+2}\right\}(t) = 2e^{3t} - e^{-2t}.
\end{aligned}$$

So,

$$y(t) = (h * g)(t) + y_k(t) = \frac{1}{5}\int_0^t \left[e^{3(t-v)} - e^{-2(t-v)}\right] g(v)\,dv + 2e^{3t} - e^{-2t}.$$

**27.** Taking the Laplace transform and assuming zero initial conditions, we find the transfer function $H(s)$.

$$s^2Y(s) - 2sY(s) + 5Y(s) = G(s) \qquad \Rightarrow \qquad H(s) = \frac{Y(s)}{G(s)} = \frac{1}{s^2 - 2s + 5}\,.$$

Therefore, the impulse response function is

$$h(t) = \mathcal{L}^{-1}\left\{H(s)\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2 + 2^2}\right\}(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2 + 2^2}\right\}(t) = \frac{1}{2}e^t \sin 2t\,.$$

Next, we find the solution $y_k(t)$ to the corresponding initial value problem for the homogeneous equation,

$$y'' - 2y' + 5y = 0, \qquad y(0) = 0, \quad y'(0) = 2.$$

Since the associated equation, $r^2 - 2r + 5 = 0$, has roots $r = 1 \pm 2i$, a general solution to the homogeneous equations is

$$y_h(t) = e^t\left(C_1 \cos 2t + C_2 \sin 2t\right).$$

We satisfy the initial conditions by solving

$$\begin{aligned} 0 &= y(0) = C_1 \\ 2 &= y'(0) = C_1 + 2C_2 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} C_1 &= 0, \\ C_2 &= 1. \end{aligned}$$

Hence, $y_k(t) = e^t \sin 2t$ and

$$y(t) = (h * g)(t) + y_k(t) = \frac{1}{2}\int_0^t e^{t-v} \sin 2(t-v) g(v)\,dv + e^t \sin 2t$$

is the desired solution.

**29.** With given data, the initial value problem becomes

$$5I''(t) + 20I'(t) + \frac{1}{0.005}I(t) = e(t), \qquad I(0) = -1, \quad I'(0) = 8.$$

Using formula (15) on page 403 of the text, we find the transfer function

$$H(s) = \frac{1}{5s^2 + 20s + 200} = \frac{1}{5}\,\frac{1}{(s+2)^2 + 6^2}\,.$$

Therefore,

$$h(t)=\mathcal{L}^{-1}\left\{\frac{1}{5}\frac{1}{(s+2)^2+6^2}\right\}(t)=\frac{1}{30}\mathcal{L}^{-1}\left\{\frac{6}{(s+2)^2+6^2}\right\}(t)=\frac{1}{30}e^{-2t}\sin 6t.$$

Next, we consider the initial value problem

$$5I''(t)+20I'(t)+200I(t)=0, \qquad I(0)=-1, \quad I'(0)=8$$

for the corresponding homogeneous equation. Its characteristic equation, $5r^2+20r+200=0$, has roots $r=-2\pm 6i$, which yield a general solution

$$I_h(t)=e^{-2t}\left(C_1\cos 6t+C_2\sin 6t\right).$$

We find constants $C_1$ and $C_2$ so that the solution satisfies the initial conditions. Thus we have

$$\begin{array}{l} -1=I(0)=C_1\,, \\ 8=I'(0)=-2C_1+6C_2 \end{array} \quad\Rightarrow\quad \begin{array}{l} C_1=-1\,, \\ C_2=1\,, \end{array}$$

and so $I_k(t)=e^{-2t}(\sin 6t-\cos 6t)$. Finally,

$$I(t)=h(t)*e(t)+I_k(t)=\frac{1}{30}\int_0^t e(v)e^{-2(t-v)}\sin 6(t-v)\,dv+e^{-2t}\left(\sin 6t-\cos 6t\right).$$



**31.** By the convolution theorem, we get

$$\mathcal{L}\{1*1*1\}(s)=\mathcal{L}\{1\}(s)\mathcal{L}\{1*1\}(s)=\mathcal{L}\{1\}(s)\mathcal{L}\{1\}(s)\mathcal{L}\{1\}(s)=\left(\frac{1}{s}\right)^3=\frac{1}{s^3}.$$

Therefore, the definition of the inverse Laplace transform yields

$$1*1*1=\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}(t)=\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}(t)=\frac{1}{2}t^2\,.$$

**33.** Using the linear property of integrals, we have

$$\begin{aligned} f*(g+h) &= \int_0^t f(t-v)[g+h](v)\,dv=\int_0^t f(t-v)[g(v)+h(v)]\,dv \\ &= \int_0^t f(t-v)g(v)\,dv+\int_0^t f(t-v)h(v)\,dv=f*g+f*h. \end{aligned}$$

**35.** Since

$$\int_0^t f(v)\,dv = \int_0^t 1 \cdot f(v)\,dv = 1 * f(t),$$

we conclude that

$$\mathcal{L}\left\{\int_0^t f(v)\,dv\right\}(s) = \mathcal{L}\{1 * f(t)\}(s) = \mathcal{L}\{1\}(s)\mathcal{L}\{f(t)\}(s) = \frac{1}{s}F(s).$$

Hence, by the definition of the inverse Laplace transform,

$$\int_0^t f(v)\,dv = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}(t).$$

(Note that the integral in the left-hand side is a continuous function.)

**37.** Actually, this statement holds for any continuously differentiable function $h(t)$ on $[0,\infty)$ satisfying $h(0) = 0$. Indeed, first of all,

$$(h * g)(0) = \left.\int_0^t h(t-v)g(v)\,dv\right|_{t=0} = \int_0^0 h(-v)g(v)\,dv = 0$$

since the interval of integration has zero length. Next, we apply the Leibniz's rule to find the derivative of $(h * g)(t)$.

$$\begin{aligned}(h * g)'(t) &= \left(\int_0^t h(t-v)g(v)\,dv\right)' = \int_0^t \frac{\partial h(t-v)g(v)}{\partial t}\,dv + h(t-v)g(v)\Big|_{v=t} \\ &= \int_0^t h'(t-v)g(v)\,dv + h(0)g(t) = \int_0^t h'(t-v)g(v)\,dv\end{aligned}$$

since $h(0) = 0$. Therefore,

$$(h * g)'(0) = \int_0^0 h'(-v)g(v)\,dv = 0,$$

again as a definite integral with equal limits of integration.

**EXERCISES 7.8: Impulses and the Dirac Delta Function, page 412**

**1.** By equation (3) on page 407 of the text,

$$\int_{-\infty}^{\infty} (t^2 - 1)\boldsymbol{\delta}(t)\,dt = \left.\left(t^2 - 1\right)\right|_{t=0} = -1.$$

**3.** By equation (3) on page 407 of the text,

$$\int_{-\infty}^{\infty} (\sin 3t)\boldsymbol{\delta}\left(t - \frac{\pi}{2}\right) dt = \sin\left(3 \cdot \frac{\pi}{2}\right) = -1.$$

**5.** Formula (6) of the Laplace transform of the Dirac delta function yields

$$\int_{0}^{\infty} e^{-2t}\boldsymbol{\delta}(t-1)\,dt = \mathcal{L}\left\{\boldsymbol{\delta}(t-1)\right\}(2) = e^{-s}\Big|_{s=2} = e^{-2}.$$

**7.** Using the linearity of the Laplace transform and (6) on page 409 of the text, we get

$$\mathcal{L}\left\{\boldsymbol{\delta}(t-1) - \boldsymbol{\delta}(t-3)\right\}(s) = \mathcal{L}\left\{\boldsymbol{\delta}(t-1)\right\}(s) - \mathcal{L}\left\{\boldsymbol{\delta}(t-3)\right\}(s) = e^{-s} - e^{-3s}.$$

**9.** Since $\boldsymbol{\delta}(t-1) = 0$ for $t < 1$,

$$\mathcal{L}\left\{t\boldsymbol{\delta}(t-1)\right\}(s) := \int_{0}^{\infty} e^{-st} t\boldsymbol{\delta}(t-1)\,dt = \int_{-\infty}^{\infty} e^{-st} t\boldsymbol{\delta}(t-1)\,dt = e^{-st}t\Big|_{t=1} = e^{-s}$$

by equation (3) on page 407 of the text.

Another way to solve this problem is to use Theorem 6 inj Section 7.3. This yields

$$\mathcal{L}\left\{t\boldsymbol{\delta}(t-1)\right\}(s) = -\frac{d}{ds}\mathcal{L}\left\{\boldsymbol{\delta}(t-1)\right\}(s) = -\frac{d\left(e^{-s}\right)}{ds} = e^{-s}.$$

**11.** Since $\boldsymbol{\delta}(t-\pi) = 0$ for $t < \pi$, we use the definition of the Laplace transform and formula (3), page 407 of the text, to conclude that

$$\mathcal{L}\left\{(\sin t)\boldsymbol{\delta}(t-\pi)\right\}(s) := \int_{0}^{\infty} e^{-st}(\sin t)\boldsymbol{\delta}(t-\pi)\,dt = \int_{-\infty}^{\infty} e^{-st}(\sin t)\boldsymbol{\delta}(t-\pi)\,dt = e^{-\pi t}\sin\pi = 0.$$

**13.** Let $W(s) := \mathcal{L}\{w\}(s)$. Using the initial conditions and Theorem 5 in Section 7.3, we find that

$$\mathcal{L}\{w''\}(s) = s^2W(s) - sw(0) - w'(0) = s^2W(s).$$

Thus, applying the Laplace transform to both sides of the given equation yields

$$s^2W(s) + W(s) = \mathcal{L}\{\boldsymbol{\delta}(t-\pi)\}(s) = e^{-\pi s} \qquad \Rightarrow \qquad W(s) = \frac{e^{-\pi s}}{s^2+1}.$$

Taking the inverse Laplace transform of both sides of the last equation and using Theorem 8 in Section 7.6, we get

$$w(t) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t-\pi)u(t-\pi) = \sin(t-\pi)u(t-\pi) = -(\sin t)u(t-\pi).$$

