

## Chapter One

# Vectors and The Geometry of Space

### 1.1 3-D Coordinate Systems

The Cartesian coordinates  $(x, y, z)$  of a point  $P$  in space are the numbers at which the planes through  $P$  perpendicular to the axes cut the axes (Figure 1.1). Cartesian coordinates for space are also called rectangular coordinates because the axes that define them meet at right angles. Points on the  $x$ -axis have  $y$ - and  $z$ -coordinates equal to zero. That is, they have coordinates of the form  $(x, 0, 0)$ . Similarly, points on the  $y$ -axis have coordinates of the form  $(0, y, 0)$ , and points on the  $z$ -axis have coordinates of the form  $(0, 0, z)$ .

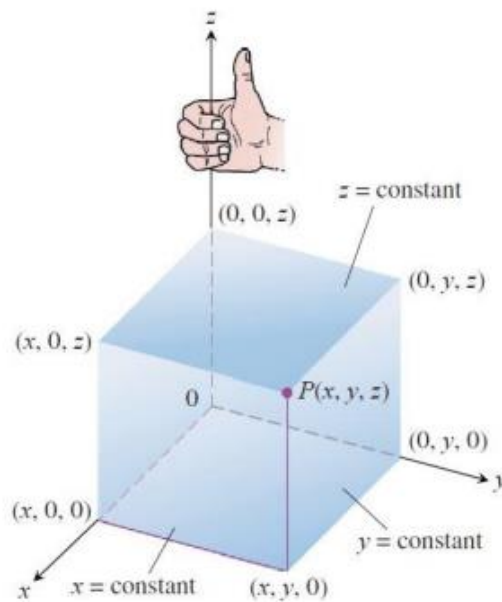


Figure 1.1: The Cartesian coordinate system is right-handed

## Distance and Spheres in Space

The formula for the distance between two points in the xy-plane extends to points in space.

The Distance Between  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$  is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

**Example 1:** Finding the Distance Between Two Points The distance between  $P_1(2, 1, 5)$  and  $P_2(-2, 3, 0)$  is

$$\begin{aligned} |P_1P_2| &= \sqrt{(-2 - 2)^2 + (3 - 1)^2 + (0 - 5)^2} \\ &= \sqrt{16 + 4 + 25} \\ &= \sqrt{45} \end{aligned}$$

### 1.2 Vectors

A vector in the plane is a directed line segment. The directed line segment  $\overline{AB}$  has initial point A and terminal point B; its length is denoted by  $|\overline{AB}|$  (see Figure 1.2). Two vectors are equal if they have the same length and direction.

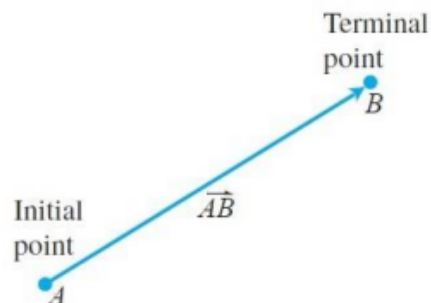


Figure 1.2: The directed line segment  $|\overline{AB}|$

We need a way to represent vectors algebraically so that we can be more precise about the direction of a vector. Let  $\mathbf{v} = \overrightarrow{PQ}$ . There is one directed line segment equal to  $\overrightarrow{PQ}$  whose initial point is the origin (Figure 1.3). It is the representative of  $\mathbf{v}$  in standard position and is the vector we normally use to represent  $\mathbf{v}$ .

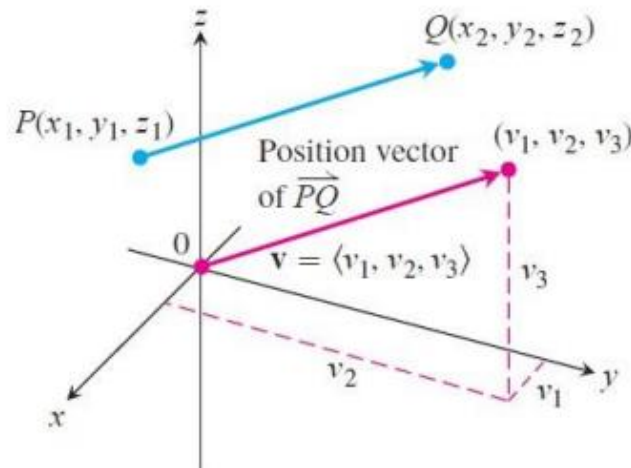


Figure 1.3: vector  $\overrightarrow{PQ}$  in standard position has its initial point at the origin. The directed line segments  $\overrightarrow{PQ}$  and  $\mathbf{v}$  are parallel and have the same length.

We can specify  $\mathbf{v}$  by writing the coordinates of its terminal point  $(v_1, v_2, v_3)$  when  $\mathbf{v}$  is in the standard position. If  $\mathbf{v}$  is a vector in the plane its terminal point  $(v_1, v_2)$  has two coordinates.

So, we can define the Component Form of the vector:

If  $\mathbf{v}$  is a two-dimensional vector in the plane equal to the vector with an initial point at the origin and terminal point  $(v_1, v_2)$ , then the component form of  $\mathbf{v}$  is

$$\mathbf{v} = (v_1, v_2)$$

If  $\mathbf{v}$  is a three-dimensional vector equal to the vector with the initial point at the origin and terminal point  $(v_1, v_2, v_3)$ , then the component form of  $\mathbf{v}$  is

$$\mathbf{v} = (v_1, v_2, v_3)$$

Given the points (Figure 1.9)  $\mathbf{P}(x_1, y_1, z_1)$  and  $\mathbf{Q}(x_2, y_2, z_2)$  the standard position vector  $\mathbf{v} = (v_1, v_2, v_3)$  equal to  $\overrightarrow{PQ}$  is

$$\mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

If  $\mathbf{v}$  is two-dimensional with  $\mathbf{P}(x_1, y_1)$  and  $\mathbf{Q}(x_2, y_2)$  as points in the plane, then  $\mathbf{v} = (x_2 - x_1, y_2 - y_1)$ . There is no third component for planar vectors.

Two vectors are equal if and only if their standard position vectors are identical. Thus  $(u_1, u_2, u_3)$  and  $(v_1, v_2, v_3)$  are equal if and only if  $u_1 = v_1$ ,  $u_2 = v_2$ , and  $u_3 = v_3$ . The **magnitude** or **length** of the vector  $\overrightarrow{PQ}$  is the length of any of its equivalent directed line segment representations. In particular, if  $\mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$  is the standard position vector for  $\overrightarrow{PQ}$  then the distance formula gives the magnitude or length of  $\mathbf{v}$ , denoted by the symbol  $|\mathbf{v}|$  or  $\|\mathbf{v}\|$ . Therefore, the magnitude or length of the vector is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

**Example 2:** Component Form and Length of a Vector Find the (a) component form and (b) length of the vector with initial point  $P$   $(-3, 4, 1)$  and terminal point  $Q$   $(-5, 2, 2)$ .

**Solution:**

(a) The standard position vector  $\mathbf{v}$  represents  $\overline{PQ}$  has components

$$v_1 = x_2 - x_1 = -5 - (-3) = -2,$$

$$v_2 = y_2 - y_1 = 2 - 4 = -2, \text{ and}$$

$$v_3 = z_2 - z_1 = 2 - 1 = 1.$$

The component form  $\overline{PQ}$  is

$$\mathbf{v} = (-2, -2, 1)$$

(b) The length or magnitude of  $\mathbf{v} = \overline{PQ}$  is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$$

## Vector Algebra Operation

Two principal operations involving vectors are vector addition and scalar multiplication.

Let  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$  vectors with  $k$  a scalar. A scalar is a real number and can be positive, negative, or zero.

**A.** Vector addition:  $\mathbf{u} \pm \mathbf{v} = (u_1 \pm v_1, u_2 \pm v_2, u_3 \pm v_3)$ ;

**B.** Scalar multiplication:  $k\mathbf{u} = (ku_1, ku_2, ku_3)$

The definition of vector addition is illustrated geometrically for planar vectors in Figure 1.4, where the initial point of one vector is placed at the terminal point of the other.