**15.** Let $Y := \mathcal{L}\{y\}$. Taking the Laplace transform of $y'' + 2y' - 3y = \boldsymbol{\delta}(t-1) - \boldsymbol{\delta}(t-2)$ and applying the initial conditions $y(0) = 2$, $y'(0) = -2$, we obtain

$$\left[s^2Y(s) - 2s + 2\right] + 2\left[sY(s) - 2\right] - 3Y(s) = \mathcal{L}\{\boldsymbol{\delta}(t-1) - \boldsymbol{\delta}(t-2)\}(s) = e^{-s} - e^{-2s}$$
$$\Rightarrow \qquad Y(s) = \frac{2s+2+e^{-s}-e^{-2s}}{s^2+2s-3} = \frac{2s+2}{(s+3)(s-1)} + \frac{e^{-s}}{(s+3)(s-1)} - \frac{e^{-2s}}{(s+3)(s-1)}$$
$$= \frac{1}{s-1} + \frac{1}{s+3} + \frac{e^{-s}}{4}\left(\frac{1}{s-1} - \frac{1}{s+3}\right) - \frac{e^{-2s}}{4}\left(\frac{1}{s-1} - \frac{1}{s+3}\right),$$

so that by Theorem 8 on page 387 of the text we get

$$y(t) = e^{t} + e^{-3t} + \frac{1}{4}\left(e^{t-1} - e^{-3(t-1)}\right)u(t-1) - \frac{1}{4}\left(e^{t-2} - e^{-3(t-2)}\right)u(t-2).$$

**17.** Let $Y := \mathcal{L}\{y\}$. We use the initial conditions to find that

$$\mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2.$$

Thus taking the Laplace transform of both sides of the given equation and using formula (6) on page 409, we get

$$\left[s^2Y(s) - 2\right] - Y(s) = 4\mathcal{L}\{\boldsymbol{\delta}(t-2)\}(s) + \mathcal{L}\left\{t^2\right\}(s) = 4e^{-2s} + \frac{2}{s^3}$$
$$\Rightarrow \qquad Y(s) = \frac{4e^{-2s}}{s^2-1} + \frac{2(s^3+1)}{s^3(s^2-1)} = 2e^{-2s}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) + \frac{2}{s-1} - \frac{2}{s^3} - \frac{2}{s}.$$

Now we can apply the inverse Laplace transform.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{2e^{-2s}\left(\frac{1}{s-1}-\frac{1}{s+1}\right)+\frac{2}{s-1}-\frac{2}{s^3}-\frac{2}{s}\right\}(t) \\ &= 2\left(\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}\right)(t-2)u(t-2) \\ &\qquad +2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t)-\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}(t)-2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) \\ &= 2\left(e^{t-2}-e^{2-t}\right)u(t-2)+2e^t-t^2-2. \end{aligned}$$

**19.** Let $W(s) := \mathcal{L}\{w\}(s)$. We apply the Laplace transform to the given equation and obtain

$$\mathcal{L}\{w''\}(s)+6\mathcal{L}\{w'\}(s)+5W(s)=\mathcal{L}\left\{e^t\boldsymbol{\delta}(t-1)\right\}(s). \tag{7.48}$$

From formula (4) on page 362 of the text we see that

$$\begin{aligned} \mathcal{L}\{w'\}(s) &= sW(s)-w(0)=sW(s), \\ \mathcal{L}\{w''\}(s) &= s^2W(s)-sw(0)-w'(0)=s^2W(s)-4. \end{aligned} \tag{7.49}$$

Also, the translation property (1), Section 7.3, of the Laplace transform yields

$$\mathcal{L}\left\{e^t\boldsymbol{\delta}(t-1)\right\}(s)=\mathcal{L}\{\boldsymbol{\delta}(t-1)\}(s-1)=e^{-(s-1)}=e^{1-s}. \tag{7.50}$$

Substituting (7.49) and (7.50) back into (7.48), we obtain

$$\begin{aligned} &\left[s^2W(s)-4\right]+6\left[sW(s)\right]+5W(s)=e^{1-s} \\ \Rightarrow\qquad & W(s)=\frac{4+e^{1-s}}{s^2+6s+5}=\frac{4+e^{1-s}}{(s+1)(s+5)}=\frac{1}{s+1}-\frac{1}{s+5}+\frac{e}{4}e^{-s}\left(\frac{1}{s+1}-\frac{1}{s+5}\right). \end{aligned}$$

Finally, the inverse Laplace transform of both sides of this equation yields

$$w(t)=e^{-t}-e^{-5t}+\frac{e}{4}\left[e^{-(t-1)}-e^{-5(t-1)}\right]u(t-1).$$

**21.** We apply the Laplace transform to the given equation, solve the resulting equation for $\mathcal{L}\{y\}(s)$, and then use the inverse Laplace transforms. This yields

$$\mathcal{L}\{y''\}(s)+\mathcal{L}\{y\}(s)=\mathcal{L}\{\boldsymbol{\delta}(t-2\pi)\}(s)$$

$$\Rightarrow \quad \left[s^2\mathcal{L}\{y\}(s) - 1\right] + \mathcal{L}\{y\}(s) = e^{-2\pi s} \quad \Rightarrow \quad \mathcal{L}\{y\}(s) = \frac{1+e^{-2\pi s}}{s^2+1}$$

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t-2\pi)u(t-2\pi)$$
$$= \sin t + [\sin(t-2\pi)]u(t-2\pi) = [1+u(t-2\pi)]\sin t.$$

The graph of the solution is shown in Figure B.49 in the answers of the text.

**23.** The solution to the initial value problem

$$y'' + y = \boldsymbol{\delta}(t-2\pi), \qquad y(0) = 0, \quad y'(0) = 1$$

is given in Problem 21, that is

$$y_1(t) = [1+u(t-2\pi)]\sin t.$$

Thus, if $y_2(t)$ is the solution to the initial value problem

$$y'' + y = -\boldsymbol{\delta}(t-\pi), \qquad y(0) = 0, \quad y'(0) = 0, \tag{7.51}$$

then, by the superposition principle (see Section 4.5), $y(t) = y_1(t) + y_2(t)$ is the desired solution. The Laplace transform of both sides in (7.51) yields

$$s^2\mathcal{L}\{y\}(s) + \mathcal{L}\{y\}(s) = -e^{-\pi s} \quad \Rightarrow \quad \mathcal{L}\{y\}(s) = -\frac{e^{-\pi s}}{s^2+1}$$

$$\Rightarrow \quad y_2(t) = -\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t-\pi)u(t-\pi) = -[\sin(t-\pi)]u(t-\pi) = u(t-\pi)\sin t.$$

(We have used zero initial conditions to express $\mathcal{L}\{y''\}$ in terms of $\mathcal{L}\{y\}$.) Therefore, the answer is

$$y(t) = y_1(t) + y_2(t) = [1+u(t-2\pi)]\sin t + u(t-\pi)\sin t = [1+u(t-\pi)+u(t-2\pi)]\sin t.$$

The sketch of this curve is given in Figure B.50 .

**25.** Taking the Laplace transform of $y'' + 4y' + 8y = \boldsymbol{\delta}(t)$ with zero initial conditions yields

$$s^2Y(s) + 4sY(s) + 8Y(s) = \mathcal{L}\{\boldsymbol{\delta}(t)\}(s) = 1.$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{1}{s^2+4s+8} = \frac{1}{(s+2)^2+4} = \frac{1}{2}\frac{2}{(s+2)^2+2^2}$$

so that

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{2}{(s+2)^2+2^2}\right\}(t) = \frac{1}{2}e^{-2t}\sin 2t\,.$$

Notice that $H(s)$ for $y''+4y'+8y = g(t)$ with $y(0) = y'(0) = 0$ is given by $H(s) = 1/(s^2+4s+8)$, so that again

$$h(t) = \mathcal{L}^{-1}\{H(s)\}(t) = \frac{1}{2}e^{-2t}\sin 2t\,.$$

**27.** The Laplace transform of both sides of the given equation, with zero initial conditions and $g(t) = \boldsymbol{\delta}(t)$, gives us

$$s^2\mathcal{L}\{y\}(s) - 2s\mathcal{L}\{y\}(s) + 5\mathcal{L}\{y\}(s) = \mathcal{L}\{\boldsymbol{\delta}(t)\}(s)$$

$$\Rightarrow \qquad \mathcal{L}\{y\}(s) = \frac{1}{s^2-2s+5} = \frac{1}{(s-1)^2+2^2}\,.$$

The inverse Laplace transform now yields

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+2^2}\right\}(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}(t) = \frac{1}{2}e^{t}\sin 2t\,.$$



**29.** We solve the given initial value problem to find the displacement $x(t)$. Let $X(s) := \mathcal{L}\{x\}(s)$. Applying the Laplace transform to the differential equation yields

$$\mathcal{L}\{x''\}(s) + 9X(s) = \mathcal{L}\left\{-3\boldsymbol{\delta}\left(t-\frac{\pi}{2}\right)\right\}(s) = -3e^{-\pi s/2}\,.$$

Since

$$\mathcal{L}\{x''\}(s) = s^2X(s) - sx(0) - x'(0) = s^2X(s) - s,$$

the above equation becomes

$$\left[s^2X(s) - s\right] + 9X(s) = -3e^{-\pi s/2} \qquad \Rightarrow \qquad X(s) = \frac{s-3e^{-\pi s/2}}{s^2+9} = \frac{s}{s^2+3^2} - e^{-\pi s/2}\frac{3}{s^2+3^2}.$$

Therefore,

$$x(t) \;=\; \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2} - e^{-\pi s/2}\frac{3}{s^2+3^2}\right\}(t)$$

$$= \cos 3t - \left[\sin 3\left(t - \frac{\pi}{2}\right)\right] u\left(t - \frac{\pi}{2}\right) = \left[1 - u\left(t - \frac{\pi}{2}\right)\right] \cos 3t\,.$$

Since, for $t > \pi/2$, $u(t - \pi/2) \equiv 1$, we conclude that

$$x(t) \equiv 0 \qquad \text{for} \quad t > \frac{\pi}{2}\,.$$

This means that the mass stops after the hit and remains in the equilibrium position thereafter.

**31.** By taking the Laplace transform of

$$ay'' + by' = cy = \boldsymbol{\delta}(t), \qquad y(0) = y'(0) = 0,$$

and solving for $Y := \mathcal{L}\{y\}$, we find that the transfer function is given by

$$H(s) = \frac{1}{as^2 + bs + c}\,.$$

If the roots of the polynomial $as^2 + bs + c$ are real and distinct, say $r_1$, $r_2$, then

$$H(s) = \frac{1}{(s - r_1)(s - r_2)} = \frac{1/(r_1 - r_2)}{s - r_1} - \frac{1/(r_1 - r_2)}{s - r_2}\,.$$

Thus

$$h(t) = \frac{1}{r_1 - r_2}\left(e^{r_1 t} - e^{r_2 t}\right)$$

and clearly $h(t)$ is bounded as $t \to \infty$ if and only if $r_1$ and $r_2$ are less than or equal to zero.