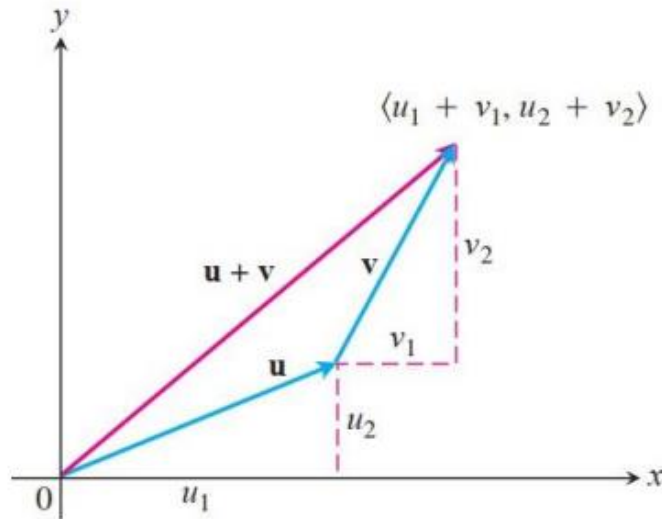


Figure 1.4: Geometric interpretation of the vector sum.

Figure 1.5 displays a geometric interpretation of the product  $ku$  of the scalar  $k$  and vector  $u$ . If  $k > 0$ , then  $ku$  has the same direction as  $u$ ; if  $k < 0$ , then the direction of  $ku$  is opposite to that of  $u$ . Comparing the lengths of  $u$  and  $ku$ , we see that

$$|ku| = \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} = |k|\sqrt{u_1^2 + u_2^2 + u_3^2} = |k||u|$$

The length of  $ku$  is the absolute value of the scalar  $k$  times the length of  $u$ . The vector  $(-1)u = -u$  has the same length as  $u$  but points in the opposite direction.

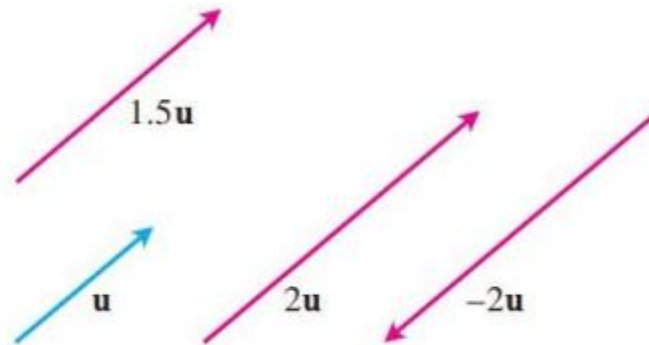


Figure 1.5: Scalar multiples of  $u$ .

By the difference  $u - v$  of two vectors, we mean  $u - v = u + (-v)$ . Note that  $(u - v) + v = u$ , so adding the vector  $(u - v)$  to  $v$  gives  $u$  (Figure 1.6).

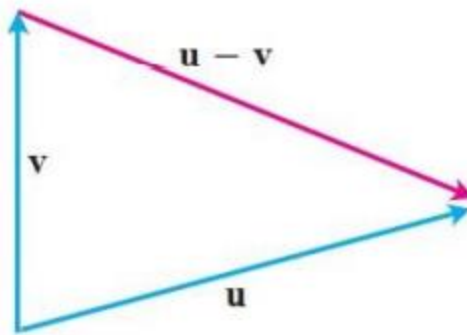


Figure 1.6: The vector  $u - v$ , when added to  $v$ , gives  $u$ .

Figure 1.7 shows the difference  $u - v$  as the sum  $u + (-v)$ .

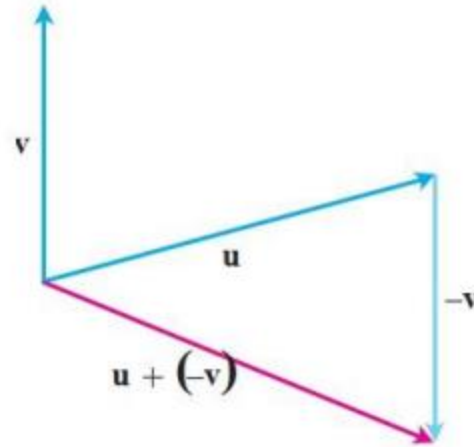


Figure 1.7:  $u - v = u + (-v)$ .