If the roots of $as^2 + bs + c$ are complex, then, by the quadratic formula, they are given by

$$-\frac{b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a}\, i$$

so that the real part of the roots is $-b/(2a)$. Now

$$\begin{aligned} H(s) &= \frac{1}{as^2 + bs + c} = \frac{1}{a} \cdot \frac{1}{s^2 + (b/a)s + (c/a)} = \frac{1}{a} \cdot \frac{1}{[s + b/(2a)]^2 + (4ac - b^2)/(4a^2)} \\ &= \frac{2}{\sqrt{4ac - b^2}} \cdot \frac{\sqrt{4ac - b^2}/(2a)}{[s + b/(2a)]^2 + [\sqrt{4ac - b^2}/(2a)]^2} \end{aligned}$$

so that

$$h(t) = \frac{2}{\sqrt{4ac - b^2}}\, e^{-(b/2a)t} \sin\left(\frac{\sqrt{4ac - b^2}}{2a}\, t\right),$$

and again it is clear that $h(t)$ is bounded if and only if $-b/(2a)$, the real part of the roots of $as^2 + bs + c$, is less than or equal to zero.

**33.** Let a function $f(t)$ be defined on $(-\infty, \infty)$ and continuous in a neighborhood of the origin, $t = 0$. Since $\boldsymbol{\delta}(t) = 0$ for any $t \neq 0$, so does the product $f(t)\boldsymbol{\delta}(t)$. Therefore,

$$\int_{-\infty}^{\infty} f(t)\boldsymbol{\delta}(t)\,dt = \int_{-\varepsilon}^{\varepsilon} f(t)\boldsymbol{\delta}(t)\,dt \qquad \text{for any } \varepsilon > 0. \tag{7.52}$$

By the mean value theorem, for any $\varepsilon$ small enough (so that $f(t)$ is continuous on $(-\varepsilon, \varepsilon)$) there exists a point $\zeta_\varepsilon$ in $(-\varepsilon, \varepsilon)$ such that

$$\int_{-\varepsilon}^{\varepsilon} f(t)\boldsymbol{\delta}(t)\,dt = f(\zeta_\varepsilon)\int_{-\varepsilon}^{\varepsilon} \boldsymbol{\delta}(t)\,dt = f(\zeta_\varepsilon)\int_{-\infty}^{\infty} \boldsymbol{\delta}(t)\,dt = f(\zeta_\varepsilon)\,.$$

Together with (7.52) this yields

$$\int_{-\infty}^{\infty} f(t)\boldsymbol{\delta}(t)\,dt = f(\zeta_\varepsilon)\,, \qquad \text{for any } \varepsilon > 0.$$

Now we take limit, as $\varepsilon \to 0$, in both sides.

$$\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{\infty} f(t)\boldsymbol{\delta}(t)\,dt\right] = \lim_{\varepsilon\to 0}\left[f(\zeta_\varepsilon)\right].$$

Note that the integral in the left-hand side does not depend on $\varepsilon$, and so the limit equals to the integral itself. In the right-hand side, since $\zeta_\varepsilon$ belongs to $(-\varepsilon, \varepsilon)$, $\zeta_\varepsilon \to 0$ as $\varepsilon \to 0$, and the continuity of $f(t)$ implies that $f(\zeta_\varepsilon)$ converges to $f(0)$, as $\varepsilon \to 0$. Combining these observations, we get the required.



**35.** Following the hint, we solve the initial value problem

$$EIy^{(4)}(x) = L\boldsymbol{\delta}(x-\lambda), \qquad y(0) = y'(0) = 0,\ y''(0) = A,\ y'''(0) = B.$$

Using these initial conditions and Theorem 5 in Section 7.3 with $n = 4$, we obtain

$$\mathcal{L}\left\{y^{(4)}(x)\right\}(s) = s^4\mathcal{L}\left\{y(x)\right\}(s) - sA - B,$$

and so the Laplace transform of the given equation yields

$$EI\left[s^4\mathcal{L}\{y(x)\}(s) - sA - B\right] = L\mathcal{L}\{\boldsymbol{\delta}(x-\lambda)\}(s) = Le^{-\lambda s}.$$

Therefore,

$$\begin{aligned}
\mathcal{L}\{y(x)\}(s) &= \frac{L}{EI}\frac{e^{-\lambda s}}{s^4} + \frac{A}{s^3} + \frac{B}{s^4} \\
\Rightarrow \qquad y(x) &= \mathcal{L}^{-1}\left\{\frac{L}{EI}\frac{e^{-\lambda s}}{s^4} + \frac{A}{s^3} + \frac{B}{s^4}\right\}(x) \\
&= \frac{L}{EI3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}(x-\lambda)u(x-\lambda) + \frac{A}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\}(x) + \frac{B}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}(x) \\
&= \frac{L}{6EI}(x-\lambda)^3u(x-\lambda) + \frac{A}{2}x^2 + \frac{B}{6}x^3. \qquad (7.53)
\end{aligned}$$

Next, we are looking for $A$ and $B$ such that $y''(2\lambda) = y'''(2\lambda) = 0$. Note that, for $x > \lambda$, $u(x-\lambda) \equiv 1$ and so (7.53) becomes

$$y(x) = \frac{L}{6EI}(x-\lambda)^3 + \frac{A}{2}x^2 + \frac{B}{6}x^3.$$

Differentiating we get

$$y''(x) = \frac{L}{EI}(x-\lambda) + A + Bx \qquad \text{and} \qquad y'''(x) = \frac{L}{EI} + B.$$

Hence, $A$ and $B$ must satisfy

$$\begin{aligned}
0 = y''(2\lambda) &= [L/(EI)](2\lambda-\lambda) + A + 2B\lambda, \\
0 = y'''(2\lambda) &= L/(EI) + B
\end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= \lambda L/(EI), \\ B &= -L/(EI). \end{aligned}$$

Substitution back into (7.53) yields the solution

$$y(x) = \frac{L}{6EI}\left[(x-\lambda)^3u(x-\lambda) + 3\lambda x^2 - x^3\right].$$

**EXERCISES 7.9: Solving Linear Systems with Laplace Transforms, page 416**

**1.** Let $X(s) = \mathcal{L}\{x\}(s)$, $Y(s) = \mathcal{L}\{y\}(s)$. Applying the Laplace transform to both sides of the given equations yields

$$\begin{aligned}
\mathcal{L}\{x'\}(s) &= 3X(s) - 2Y(s), \\
\mathcal{L}\{y'\}(s) &= 3Y(s) - 2X(s).
\end{aligned} \qquad (7.54)$$

Since

$$\mathcal{L}\{x'\}(s) = sX(s) - x(0) = sX(s) - 1,$$
$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 1,$$

the system (7.54) becomes

$$\begin{aligned} sX(s) - 1 &= 3X(s) - 2Y(s), \\ sY(s) - 1 &= 3Y(s) - 2X(s) \end{aligned} \quad \Rightarrow \quad \begin{aligned} (s-3)X(s) + 2Y(s) &= 1, \\ 2X(s) + (s-3)Y(s) &= 1. \end{aligned} \tag{7.55}$$

Subtracting the second equation from the first equation yields

$$(s-5)X(s) + (5-s)Y(s) = 0 \quad \Rightarrow \quad X(s) = Y(s).$$

So, from the first equation in (7.55) we get

$$(s-3)X(s) + 2X(s) = 1 \quad \Rightarrow \quad X(s) = \frac{1}{s-1} \quad \Rightarrow \quad x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) = e^t.$$

Since $Y(s) = X(s)$, $y(t) = x(t) = e^t$.

**3.** Let $Z(s) = \mathcal{L}\{z\}(s)$, $W(s) = \mathcal{L}\{w\}(s)$. Using the initial conditions we conclude that

$$\mathcal{L}\{z'\}(s) = sZ(s) - z(0) = sZ(s) - 1, \quad \mathcal{L}\{w'\}(s) = sW(s) - w(0) = sW(s).$$

Using these equations and taking the Laplace transform of the equations in the given system, we obtain

$$\begin{aligned} [sZ(s) - 1] + [sW(s)] &= Z(s) - W(s), \\ sZ(s) - 1] - [sW(s)] &= Z(s) - W(s) \end{aligned} \quad \Rightarrow \quad \begin{aligned} (s-1)W(s) + (s+1)W(s) &= 1, \\ (s-1)W(s) - (s-1)W(s) &= 1. \end{aligned} \tag{7.56}$$

Subtracting equations yields

$$2sW(s) = 0 \quad \Rightarrow \quad W(s) = 0 \quad \Rightarrow \quad w(t) = \mathcal{L}^{-1}\{0\}(t) \equiv 0.$$

Substituting $W(s)$ into either equation in (7.56), we obtain

$$(s-1)Z(s) = 1 \quad \Rightarrow \quad Z(s) = \frac{1}{s-1} \quad \Rightarrow \quad z(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) = e^t.$$

**5.** Denote $X(s) = \mathcal{L}\{x\}(s)$, $Y(s) = \mathcal{L}\{y\}(s)$. The Laplace transform of the given equations yields

$$\mathcal{L}\{x'\}(s) = Y(s) + \mathcal{L}\{\sin t\}(s),$$
$$\mathcal{L}\{y'\}(s) = X(s) + 2\mathcal{L}\{\cos t\}(s),$$

which becomes

$$\begin{aligned} sX(s) - 2 &= Y(s) + 1/(s^2+1), \\ sY(s) &= X(s) + 2s/(s^2+1) \end{aligned} \quad \Rightarrow \quad \begin{aligned} sX(s) - Y(s) &= (2s^2+3)/(s^2+1), \\ -X(s) + sY(s) &= 2s/(s^2+1) \end{aligned}$$

after expressing $\mathcal{L}\{x'\}$ and $\mathcal{L}\{y'\}$ in terms of $X(s)$ and $Y(s)$. Multiplying the second equation by $s$ and adding the result to the first equation, we get

$$\left(s^2-1\right)Y(s) = \frac{4s^2+3}{s^2+1} \quad \Rightarrow \quad Y(s) = \frac{4s^2+3}{(s-1)(s+1)(s^2+1)}.$$

Since the partial fractions decomposition for $Y(s)$ is

$$\frac{4s^2+3}{(s-1)(s+1)(s^2+1)} = \frac{7/4}{s-1} - \frac{7/4}{s+1} + \frac{1/2}{s^2+1},$$

taking the inverse Laplace transform yields

$$y(t) = \mathcal{L}^{-1}\left\{\frac{7/4}{s-1} - \frac{7/4}{s+1} + \frac{1/2}{s^2+1}\right\}(t) = \frac{7}{4}e^t - \frac{7}{4}e^{-t} + \frac{1}{2}\sin t.$$

From the second equation in the original system,

$$x(t) = y' - 2\cos t = \frac{7}{4}e^t + \frac{7}{4}e^{-t} - \frac{3}{2}\cos t.$$

**7.** We will first write this system without using operator notation. Thus, we have

$$\begin{aligned} x' - 4x + 6y &= 9e^{-3t}, \\ x - y' + y &= 5e^{-3t}. \end{aligned} \tag{7.57}$$

By taking the Laplace transform of both sides of both of these differential equations and using the linearity of the Laplace transform, we obtain

$$\begin{aligned} \mathcal{L}\{x'\}(s) - 4X(s) + 6Y(s) &= 9/(s+3), \\ X(s) - \mathcal{L}\{y'\}(s) + Y(s) &= 5/(s+3), \end{aligned} \tag{7.58}$$

where $X(s)$ and $Y(s)$ are the Laplace transforms of $x(t)$ and $y(t)$, respectively. Using the initial conditions $x(0) = -9$ and $y(0) = 4$, we can express

$$\mathcal{L}\{x'\}(s) = sX(s) - x(0) = sX(s) + 9,$$
$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 4.$$

Substituting these expressions into the system given in (7.58) and simplifying yields

$$(s-4)X(s) + 6Y(s) = -9 + \frac{9}{s+3} = \frac{-9s-18}{s+3},$$
$$X(s) + (-s+1)Y(s) = -4 + \frac{5}{s+3} = \frac{-4s-7}{s+3}.$$

By multiplying the second equation above by $-(s-4)$, adding the resulting equations, and simplifying, we obtain

$$\left(s^2 - 5s + 10\right)Y(s) = \frac{(4s+7)(s-4)}{s+3} + \frac{-9s-18}{s+3} = \frac{4s^2-18s-46}{s+3}$$
$$\Rightarrow \qquad Y(s) = \frac{4s^2-18s-46}{(s+3)(s^2-5s+10)}.$$

Note that the quadratic $s^2 - 5s + 10 = (s-5/2)^2 + 15/4$ is irreducible. The partial fractions decomposition yields

$$\begin{aligned} Y(s) &= \frac{1}{17}\left[\frac{46s-334}{(s-5/2)^2+15/4} + \frac{22}{s+3}\right] \\ &= \frac{1}{17}\left[46\left(\frac{s-5/2}{(s-5/2)^2+15/4}\right) - \frac{146\sqrt{15}}{5}\left(\frac{\sqrt{15}/2}{(s-5/2)^2+15/4}\right) + 22\,\frac{1}{s+3}\right], \end{aligned}$$

and so

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = \frac{46}{17}e^{5t/2}\cos\left(\frac{\sqrt{15}t}{2}\right) - \frac{146\sqrt{15}}{85}e^{5t/2}\sin\left(\frac{\sqrt{15}t}{2}\right) + \frac{22}{17}e^{-3t}.$$

From the second equation in the system (7.57) above, we find that

$$\begin{aligned} x(t) &= 5e^{-3t} + y'(t) - y(t) = 5e^{-3t} + \frac{115}{17}e^{5t/2}\cos\left(\frac{\sqrt{15}t}{2}\right) \\ &\quad - \left(\frac{23\sqrt{15}}{17} + \frac{73\sqrt{15}}{17}\right)e^{5t/2}\sin\left(\frac{\sqrt{15}t}{2}\right) - \frac{219}{17}e^{5t/2}\cos\left(\frac{\sqrt{15}t}{2}\right) - \frac{66}{17}e^{-3t} \end{aligned}$$

$$= -\frac{150}{17} e^{5t/2} \cos\left(\frac{\sqrt{15}t}{2}\right) - \frac{334\sqrt{15}}{85} e^{5t/2} \sin\left(\frac{\sqrt{15}t}{2}\right) - \frac{3}{17} e^{-3t} .$$

**9.** Taking the Laplace transform of both sides of both of these differential equations yields the system

$$\begin{aligned} &\mathcal{L}\{x''\}(s) + X(s) + 2\mathcal{L}\{y'\}(s) = 0, \\ &-3\mathcal{L}\{x''\}(s) - 3X(s) + 2\mathcal{L}\{y''\}(s) + 4Y(s) = 0, \end{aligned}$$

where $X(s) = \mathcal{L}\{x\}(s)$, $Y(s) = \mathcal{L}\{y\}(s)$. Using the initial conditions $x(0) = 2$, $x'(0) = -7$ and $y(0) = 4$, $y'(0) = -9$, we see that

$$\begin{aligned} \mathcal{L}\{x''\}(s) &= s^2X(s) - sx(0) - x'(0) = s^2X(s) - 2s + 7, \\ \mathcal{L}\{y'\}(s) &= sY(s) - y(0) = sY(s) - 4, \\ \mathcal{L}\{y''\}(s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 4s + 9. \end{aligned}$$

Substituting these expressions into the system given above yields

$$\begin{aligned} &[s^2X(s) - 2s + 7] + X(s) + 2[sY(s) - 4] = 0, \\ &-3[s^2X(s) - 2s + 7] - 3X(s) + 2[s^2Y(s) - 4s + 9] + 4Y(s) = 0, \end{aligned}$$

which simplifies to

$$\begin{aligned} &(s^2 + 1) X(s) + 2sY(s) = 2s + 1, \\ &-3(s^2 + 1) X(s) + 2(s^2 + 2) Y(s) = 2s + 3. \end{aligned} \tag{7.59}$$

Multiplying the first equation by 3 and adding the two resulting equations eliminates the function $X(s)$. Thus, we obtain

$$\left(2s^2 + 6s + 4\right) Y(s) = 8s + 6 \qquad \Rightarrow \qquad Y(s) = \frac{4s + 3}{(s + 2)(s + 1)} = \frac{5}{s + 2} - \frac{1}{s + 1} ,$$

where we have factored the expression $2s^2 + 6s + 4$ and used the partial fractions expansion. Taking the inverse Laplace transform, we obtain

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = 5\mathcal{L}^{-1}\left\{\frac{1}{s + 2}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\}(t) = 5e^{-2t} - e^{-t} .$$

To find the solution $x(t)$, we again examine the system given in (7.59) above. This time we will eliminate the function $Y(s)$ by multiplying the first equation by $s^2 + 2$ and the second equation by $-s$ and adding the resulting equations. Thus, we have

$$\begin{aligned} \left(s^2+3s+2\right)\left(s^2+1\right)X(s) &= 2s^3-s^2+s+2 \\ \Rightarrow \qquad X(s) &= \frac{2s^3-s^2+s+2}{(s+2)(s+1)(s^2+1)}\,. \end{aligned}$$

Expressing $X(s)$ in a partial fractions expansion, we find that

$$X(s)=\frac{4}{s+2}-\frac{1}{s+1}-\frac{s}{s^2+1}$$

and so

$$x(t)=\mathcal{L}^{-1}\left\{\frac{4}{s+2}-\frac{1}{s+1}-\frac{s}{s^2+1}\right\}(t)=4e^{-2t}-e^{-t}-\cos t.$$

Hence, the solution to this initial value problem is

$$x(t)=4e^{-2t}-e^{-t}-\cos t \quad \text{and} \quad y(t)=5e^{-2t}-e^{-t}\,.$$



**11.** Since

$$\begin{aligned} \mathcal{L}\left\{x'\right\}(s) &= sX(s)-x(0)=sX(s) \qquad \text{and} \\ \mathcal{L}\left\{y'\right\}(s) &= sY(s)-y(0)=sY(s)\,, \end{aligned}$$

applying the Laplace transform to the given equations yields

$$\begin{aligned} sX(s)+Y(s) &= \mathcal{L}\left\{1-u(t-2)\right\}(s)=\frac{1}{s}-\frac{e^{-2s}}{s}=\frac{1-e^{-2s}}{s}\,, \\ X(s)+sY(s) &= \mathcal{L}\left\{0\right\}(s)=0\,. \end{aligned}$$

From the second equation, $X(s)=-sY(s)$. Substituting this into the first equation, we eliminate $X(s)$ and obtain

$$\begin{aligned} -s^2Y(s)+Y(s) &= \frac{1-e^{-2s}}{s} \\ \Rightarrow \qquad Y(s) &= \frac{1-e^{-2s}}{s(1-s^2)}=\left(1-e^{-2s}\right)\left(\frac{1}{s}-\frac{1/2}{s-1}-\frac{1/2}{s+1}\right). \end{aligned}$$

Using now the linear property of the inverse Laplace transform and formula (6) on page 387, we get

$$y(t) \;=\; \mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1/2}{s-1}-\frac{1/2}{s+1}\right\}(t)-\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1/2}{s-1}-\frac{1/2}{s+1}\right\}(t-2)u(t-2)$$

$$= 1 - \frac{e^t + e^{-t}}{2} - \left[1 - \frac{e^{t-2} + e^{-(t-2)}}{2}\right] u(t-2).$$

Since, from the second equation in the original system, $x = -y'$, we have

$$\begin{aligned} x(t) &= -\left\{1 - \frac{e^t + e^{-t}}{2} - \left[1 - \frac{e^{t-2} + e^{-(t-2)}}{2}\right] u(t-2)\right\} \\ &= \frac{e^t - e^{-t}}{2} - \left[\frac{e^{t-2} - e^{-(t-2)}}{2}\right] u(t-2). \end{aligned}$$

**13.** Since, by formula (8) on page 387 of the text,

$$\mathcal{L}\{(\sin t)u(t-\pi)\}(s) = e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\}(s) = e^{-\pi s}\mathcal{L}\{-\sin t\}(s) = -\frac{e^{-\pi s}}{s^2+1},$$

applying the Laplace transform to the given system yields

$$\begin{aligned} &\mathcal{L}\{x'\}(s) - \mathcal{L}\{y'\}(s) = \mathcal{L}\{(\sin t)u(t-\pi)\}(s), \\ &\mathcal{L}\{x\}(s) + \mathcal{L}\{y'\}(s) = \mathcal{L}\{0\}(s) \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} &[sX(s) - 1] - [sY(s) - 1] = -\frac{e^{-\pi s}}{s^2+1}, \\ &X(s) + [sY(s) - 1] = 0, \end{aligned}$$

where we have used the initial conditions, $x(0) = 1$ and $y(0) = 1$, and Theorem 4, Section 7.3, to express $\mathcal{L}\{x'\}(s)$ and $\mathcal{L}\{y'\}(s)$ in terms of $X(s) = \mathcal{L}\{x\}(s)$ and $Y(s) = \mathcal{L}\{y\}(s)$. The above system simplifies to

$$\begin{aligned} &X(s) - Y(s) = -\frac{e^{-\pi s}}{s(s^2+1)}, \\ &X(s) + sY(s) = 1. \end{aligned}$$