### Example 3: Performing operations on vectors

Let  $\mathbf{u} = \langle -1, 3, 1 \rangle$  and  $\mathbf{v} = \langle 4, 7, 0 \rangle$ . Find (a)  $2\mathbf{u} + 3\mathbf{v}$ ; (b)  $\mathbf{u} - \mathbf{v}$ ; (c)  $\left| \frac{1}{2}\mathbf{u} \right|$ .

**Solution:** (a)  $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1 \rangle + 3\langle 4, 7, 0 \rangle = \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle = \langle 10, 27, 2 \rangle$

(b)  $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1 \rangle - \langle 4, 7, 0 \rangle = \langle -1 - 4, 3 - 7, 1 - 0 \rangle = \langle -5, -4, 1 \rangle$

(c)  $\left| \frac{1}{2}\mathbf{u} \right| = \left| \left\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\rangle \right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}$ .

### Properties of Vector Operations

Let  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  be vectors and  $a, b$  be scalars.

- |  |  |
|--|--|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ |
| 3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$              | 4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$   |
| 5. $0\mathbf{u} = \mathbf{0}$                          | 6. $1\mathbf{u} = \mathbf{u}$  |
| 7. $a(b\mathbf{u}) = (ab)\mathbf{u}$                   | 8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$                          |
| 9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$     |  |



## Unit Vectors

A vector  $\mathbf{v}$  of length 1 is called a unit vector. The standard unit vectors are  $\mathbf{i} = (1, 0, 0)$ ,  $\mathbf{j} = (0, 1, 0)$ , and  $\mathbf{k} = (0, 0, 1)$ .

Any vector can be written as follows:  $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ .

We call the scalar (or number)  $v_1$  component of the vector  $\mathbf{v}$ ,  $v_2$  the  $\mathbf{j}$ -component, and  $v_3$  the  $\mathbf{k}$ -component.

In component form, the vector from  $P_1(x_1, y_1, z_1)$  to  $P_2(x_2, y_2, z_2)$  is seen in Figure 1.8.

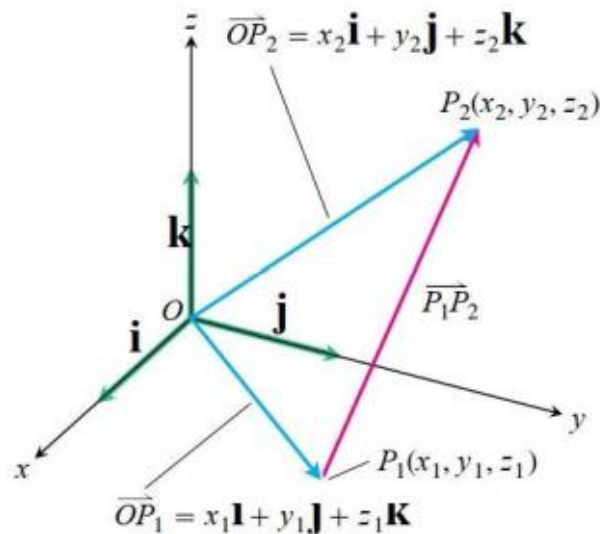


Figure 1.8: The vector from  $P_1$  to  $P_2$ .

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

Whenever  $\mathbf{v} \neq 0$ , its length  $|\mathbf{v}|$  is not zero and

$$\left| \frac{1}{|\mathbf{v}|} \mathbf{v} \right| = \frac{1}{|\mathbf{v}|} |\mathbf{v}| = 1$$

That is,  $\mathbf{v}/|\mathbf{v}|$  is a unit vector in the direction of  $\mathbf{v}$ .

**Example 4:** Finding a Vector's Direction Find a unit vector  $\mathbf{u}$  in the direction of the vector from  $P_1(1, 0, 1)$  to  $P_2(3, 2, 0)$ .

**Solution:** we divide  $\overrightarrow{P_1P_2}$  by its length:

$$\begin{aligned}\overrightarrow{P_1P_2} &= (3 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (0 - 1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} \\ |\overrightarrow{P_1P_2}| &= \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3 \\ \mathbf{u} &= \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}\end{aligned}$$

The unit vector  $\mathbf{u}$  is the direction of  $\overrightarrow{P_1P_2}$ .