From the second equation, $X(s) = 1 - sY(s)$, and with this substitution the first equation becomes

$$1 - sY(s) - Y(s) = -\frac{e^{-\pi s}}{s(s^2+1)} \quad \Rightarrow \quad Y(s) = \left[1 + \frac{e^{-\pi s}}{s(s^2+1)}\right]\frac{1}{s+1} = \frac{1}{s+1} + \frac{e^{-\pi s}}{s(s+1)(s^2+1)}.$$

Using partial fractions we express

$$Y(s) = \frac{1}{s+1} + e^{-\pi s}\left[\frac{1}{s} - \frac{1/2}{s+1} - \frac{(1/2)s}{s^2+1} - \frac{1/2}{s^2+1}\right]$$

and so

$$\begin{aligned} y(t) &= e^{-t} + \left[1 - \frac{1}{2}\,e^{-(t-\pi)} - \frac{1}{2}\,\cos(t-\pi) - \frac{1}{2}\,\sin(t-\pi)\right] u(t-\pi) \\ &= e^{-t} + \left[1 - \frac{1}{2}\,e^{-(t-\pi)} + \frac{1}{2}\,\cos t + \frac{1}{2}\,\sin t\right] u(t-\pi). \end{aligned}$$

Finally,

$$x(t) = -y'(t) = e^{-t} - \left[\frac{1}{2}\,e^{-(t-\pi)} - \frac{1}{2}\,\sin t + \frac{1}{2}\,\cos t\right] u(t-\pi).$$

**15.** First, note that the initial conditions are given at the point $t = 1$. Thus, for the Laplace transform method, we have to shift the argument to get zero initial point. Let us denote

$$u(t) := x(t+1) \qquad \text{and} \qquad v(t) := y(t+1).$$

The chain rule yields

$$u'(t) = x'(t+1)(t+1)' = x'(t+1), \quad v'(t) = y'(t+1)(t+1)' = y'(t+1).$$

In the original system, we substitute $t+1$ for $t$ to get

$$\begin{aligned} &x'(t+1) - 2y(t+1) = 2, \\ &x'(t+1) + x(t+1) - y'(t+1) = (t+1)^2 + 2(t+1) - 1, \end{aligned}$$

and make $u$ and $v$ substitution. This yields

$$\begin{aligned} &u'(t) - 2v(t) = 2, \\ &u'(t) + u(t) - v'(t) = (t+1)^2 + 2(t+1) - 1 = t^2 + 4t + 2 \end{aligned}$$

with initial conditions $u(0) = 1$, $v(0) = 0$. Taking the Laplace transform and using formula (2) on page 361 of the text, we obtain the system

$$\begin{aligned} &[sU(s) - 1] - 2V(s) = \frac{2}{s}\,, \\ &[sU(s) - 1] + U(s) - sV(s) = \frac{2}{s^3} + \frac{4}{s^2} + \frac{2}{s}\,, \end{aligned}$$

where $U(s) = \mathcal{L}\{u\}(s)$, $V(s) = \mathcal{L}\{v\}(s)$. Expressing

$$U(s) = \frac{2V(s)}{s} + \frac{2}{s^2} + \frac{1}{s}$$

from the first equation and substituting this into the second equation, we obtain

$$\left[\frac{2}{s}+2V(s)\right]+\left[\frac{2V(s)}{s}+\frac{2}{s^2}+\frac{1}{s}\right]-sV(s)=\frac{2}{s^3}+\frac{4}{s^2}+\frac{2}{s},$$

which yields

$$V(s)=\frac{1}{s^2} \qquad \Rightarrow \qquad U(s)=\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}.$$

Applying now inverse Laplace transforms yields

$$u(t)=t^2+2t+1=(t+1)^2, \qquad v(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t)=t.$$

Finally,

$$x(t)=u(t-1)=t^2 \quad \text{and} \quad y(t)=v(t-1)=t-1.$$

**17.** As in Problem 15, first we make a shift in $t$ to move the initial conditions to $t=0$. Let

$$u(t):=x(t+2) \qquad \text{and} \qquad v(t):=y(t+2).$$

With $t$ replaced by $t+2$, the original system becomes

$$x'(t+2)+x(t+2)-y'(t+2)=2te^t,$$
$$x''(t+2)-x'(t+2)-2y(t+2)=-e^t$$

or

$$\begin{aligned} u'(t)+u(t)-v'(t)&=2te^t, \\ u''(t)-u'(t)-2v(t)&=-e^t, \end{aligned} \qquad \text{with} \qquad \begin{aligned} u(0)&=0, \\ u'(0)&=1, \\ v(0)&=1. \end{aligned}$$

Applying the Laplace transform to these equations and expressing $\mathcal{L}\{u''\}$, $\mathcal{L}\{u'\}$, and $\mathcal{L}\{v'\}$ in terms of $U=\mathcal{L}\{u\}$ and $V=\mathcal{L}\{v\}$ (see formula (4) on page 362 of the text, we obtain

$$[sU(s)]+U(s)-[sV(s)-1]=2\mathcal{L}\left\{te^t\right\}(s)=\frac{2}{(s-1)^2},$$
$$\left[s^2U(s)-1\right]-[sU(s)]-2V(s)=-\frac{1}{s-1}.$$

We multiply the first equation by 2, the second equation by $s$, and subtract the resulting equations in order to eliminate $V(s)$. Thus we get

$$\left[s(s^2-s)-2(s+1)\right]U(s)=s-\frac{s}{s-1}-\frac{4}{(s-1)^2}+2$$

$$\Rightarrow \qquad \left(s^3 - s^2 - 2s - 2\right) U(s) = \frac{s^3 - s^2 - 2s - 2}{(s-1)^2} \qquad \Rightarrow \qquad U(s) = \frac{1}{(s-1)^2}\,.$$

The inverse Laplace transform then yields

$$u(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}(t) = te^t \qquad \Rightarrow \qquad x(t) = u(t-2) = (t-2)e^{t-2}\,.$$

We find $y(t)$ from the second equation in the original system.

$$y(t) = \frac{x''(t) - x'(t) + e^{t-2}}{2} = \frac{te^{t-2} - (t-1)e^{t-2} + e^{t-2}}{2} = e^{t-2}\,.$$

**19.** We first take the Laplace transform of both sides of all three of these equations and use the initial conditions to obtain a system of equations for the Laplace transforms of the solution functions:

$$\begin{aligned} sX(s) + 6 &= 3X(s) + Y(s) - 2Z(s), \\ sY(s) - 2 &= -X(s) + 2Y(s) + Z(s), \\ sZ(s) + 12 &= 4X(s) + Y(s) - 3Z(s). \end{aligned}$$

Simplifying yields

$$\begin{aligned} &(s-3)X(s) - Y(s) + 2Z(s) = -6, \\ &X(s) + (s-2)Y(s) - Z(s) = 2, \\ &-4X(s) - Y(s) + (s+3)Z(s) = -12. \end{aligned} \tag{7.60}$$

To solve this system, we will use substitution to eliminate the function $Y(s)$. Therefore, we solve for $Y(s)$ in the first equation in (7.60) to obtain

$$Y(s) = (s-3)X(s) + 2Z(s) + 6.$$

Substituting this expression into the two remaining equations in (7.60) and simplifying yields

$$\begin{aligned} &(s^2 - 5s + 7)X(s) + (2s-5)Z(s) = -6s + 14, \\ &-(s+1)X(s) + (s+1)Z(s) = -6. \end{aligned} \tag{7.61}$$

Next we will eliminate the function $X(s)$ from the system given in (7.61). To do this we can either multiply the first equation by $(s+1)$ and the second by $(s^2 - 5s + 7)$ and add, or we can solve the last equation given in (7.61) for $X(s)$ to obtain

$$X(s) = Z(s) + \frac{6}{s+1}\,, \tag{7.62}$$

and substitute this into the first equation in (7.61). By either method we see that

$$Z(s) = \frac{-12s^2 + 38s - 28}{(s+1)(s^2 - 3s + 2)} = \frac{-12s^2 + 38s - 28}{(s+1)(s-2)(s-1)}\,.$$

Now, $Z(s)$ has the partial fraction expansion

$$Z(s) = \frac{-13}{s+1} + \frac{1}{s-1}.$$

Therefore, by taking inverse Laplace transforms of both sides of this equation, we obtain

$$z(t) = \mathcal{L}^{-1}\{Z(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{-13}{s+1} + \frac{1}{s-1}\right\}(t) = -13e^{-t} + e^{t}\,.$$

To find $X(s)$, we will use equation (7.62) and the expression found above for $Z(s)$. Thus, we have

$$X(s) = Z(s) + \frac{6}{s+1} = \frac{-13}{s+1} + \frac{1}{s-1} + \frac{6}{s+1} = \frac{-7}{s+1} + \frac{1}{s-1}$$

$$\Rightarrow \qquad x(t) = \mathcal{L}^{-1}\{X(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{-7}{s+1} + \frac{1}{s-1}\right\}(t) = -7e^{-t} + e^{t}\,.$$



To find $y(t)$, we could substitute the expressions that we have already found for $X(s)$ and $Z(s)$ into the $Y(s) = (s-3)X(s) + 2Z(s) + 6$, which we found above, or we could return to the original system of differential equations and use $x(t)$ and $z(t)$ to solve for $y(t)$. For the latter method, we solve the first equation in the original system for $y(t)$ to obtain

$$\begin{aligned} y(t) &= x'(t) - 3x(t) + 2z(t) \\ &= 7e^{-t} + e^{t} + 21e^{-t} - 3e^{t} - 26e^{-t} + 2e^{t} = 2e^{-t}\,. \end{aligned}$$

Therefore, the solution to the initial value problem is

$$x(t) = -7e^{-t} + e^{t}\,, \qquad y(t) = 2e^{-t}\,, \qquad z(t) = -13e^{-t} + e^{t}\,.$$

**21.** We refer the reader to the discussion in Section 5.1 in obtaining the system (1) on page 242 of the text governing interconnected tanks. All the arguments provided remain in force except for the one affected by the new "valve condition", which the formula for the input rate for the tank A. In Section 5.1, just fresh water was pumped into the tank A and so there was no salt coming from outside of the system into the tank A . Now we have more complicated rule: the incoming liquid is fresh water for the first 5 min, but then it changes to a solution having a concentration 2 kg/L. This solution contributes additional

$$2\,(\text{kg/L}) \times 6\,(\text{L/min}) = 12\,(\text{kg/min})$$

to the input rate into the tank A. Thus, from the valve, we have

$$\begin{cases} 0, & t < 5, \\ 12, & t > 5 \end{cases} \quad = 12u(t-5)\ (\text{kg/min})$$

of salt coming to the tank A. With this change, the system (1) in the text becomes

$$\begin{aligned} x' &= -x/3 + y/12 + 12u(t-5), \\ y' &= x/3 - y/3. \end{aligned} \tag{7.63}$$

Also, we have the initial conditions $x(0) = x_0 = 0$, $y(0) = y_0 = 4$. Let $X := \mathcal{L}\{x\}$ and $Y := \mathcal{L}\{y\}$. Taking the Laplace transform of both equations in the system above, we get

$$\begin{aligned} \mathcal{L}\{x'\}(s) &= -\frac{1}{3}X(s) + \frac{1}{12}Y(s) + 12\mathcal{L}\{u(t-5)\}(s), \\ \mathcal{L}\{y'\}(s) &= \frac{1}{3}X(s) - \frac{1}{3}Y(s). \end{aligned}$$

Since $\mathcal{L}\{u(t-5)\}(s) = e^{-5s}/s$ and

$$\begin{aligned} \mathcal{L}\{x'\}(s) &= sX(s) - x(0) = sX(s), \\ \mathcal{L}\{y'\}(s) &= sY(s) - y(0) = sY(s) - 4, \end{aligned}$$

we obtain

$$\begin{aligned} sX(s) &= -\frac{1}{3}X(s) + \frac{1}{12}Y(s) + \frac{12e^{-5s}}{s}, \\ sY(s) - 4 &= \frac{1}{3}X(s) - \frac{1}{3}Y(s) \end{aligned}$$

which simplifies to

$$\begin{aligned} 4(3s+1)X(s) - Y(s) &= \frac{144e^{-5s}}{s}, \\ -X(s) + (3s+1)Y(s) &= 12. \end{aligned}$$

From the second equation in this system, we have $X(s) = (3s+1)Y(s) - 12$. Substitution into the first equation yields

$$\begin{aligned} 4(3s+1)\left[(3s+1)Y(s) - 12\right] - Y(s) &= \frac{144e^{-5s}}{s} \\ \Rightarrow \qquad \left[4(3s+1)^2 - 1\right]Y(s) &= 48(3s+1) + \frac{144e^{-5s}}{s}. \end{aligned}$$

Note that

$$4(3s+1)^2 - 1 = [2(3s+1)+1]\cdot[2(3s+1)-1] = (6s+3)(6s+1) = 36\left(s+\frac{1}{2}\right)\left(s+\frac{1}{6}\right).$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{4(3s+1)}{3(s+1/2)(s+1/6)} + \frac{4e^{-5s}}{s(s+1/2)(s+1/6)} \\ &= \frac{2}{(s+1/2)} + \frac{2}{(s+1/6)} + e^{-5s}\left[\frac{48}{s} + \frac{24}{s+1/2} - \frac{72}{s+1/6}\right], \end{aligned}$$

where we have applied the partial fractions decomposition. Taking the inverse Laplace transform and using Theorem 8 in Section 7.6 for the inverse Laplace transform of the term having the exponential factor, we get

$$\begin{aligned} y(t) &= 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1/2)}\right\}(t) + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1/6)}\right\}(t) \\ &\quad + \left[48\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 24\mathcal{L}^{-1}\left\{\frac{1}{s+1/2}\right\} - 72\mathcal{L}^{-1}\left\{\frac{1}{s+1/6}\right\}\right](t-5)u(t-5) \\ &= 2e^{-t/2} + 2e^{-t/6} + \left[48 + 24e^{-(t-5)/2} - 72e^{-(t-5)/6}\right]u(t-5). \end{aligned}$$

From the second equation in (7.63), after some algebra, we find $x(t)$.

$$x(t) = 3y'(t) + y = -e^{-t/2} + e^{-t/6} + \left[48 - 12e^{-(t-5)/2} - 36e^{-(t-5)/6}\right]u(t-5).$$

**23.** Recall that Kirchhoff's voltage law says that, in an electrical circuit consisting of an inductor of $L$ H, a resistor of $R\,\Omega$, a capacitor of $C$ F, and a voltage source of $E$ V,

$$E_L + E_R + E_C = E, \qquad (7.64)$$

where $E_L$, $E_R$, and $E_C$ denote the voltage drops across the inductor, resistor, and capacitor, respectively. These voltage grops are given by

$$E_L = L\frac{dI}{dt}, \qquad E_R := RI, \qquad E_C := \frac{q}{C}, \tag{7.65}$$

where $I$ denotes the current passing through the correspondent element.

Also, Kirchhoff's current law states that the algebraic sum of currents passing through any point in an electrical network equals to zero.

The electrical network shown in Figure 7.28 consists of three closed circuits: loop 1 through the battery, $R_1 = 2\,\Omega$ resistor, $L_1 = 0.1\,\text{H}$ inductor, and $L_2 = 0.2\,\text{H}$ inductor; loop 2 through the inductor $L_1$ and $R_2 = 1\,\Omega$ resistor; loop 3 through the battery, resistors $R_1$ and $R_2$, and inductor $L_2$. We apply Kirchhoff's voltage law (7.64) to two of these loops, say, the loop 1 and the loop 2, and (since the equation obtained from Kirchhoff's voltage law for the third loop is a linear combination of the other two) Kirchhoff's current law to one of the junction points, say, the upper one. Thus, choosing the clockwise direction in the loops and using formulas (7.65), we obtain



Loop 1:

$$E_{R_1} + E_{L_1} + E_{L_2} = E \qquad \Rightarrow \qquad 2I_1 + 0.1I_3' + 0.2I_1' = 6;$$

Loop 2:

$$E_{L_1} + E_{R_2} = 0 \qquad \Rightarrow \qquad 0.1I_3' - I_2 = 0$$

with the negative sign due to the counterclockwise direction of the current $I_2$ in this loop;

Upper junction point:

$$I_1 - I_2 - I_3 = 0.$$

Therefore, we have the following system for the currents $I_1$, $I_2$, and $I_3$:

$$\begin{aligned} &2I_1 + 0.1I_3' + 0.2I_1' = 6,\\ &0.1I_3' - I_2 = 0,\\ &I_1 - I_2 - I_3 = 0 \end{aligned} \tag{7.66}$$

with initial conditions $I_1(0) = I_2(0) = I_3(0) = 0$.

Let $\mathbf{I}_1(s) := \mathcal{L}\{I_1\}(s)$, $\mathbf{I}_2(s) := \mathcal{L}\{I_2\}(s)$, and $\mathbf{I}_3(s) := \mathcal{L}\{I_3\}(s)$. Using the initial conditions, we conclude that

$$\begin{aligned}\mathcal{L}\{I_1'\}(s) &= s\mathbf{I}_1(s) - I_1(0) = s\mathbf{I}_1(s),\\ \mathcal{L}\{I_3'\}(s) &= s\mathbf{I}_3(s) - I_3(0) = s\mathbf{I}_3(s).\end{aligned}$$

Using these equations and taking the Laplace transform of the equations in (7.66), we come up with

$$\begin{aligned}&(0.2s+2)\mathbf{I}_1(s) + 0.1s\mathbf{I}_3(s) = \frac{6}{s},\\ &0.1s\mathbf{I}_3(s) - \mathbf{I}_2(s) = 0,\\ &\mathbf{I}_1(s) - \mathbf{I}_2(s) - \mathbf{I}_3(s) = 0\end{aligned}$$

Expressing $\mathbf{I}_2(s) = 0.1s\mathbf{I}_3(s)$ from the second equation and substituting this into the third equation, we get

$$\mathbf{I}_1(s) - 0.1s\mathbf{I}_3(s) - \mathbf{I}_3(s) = 0 \quad \Rightarrow \quad \mathbf{I}_1(s) = (0.1s+1)\mathbf{I}_3(s).$$

The latter, when substituted into the first equation, yields

$$\begin{aligned}&(0.2s+2)(0.1s+1)\mathbf{I}_3(s) + 0.1s\mathbf{I}_3(s) = \frac{6}{s}\\ \Rightarrow \quad &\left[2(0.1s+1)^2 + 0.1s\right]\mathbf{I}_3(s) = \frac{6}{s}\\ \Rightarrow \quad &\mathbf{I}_3(s) = \frac{6}{s[2(0.1s+1)^2+0.1s]} = \frac{300}{s(s+20)(s+5)}.\end{aligned}$$

We use the partial fractions decomposition to find that

$$\mathbf{I}_3(s) = \frac{3}{s} + \frac{1}{s+20} - \frac{4}{s+5}$$

and so

$$I_3(t) = \mathcal{L}^{-1}\left\{\frac{3}{s} + \frac{1}{s+20} - \frac{4}{s+5}\right\}(t) = 3 + e^{-20t} - 4e^{-5t}.$$

Now we can find $I_2(t)$ using the second equation in (7.66).

$$I_2(t) = 0.1I_3'(t) = 0.1\left(3 + e^{-20t} - 4e^{-5t}\right)' = -2e^{-20t} + 2e^{-5t}.$$

Finally, the third equation in (7.66) yields

$$I_1(t) = I_2(t) + I_3(t) = 3 - e^{-20t} - 2e^{-5t}.$$

## REVIEW PROBLEMS: page 418

**1.** By the definition of Laplace transform,

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^2 e^{-st}(3)\, dt + \int_2^\infty e^{-st}(6-t)\, dt.$$

For the first integral, we have

$$\int_0^2 e^{-st}(3)\, dt = \left.\frac{3e^{-st}}{-s}\right|_{t=0}^{t=2} = \frac{3(1-e^{-2s})}{s}.$$

The second integral is an improper integral. Using integration by parts, we obtain

$$\begin{aligned}
\int_2^\infty e^{-st}(6-t)\, dt &= \lim_{M\to\infty} \int_2^M e^{-st}(6-t)\, dt = \lim_{M\to\infty} \left[ (6-t)\left.\frac{e^{-st}}{-s}\right|_{t=2}^{t=M} - \int_2^M \frac{e^{-st}}{-s}(-1)dt \right] \\
&= \lim_{M\to\infty} \left[ \frac{4e^{-2s}}{s} - \frac{(6-M)e^{-sM}}{s} + \left.\frac{e^{-st}}{s^2}\right|_{t=2}^{t=M} \right] \\
&= \lim_{M\to\infty} \left[ \frac{4e^{-2s}}{s} - \frac{(6-M)e^{-sM}}{s} + \frac{e^{-sM}}{s^2} - \frac{e^{-2s}}{s^2} \right] = \frac{4e^{-2s}}{s} - \frac{e^{-2s}}{s^2}.
\end{aligned}$$