If  $\mathbf{v} \neq 0$ , then  $\mathbf{v}/|\mathbf{v}|$  is a unit vector in the direction of  $\mathbf{v}$ ; the equation  $\mathbf{v} = |\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$  expresses  $\mathbf{v}$  in terms of its length and direction.

## Midpoint of a Line Segment

Vectors are often useful in geometry. For example, the coordinates of the midpoint of a line segment are found by averaging.

The midpoint  $M$  of the line segment joining points  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$  is the point (see Figure 1.9)

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

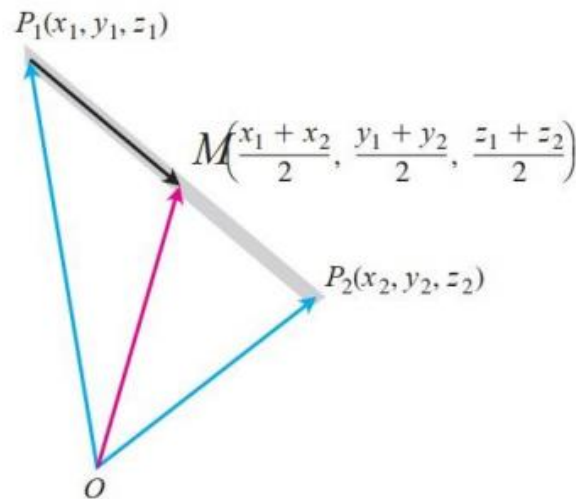


Figure 1.9: Midpoint of line segment.

### Example 5: Finding Midpoints

The midpoint of the segment joining  $P_1(3, -2, 0)$  and  $P_2(7, 4, 4)$  is

$$\left( \frac{3 + 7}{2}, \frac{-2 + 4}{2}, \frac{0 + 4}{2} \right) = (5, 1, 2)$$

### H.W (1)

1- Given  $\vec{A} = 3i - 4j - k$ ,  $\vec{B} = 3i - 4j - k$ , find

(i)  $\vec{A} + \vec{B}$ ,  $\vec{A} - \vec{B}$ .

(ii) A vector with the same direction as  $\vec{A}$  but with half-length of  $\vec{A}$ .

(iii) A vector oppositely directed to  $\vec{B}$  but with twice the length of  $\vec{B}$ .

2- Let  $\vec{u} = (1, 3)$ ,  $\vec{v} = (2, 1)$ ,  $\vec{w} = (4, -1)$ . Find the vector ( $x$ ) that satisfies  $\vec{u} - \vec{v} + x = 7x + \vec{w}$ .

3- Use vectors to find the four vertexes of a parallelogram, three of whose vertexes are  $(0, 0)$ ,  $(1, 3)$ , and  $(2, 4)$ .

4- Find the vector  $\vec{OP}$ , if ( $O$ ) is the origin, and ( $P$ ) is the midpoint of the vector  $\vec{AB}$  joining  $A(2, 1, -2)$  and  $B(6, 0, -4)$ .

5- Find a vector of length (two) and make an angle of  $30^\circ$  with the positive  $x$ -axis.

6- Find the vector tangent to the curve  $y = \frac{x^3}{2} + \frac{1}{2}$  at point  $(1, 1)$ .

7- Find the vector tangent to the curve  $y = x^2$  at point  $(2, 4)$ .

8- Find the vector of  $\vec{v} = ai + bj$  and sketch your results and prove that  $\vec{u} = \cos \theta i + \sin \theta j$

where  $\theta = \tan^{-1} \left[ \frac{b}{a} \right]$ .

### 1.3 The Dot Product

Dot products are also called inner or scalar products because the product results in a scalar, not a vector. It is used to calculate the angle between two vectors directly from their components; show whether two vectors are orthogonal or not; find the projection vector.