Thus

$$\mathcal{L}\{f\}(s) = \frac{3(1-e^{-2s})}{s} + \frac{4e^{-2s}}{s} - \frac{e^{-2s}}{s^2} = \frac{3}{s} + e^{-2s}\left(\frac{1}{s} - \frac{1}{s^2}\right).$$

**3.** From Table 7.1 on page 358 of the text, using the formula for the Laplace transform of $e^{at}t^n$ with $n = 2$ and $a = -9$, we get

$$\mathcal{L}\left\{t^2 e^{-9t}\right\}(s) = \frac{2!}{[s-(-9)]^3} = \frac{2}{(s+9)^3}.$$

**5.** We use the linearity of the Laplace transform and Table 7.1 to obtain

$$\begin{aligned}
\mathcal{L}\left\{e^{2t} - t^3 + t^2 - \sin 5t\right\}(s) &= \mathcal{L}\left\{e^{2t}\right\}(s) - \mathcal{L}\left\{t^3\right\}(s) + \mathcal{L}\left\{t^2\right\}(s) - \mathcal{L}\left\{\sin 5t\right\}(s) \\
&= \frac{1}{s-2} - \frac{3!}{s^4} + \frac{2!}{s^3} - \frac{5}{s^2+5^2} = \frac{1}{s-2} - \frac{6}{s^4} + \frac{2}{s^3} - \frac{5}{s^2+25}.
\end{aligned}$$

**7.** We apply Theorem 6 in Section 7.3 and obtain

$$\mathcal{L}\left\{t\cos 6t\right\}(s) = -\frac{d}{ds}\mathcal{L}\left\{\cos 6t\right\}(s) = -\frac{d}{ds}\left[\frac{s}{s^2+6^2}\right] = -\frac{(s^2+36)-s(2s)}{(s^2+36)^2} = \frac{s^2-36}{(s^2+36)^2}\,.$$

**9.** We apply formula (8), Section 7.6, on page 387 of the text and the linear property of the Laplace transform to get

$$\begin{aligned}\mathcal{L}\left\{t^2u(t-4)\right\}(s) &= e^{-4s}\mathcal{L}\left\{(t+4)^2\right\}(s) = e^{-4s}\mathcal{L}\left\{t^2+8s+16\right\}(s)\\ &= e^{-4s}\left(\frac{2}{s^3}+\frac{8}{s^2}+\frac{16}{s}\right) = 2e^{-4s}\left(\frac{1}{s^3}+\frac{4}{s^2}+\frac{8}{s}\right).\end{aligned}$$

**11.** Using the linearity of the inverse Laplace transform and Table 7.1 we find

$$\mathcal{L}^{-1}\left\{\frac{7}{(s+3)^3}\right\}(t) = \frac{7}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{[s-(-3)]^3}\right\}(t) = \frac{7}{2}t^2e^{-3t}\,.$$

**13.** We apply partial fractions to find the inverse Laplace transform. Since the quadratic polynomial $s^2+4s+13=(s+2)^2+3^2$ is irreducible, the partial fraction decomposition for the given function has the form

$$\frac{4s^2+13s+19}{(s-1)(s^2+4s+13)} = \frac{A}{s-1} + \frac{B(s+2)+C(3)}{(s+2)^2+3^2}\,.$$

Clearing fractions yields

$$4s^2+13s+19 = A[(s+2)^2+3^2] + [B(s+2)+C(3)](s-1)\,.$$

With $s=1$, this gives $36=18A$ or $A=2$. Substituting $s=-2$, we get

$$9 = 9A - 9C \qquad \Rightarrow \qquad C = A-1 = 1.$$

Finally, with $s=0$, we compute

$$19 = 13A + (2B+3C)(-1) \qquad \Rightarrow \qquad B = 2.$$

Thus

$$\frac{4s^2+13s+19}{(s-1)(s^2+4s+13)} = \frac{2}{s-1} + \frac{2(s+2)+(1)(3)}{(s+2)^2+3^2}\,,$$

and so

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{4s^2+13s+19}{(s-1)(s^2+4s+13)}\right\}(t) &= 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t)+2\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\}(t) \\ &\quad +\mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\}(t) \\ &= 2e^t+2e^{-2t}\cos 3t+e^{-2t}\sin 3t\,.\end{aligned}$$

**15.** The partial fraction decomposition for the given function has the form

$$\frac{2s^2+3s-1}{(s+1)^2(s+2)}=\frac{A}{(s+1)^2}+\frac{B}{s+1}+\frac{C}{s+2}=\frac{A(s+2)+B(s+1)(s+2)+C(s+1)^2}{(s+1)^2(s+2)}\,.$$

Thus

$$2s^2+3s-1=A(s+2)+B(s+1)(s+2)+C(s+1)^2\,.$$

We evaluate both sides of this equation at $s=-2$, $-1$, and 0. This yields

$$\begin{aligned} s=-2: &\quad 2(-2)^2+3(-2)-1=C(-2+1)^2 &\Rightarrow&\quad C=1, \\ s=-1: &\quad 2(-1)^2+3(-1)-1=A(-1+2) &\Rightarrow&\quad A=-2, \\ s=0: &\quad -1=2A+2B+C &\Rightarrow&\quad B=(-1-2A-C)/2=1. \end{aligned}$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{2s^2+3s-1}{(s+1)^2(s+2)}\right\}(t)=\mathcal{L}^{-1}\left\{\frac{-2}{(s+1)^2}+\frac{1}{s+1}+\frac{1}{s+2}\right\}(t)=-2te^{-t}+e^{-t}+e^{-2t}\,.$$

**17.** First we apply Theorem 8 in Section 7.6 to get

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}(4s+2)}{(s-1)(s+2)}\right\}(t)=\mathcal{L}^{-1}\left\{\frac{4s+2}{(s-1)(s+2)}\right\}(t-2)u(t-2). \tag{7.67}$$

Applying partial fractions yields

$$\frac{4s+2}{(s-1)(s+2)}=\frac{2}{s-1}+\frac{2}{s+2} \quad\Rightarrow\quad \mathcal{L}^{-1}\left\{\frac{4s+2}{(s-1)(s+2)}\right\}(t)=2e^t+2e^{-2t}\,.$$

Therefore, it follows from (7.67) that

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}(4s+2)}{(s-1)(s+2)}\right\}(t)=\left[2e^{t-2}+2e^{-2(t-2)}\right]u(t-2)=\left(2e^{t-2}+2e^{4-2t}\right)u(t-2).$$

**19.** Applying the Laplace transform to both sides of the given equation and using the linearity of the Laplace transform yields

$$\mathcal{L}\left\{y''-7y'+10y\right\}(s)=\mathcal{L}\left\{y''\right\}(s)-7\mathcal{L}\left\{y'\right\}(s)+10\mathcal{L}\left\{y\right\}(s)=0. \tag{7.68}$$

By Theorem 5 in Section 7.3,

$$\begin{aligned}\mathcal{L}\left\{y'\right\}(s)&=s\mathcal{L}\left\{y\right\}(s)-y(0)=s\mathcal{L}\left\{y\right\}(s),\\ \mathcal{L}\left\{y''\right\}(s)&=s^2\mathcal{L}\left\{y\right\}(s)-sy(0)-y'(0)=s^2\mathcal{L}\left\{y\right\}(s)+3,\end{aligned}$$

where we have used the initial conditions, $y(0)=0$ and $y'(0)=-3$. Substituting these expressions into (7.68), we get

$$\begin{aligned}&\left[s^2\mathcal{L}\left\{y\right\}(s)+3\right]-7\left[s\mathcal{L}\left\{y\right\}(s)\right]+10\mathcal{L}\left\{y\right\}(s)=0\\ \Rightarrow\quad&(s^2-7s+10)\mathcal{L}\left\{y\right\}(s)+3=0\\ \Rightarrow\quad&\mathcal{L}\left\{y\right\}(s)=\frac{-3}{s^2-7s+10}=\frac{-3}{(s-2)(s-5)}=\frac{1}{s-2}-\frac{1}{s-5}.\end{aligned}$$

Thus

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{s-2}-\frac{1}{s-5}\right\}(t)=\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t)-\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}(t)=e^{2t}-e^{5t}.$$



**21.** Let $Y(s):=\mathcal{L}\left\{y\right\}(s)$. Taking the Laplace transform of the given equation and using properties of the Laplace transform, we obtain

$$\mathcal{L}\left\{y''+2y'+2y\right\}(s)=\mathcal{L}\left\{t^2+4t\right\}(s)=\frac{2}{s^3}+\frac{4}{s^2}=\frac{2+4s}{s^3}.$$

Since

$$\mathcal{L}\left\{y'\right\}(s)=sY(s)-y(0)=sY(s),\qquad \mathcal{L}\left\{y''\right\}(s)=s^2Y(s)-sy(0)-y'(0)=s^2Y(s)+1,$$

we have

$$\begin{aligned}&\left[s^2Y(s)+1\right]+2\left[sY(s)\right]+2Y(s)=\frac{2+4s}{s^3}\\ \Rightarrow\quad&(s^2+2s+2)Y(s)=\frac{2+4s}{s^3}-1=\frac{2+4s-s^3}{s^3}\end{aligned}$$

$$\Rightarrow \qquad Y(s) = \frac{2+4s-s^3}{s^3(s^2+2s+2)} = \frac{2+4s-s^3}{s^3[(s+1)^2+1^2]}.$$

The partial fraction decomposition for $Y(s)$ has the form

$$\frac{2+4s-s^3}{s^3[(s+1)^2+1^2]} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D(s+1)+E(1)}{(s+1)^2+1^2}.$$

Clearing fractions, we obtain

$$2+4s-s^3 = A[(s+1)^2+1] + Bs[(s+1)^2+1] + Cs^2[(s+1)^2+1] + [D(s+1)+E]s^3.$$

Comparing coefficients at the corresponding power of $s$ in both sides of this equation yields

$$\begin{aligned}
s^0: &\quad 2 = 2A && \Rightarrow && A = 1,\\
s^1: &\quad 4 = 2A + 2B && \Rightarrow && B = (4-2A)/2 = 1,\\
s^2: &\quad 0 = A + 2B + 2C && \Rightarrow && C = -(A+2B)/2 = -3/2,\\
s^4: &\quad 0 = C + D && \Rightarrow && D = -C = 3/2,\\
s^3: &\quad -1 = B + 2C + D + E && \Rightarrow && E = -1 - B - 2C - D = -1/2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
Y(s) &= \frac{1}{s^3} + \frac{1}{s^2} - \frac{3/2}{s} + \frac{(3/2)(s+1)}{(s+1)^2+1^2} - \frac{(1/2)(1)}{(s+1)^2+1^2}\\
\Rightarrow \qquad y(t) &= \mathcal{L}^{-1}\{Y(s)\}(t) = \frac{t^2}{2} + t - \frac{3}{2} + \frac{3}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t.
\end{aligned}$$