The dot product  $\mathbf{u} \bullet \mathbf{v}$  ( $\mathbf{u}$  dot  $\mathbf{v}$ ) of vectors  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$  is

$$\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

#### Example 6: Finding Dot Product

(a)  $\langle 1, -2, -1 \rangle \bullet \langle -6, 2, -3 \rangle = (1)(-6) + (-2)(2) + (-1)(-3) = -7$

(b)  $\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \bullet (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1.$

#### Properties of the Dot Product

If  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are any vectors and  $c$  is a scalar, then

1.  $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$
2.  $(c\mathbf{u}) \bullet \mathbf{v} = \mathbf{u} \bullet (c\mathbf{v}) = c(\mathbf{u} \bullet \mathbf{v})$
3.  $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$
4.  $\mathbf{u} \bullet \mathbf{u} = |\mathbf{u}|^2$
5.  $\mathbf{0} \bullet \mathbf{u} = 0.$

## Angle Between Vectors

As seen in Figure 1.10

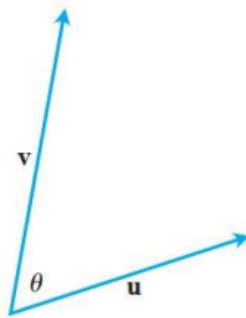


Figure 1.10: The angle between  $\mathbf{u}$  and  $\mathbf{v}$ .

Let  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$  and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ ,

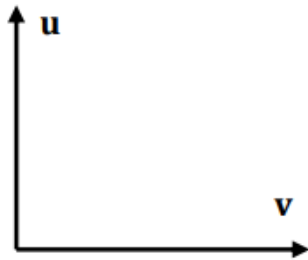
$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$$

The angle  $\theta$  between two nonzero vectors  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$  and  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$  is given by

$$\theta = \cos^{-1} \left( \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|} \right) = \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \right)$$
$$0 \leq \theta \leq \pi$$

## Perpendicular (Orthogonal) Vectors

Vectors  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal (or perpendicular) if and only if  $\mathbf{u} \bullet \mathbf{v} = 0$ .

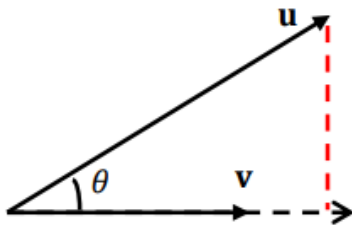


If we have two vectors  $\mathbf{u}$  and  $\mathbf{v}$ , from **dot product**, we can know:

1.  $\mathbf{u} \bullet \mathbf{v} = (+)$ , acute angle;
2.  $\mathbf{u} \bullet \mathbf{v} = (-)$ , obtuse angle;
3.  $\mathbf{u} \bullet \mathbf{v} = (0)$ , right angle.

## Vector Projections

The vector projection of  $\mathbf{u}$  onto a nonzero vector  $\mathbf{v}$  is



$$\begin{aligned} \text{proj}_{\mathbf{v}} \mathbf{u} &= (|\mathbf{u}| \cos \theta) \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \end{aligned}$$

Scalar component of  $\mathbf{u}$  in the direction of  $\mathbf{v}$ :

$$|\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$$

### Example 7: Finding the Vector Projection

Find the vector projection of  $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$  onto  $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$  and the scalar component of  $\mathbf{u}$  in the direction of  $\mathbf{v}$ .

**Solution:**

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} = \frac{6 - 6 - 4}{1 + 4 + 4} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = \frac{-4}{9} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = \left( -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k} \right)$$

We find the scalar component

$$|\mathbf{u}| \cos \theta = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \cdot \left( \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \right) = 2 - 2 - \frac{4}{3} = -\frac{4}{3}$$



## 1.4 The Cross Product

The Cross products are widely used to describe the effects of forces in studies of electricity, magnetism, fluid flows, and orbital mechanics.

We start with two nonzero vectors  $\mathbf{u}$  and  $\mathbf{v}$  in space. If  $\mathbf{u}$  and  $\mathbf{v}$  are not parallel, they determine a plane. We select a unit vector  $\mathbf{n}$  perpendicular to the plane by the right-hand rule. This means that we choose  $\mathbf{n}$  to be the unit (normal) vector that points the way your right thumb points when your fingers curl through the angle from  $\mathbf{u}$  to  $\mathbf{v}$  (Figure 1.11). Then the cross product  $\mathbf{u} \times \mathbf{v}$  (“ $\mathbf{u}$  cross  $\mathbf{v}$ ”) is the vector defined as follows:

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin \theta)\mathbf{n}$$

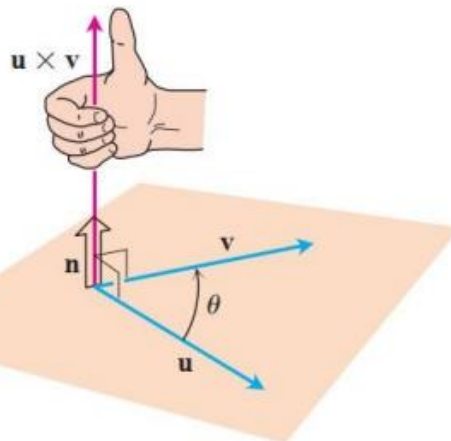


Figure 1.11: The construction  $\mathbf{u} \times \mathbf{v}$ .