**23.** By formula (4) in Section 7.6,

$$\mathcal{L}\{u(t-1)\}(s) = \frac{e^{-s}}{s}.$$

Thus, applying the Laplace transform to both sides of the given equation and using the initial conditions, we get

$$\begin{aligned}
&\mathcal{L}\{y'' + 3y' + 4y\}(s) = \frac{e^{-s}}{s}\\
\Rightarrow \qquad & \left[s^2Y(s) - 1\right] + 3\left[sY(s)\right] + 4Y(s) = \frac{e^{-s}}{s}\\
\Rightarrow \qquad & Y(s) = \frac{1}{s^2+3s+4} + \frac{e^{-s}}{s(s^2+3s+4)}
\end{aligned}$$

$$\Rightarrow \qquad Y(s) = \frac{1}{(s+3/2)^2 + (\sqrt{7}/2)^2} + e^{-s}\frac{1}{s[(s+3/2)^2 + (\sqrt{7}/2)^2]},$$

where $Y(s) := \mathcal{L}\{y\}(s)$. To apply the inverse Laplace transform, we need the partial fraction decomposition of the last fraction above.

$$\frac{1}{s[(s+3/2)^2 + (\sqrt{7}/2)^2]} = \frac{A}{s} + \frac{B(s+3/2) + C(\sqrt{7}/2)}{(s+3/2)^2 + (\sqrt{7}/2)^2}.$$

Solving for $A$, $B$, and $C$ yields

$$A = \frac{1}{4}, \qquad B = -\frac{1}{4}, \qquad C = -\frac{3}{4\sqrt{7}}.$$

Therefore,

$$Y(s) = \frac{1}{(s+3/2)^2 + (\sqrt{7}/2)^2} + e^{-s}\left[\frac{1/4}{s} - \frac{(1/4)(s+3/2)}{(s+3/2)^2 + (\sqrt{7}/2)^2} - \frac{(3/4\sqrt{7})(\sqrt{7}/2)}{(s+3/2)^2 + (\sqrt{7}/2)^2}\right]$$

and the inverse Laplace transform gives

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+3/2)^2 + (\sqrt{7}/2)^2}\right\}(t) \\ &\quad + \mathcal{L}^{-1}\left\{\frac{1/4}{s} - \frac{(1/4)(s+3/2)}{(s+3/2)^2 + (7/4)} - \frac{(3/4\sqrt{7})(\sqrt{7}/2)}{(s+3/2)^2 + (7/4)}\right\}(t-1)u(t-1) \\ &= \frac{2}{\sqrt{7}}e^{-3t/2}\sin\left(\frac{\sqrt{7}t}{2}\right) \\ &\quad + \left[\frac{1}{4} - \frac{1}{4}e^{-3(t-1)/2}\cos\left(\frac{\sqrt{7}(t-1)}{2}\right) - \frac{3}{4\sqrt{7}}e^{-3(t-1)/2}\sin\left(\frac{\sqrt{7}(t-1)}{2}\right)\right]u(t-1). \end{aligned}$$

**25.** Let $Y(s) := \mathcal{L}\{y\}(s)$. Then, from the initial conditions, we have

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s), \qquad \mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s).$$

Moreover, Theorem 6 in Section 7.3 yields

$$\begin{aligned} \mathcal{L}\{ty'\}(s) &= -\frac{d}{ds}\mathcal{L}\{y'\}(s) = -\frac{d}{ds}[sY(s)] = -sY'(s) - Y(s), \\ \mathcal{L}\{ty''\}(s) &= -\frac{d}{ds}\mathcal{L}\{y''\}(s) = -\frac{d}{ds}\left[s^2Y(s)\right] = -s^2Y'(s) - 2sY(s). \end{aligned}$$

Hence, applying the Laplace transform to the given equation and using the linearity of the Laplace transform, we obtain

$$\mathcal{L}\{ty''+2(t-1)y'-2y\}(s)=\mathcal{L}\{ty''\}(s)+2\mathcal{L}\{ty'\}(s)-2\mathcal{L}\{y'\}(s)-2\mathcal{L}\{y\}(s)=0$$
$$\Rightarrow\qquad\left[-s^2Y'(s)-2sY(s)\right]+2\left[-sY'(s)-Y(s)\right]-2\left[sY(s)\right]-2Y(s)=0$$
$$\Rightarrow\qquad -s(s+2)Y'(s)-4(s+1)Y(s)=0\qquad\Rightarrow\qquad Y'(s)+\frac{4(s+1)}{s(s+2)}Y(s)=0.$$

Separating variables and integrating yields

$$\frac{dY}{Y}=-\frac{4(s+1)}{s(s+2)}\,ds=-2\left(\frac{1}{s}+\frac{1}{s+2}\right)ds$$
$$\Rightarrow\qquad \ln|Y|=-2(\ln|s|+\ln|s+2|)+C$$
$$\Rightarrow\qquad Y(s)=\pm\frac{e^C}{s^2(s+2)^2}=\frac{c_1}{s^2(s+2)^2},$$

where $c_1\neq 0$ is an arbitrary constant. Allowing $c_1=0$, we also get the solution $Y(s)\equiv 0$, which was lost in separation of variables. Thus

$$Y(s)=\frac{c_1}{s^2(s+2)^2}=\frac{c_1}{4}\left[\frac{1}{s^2}-\frac{1}{s}+\frac{1}{(s+2)^2}+\frac{1}{s+2}\right]$$

and so

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}(t)=\frac{c_1}{4}\left(t-1+te^{-2t}+e^{-2t}\right)=c\left(t-1+te^{-2t}+e^{-2t}\right),$$

where $c=c_1/4$ is an arbitrary constant.

**27.** Note that the original equation can be written in the form

$$y(t)+t*y(t)=e^{-3t}.$$

Let $Y(s):=\mathcal{L}\{y\}(s)$. Applying the Laplace transform to both sides of this equation and using Theorem 11 in Section 7.7, we obtain

$$\mathcal{L}\{y(t)+t*y(t)\}(s)=Y(s)+\mathcal{L}\{t\}(s)Y(s)=\mathcal{L}\left\{e^{-3t}\right\}(s)$$
$$\Rightarrow\qquad Y(s)+\frac{1}{s^2}Y(s)=\frac{1}{s+3}\qquad\Rightarrow\qquad Y(s)=\frac{s^2}{(s+3)(s^2+1)}.$$

The partial fraction decomposition for $Y(s)$ has the form

$$\frac{s^2}{(s+3)(s^2+1)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+1} = \frac{A(s^2+1)+(Bs+C)(s+3)}{(s+3)(s^2+1)} .$$

Thus

$$s^2 = A(s^2+1) + (Bs+C)(s+3).$$

Evaluating both sides of this equation at $s = -3$, 0, and $-2$ yields

$$\begin{aligned}
s=-3: \quad &\Rightarrow \quad 9 = A(10) &&\Rightarrow \quad A = 9/10, \\
s=0: \quad &\Rightarrow \quad 0 = A + 3C &&\Rightarrow \quad C = -A/3 = -3/10, \\
s=-2: \quad &\Rightarrow \quad 4 = 5A - 2B + C &&\Rightarrow \quad B = (5A + C - 4)/2 = 1/10.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&Y(s) = \frac{9/10}{s+3} + \frac{(1/10)s}{s^2+1} - \frac{3/10}{s^2+1} \\
\Rightarrow \quad &y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = \frac{9}{10}e^{-3t} + \frac{1}{10}\cos t - \frac{3}{10}\sin t .
\end{aligned}$$



**29.** To find the transfer function, we use formula (15) on page 403 of the text. Comparing given equation with (14), we find that $a = 1$, $b = -5$, and $c = 6$. Thus (15) yields

$$H(s) = \frac{1}{as^2 + bs + c} = \frac{1}{s^2 - 5s + 6} .$$

The impulse response function $h(t)$ is defined as $\mathcal{L}^{-1}\{H\}(t)$. Using partial fractions, we see that

$$\begin{aligned}
&H(s) = \frac{1}{s^2 - 5s + 6} = \frac{1}{(s-3)(s-2)} = \frac{1}{s-3} - \frac{1}{s-2} \\
\Rightarrow \quad &h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-3} - \frac{1}{s-2}\right\}(t) = e^{3t} - e^{2t} .
\end{aligned}$$

**31.** Let $X(s) := \mathcal{L}\{x\}(s)$, $Y(s) := \mathcal{L}\{y\}(s)$. Using the initial condition, we obtain

$$\mathcal{L}\{x'\}(s) = sX(s) - x(0) = sX(s), \qquad \mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s).$$

Therefore, applying the Laplace transform to both sides of the equations in the given system yields

$$\begin{aligned} sX(s) + Y(s) &= \mathcal{L}\{0\}(s) = 0, \\ X(s) + sY(s) &= \mathcal{L}\{1 - u(t-2)\}(s) = \frac{1}{s} - \frac{e^{-2s}}{s} = \frac{1 - e^{-2s}}{s}. \end{aligned}$$

Expressing $Y(s) = -sX(s)$ from the first equation and substituting this into the second equation, we eliminate $Y(s)$:

$$\begin{aligned} X(s) - s^2 X(s) &= \frac{1 - e^{-2s}}{s} \\ \Rightarrow \qquad X(s) &= -\frac{1 - e^{-2s}}{s(s^2 - 1)} = -\frac{1 - e^{-2s}}{s(s-1)(s+1)}. \end{aligned}$$

Since

$$-\frac{1}{s(s-1)(s+1)} = \frac{1}{s} - \frac{1/2}{s-1} - \frac{1/2}{s+1},$$

the inverse Laplace transform yields

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{ \left(1 - e^{-2s}\right) \left( \frac{1}{s} - \frac{1/2}{s-1} - \frac{1/2}{s+1} \right) \right\}(t) \\ &= \mathcal{L}^{-1}\left\{ \frac{1}{s} - \frac{1/2}{s-1} - \frac{1/2}{s+1} \right\}(t) - \mathcal{L}^{-1}\left\{ \frac{1}{s} - \frac{1/2}{s-1} - \frac{1/2}{s+1} \right\}(t-2)u(t-2) \\ &= 1 - \frac{e^t + e^{-t}}{2} - \left[ 1 - \frac{e^{t-2} + e^{-(t-2)}}{2} \right] u(t-2). \end{aligned}$$

We now find $y(t)$ from the first equation in the original system.

$$y(t) = -x'(t) = \frac{e^t - e^{-t}}{2} - \frac{e^{t-2} - e^{-(t-2)}}{2} u(t-2).$$