Unlike the dot product, the cross product is a vector. For this reason, it is also called the vector product of  $\mathbf{u}$  and  $\mathbf{v}$  and applies only to vectors in space. The vector is orthogonal to both  $\mathbf{u}$  and  $\mathbf{v}$  because it is a scalar multiple of  $\mathbf{n}$ .

**Parallel Vectors:** Nonzero vectors  $\mathbf{u}$  and  $\mathbf{v}$  are parallel if and only if  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ .



**Properties of the Cross Product:** If  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  are any vectors and  $r$ ,  $s$  are scalars, the

1.  $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2.  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3.  $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
4.  $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
5.  $\mathbf{0} \times \mathbf{u} = \mathbf{0}$

Figure 1.12 visualizes property 4.

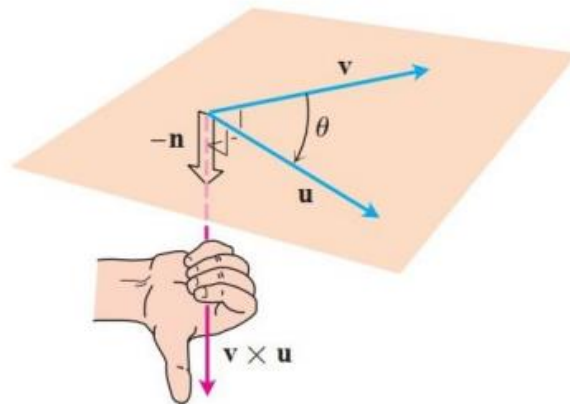


Figure 1.12: The construction  $\mathbf{v} \times \mathbf{u}$ .

When we apply the definition to calculate the pairwise cross products of  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$ , we find (Figure 1.13)

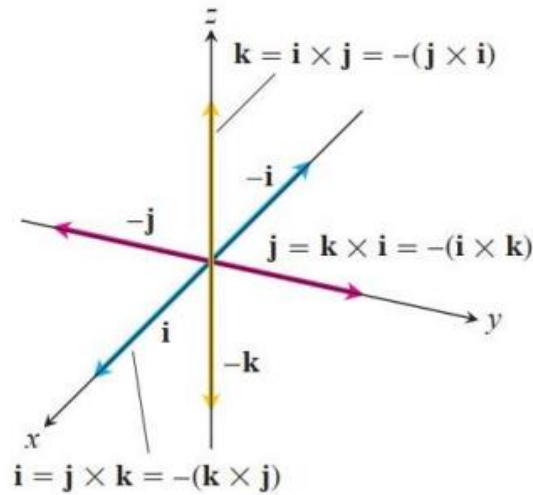
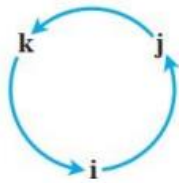


Figure 1.13: The pairwise cross product of  $i$ ,  $j$ , and  $k$ .

and

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= -(\mathbf{j} \times \mathbf{i}) = \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= -(\mathbf{k} \times \mathbf{j}) = \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= -(\mathbf{i} \times \mathbf{k}) = \mathbf{j} \\ \mathbf{i} \times \mathbf{i} &= \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0} \end{aligned}$$



### Calculating Cross Product Using Determinants:

If  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$  and  $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ , then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

This is the area of the parallelogram determined by  $\mathbf{u}$  and  $\mathbf{v}$  (Figure 1.14),  $|\mathbf{u}|$  being the base of the parallelogram and  $|\mathbf{v}|\sin \theta$  the height. Because  $\mathbf{n}$  is a unit vector, the area of a parallelogram is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin \theta |\mathbf{n}| = |\mathbf{u}||\mathbf{v}| \sin \theta$$

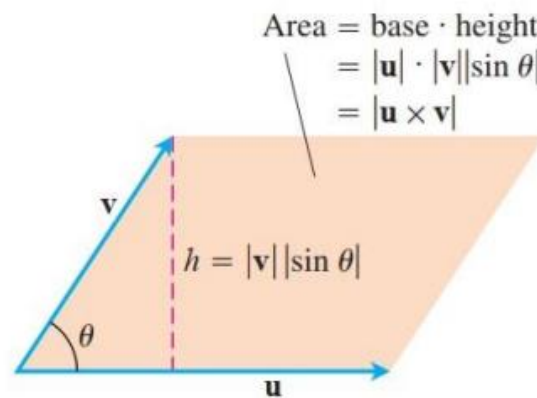


Figure 1.14: The parallelogram is determined by  $\mathbf{u}$  and  $\mathbf{v}$ .

**Example 8:** Find  $\mathbf{u} \times \mathbf{v}$  and  $\mathbf{v} \times \mathbf{u}$  if  $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

**Solution:**

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$$

**Example 9:** Find unit vector orthogonal to the vectors  $\mathbf{u} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$  and  $\mathbf{v} = 4\mathbf{j} + \mathbf{k}$

**Solution:**

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 0 & 4 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 4 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -1 \\ 0 & 4 \end{vmatrix} \mathbf{k} = -5\mathbf{i} - 3\mathbf{j} + 12\mathbf{k}$$

$$|\mathbf{u} \times \mathbf{v}| = \sqrt{(-5)^2 + (-3)^2 + (12)^2} = \sqrt{178}$$

$$\mathbf{z} = \frac{1}{\sqrt{178}}(-5\mathbf{i} - 3\mathbf{j} + 12\mathbf{k})$$

**H.W (2)**

- 1- If  $\vec{u} = i - 2j + 3k$  and  $\vec{v} = 3i + 6j + 3k$ , show that  $\vec{u}$  is perpendicular to  $\vec{v}$ .
- 2- Given  $\vec{v} = 2i - 4j + 4k$ , find the direction cosines, and direction angles.
- 3- Let  $\vec{A} = 2i - j + 3k$  and  $\vec{B} = 4i - j + 2k$ . Find
  - (a) The vector component of  $\vec{A}$  along  $\vec{B}$ .
  - (b) The vector component of  $\vec{A}$  orthogonal to  $\vec{B}$ .
- 4- Find the vector of  $(x)$ , so that the vector from  $A(1, -1, 3)$  to  $B(6, 0, -4)$  is perpendicular to the vector from  $A$  to the point  $P(x, x, x)$ .
- 5- Let  $\vec{v} = ai + j$  and  $\vec{u} = 4i + 3j$ , find (a) so that
  - (i)  $\vec{v}$  and  $\vec{u}$  are orthogonal.
  - (ii)  $\vec{v}$  and  $\vec{u}$  are parallel.
  - (ii) The angle between  $\vec{v}$  and  $\vec{u}$  is  $30^\circ$ .

- 6- Calculate the distance between the point  $(4, 3)$  and the line  $x + 3y = 0$ .
- 7- Given  $\vec{A} = i + 2j - 2k$  and  $\vec{B} = 3i + k$ . Find  $\vec{A} \times \vec{B}$  and  $\vec{B} \times \vec{A}$ .
- 8- If  $\vec{u} = 2i + j + k$  and  $\vec{v} = -4i + 3j + k$ . Find  $\vec{u} \times \vec{v}$  and  $\vec{v} \times \vec{u}$ .
- 9- Find a unit vector perpendicular to both  $\vec{A} = 2i + j - k$  and  $\vec{B} = i - j + 2k$ .
- 10- Show if  $\vec{u} = 5i + k - j$  and  $\vec{v} = -15i + 3j - 3k$  are parallel or not.
- 11- Use the cross product to find the sine of the angle between the vectors  $\vec{A} = 2i + 3j - 6k$  and  $\vec{B} = 2i + 3j + 6k$ .
- 12- Find the area of triangle whose vertices are  $A(1, -1, 0)$ ,  $B(2, 1, -1)$ , and  $C(-1, 1, 2)$ .