Chapter Four

Lines and Planes in Space

Lines: In the plane, a line is determined by a point and a number giving the slope of the line. In space, a line is determined by a point and a vector giving the direction of the line.

Suppose that ***L*** is a line in space passing through a point $\boldsymbol{P_o}$ (x_o, y_o, z_o) parallel to a vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then ***L*** is the set of all points P (x, y, z) for which $\overrightarrow{PP_o}$ parallel to ***v*** (Figure 4.1). Thus, $\overrightarrow{PP_o} = t\mathbf{v}$, t is a scalar parameter $(-\infty, \infty)$.

$$(x - x_o)\mathbf{i} + (y - y_o)\mathbf{j} + (z - z_o)\mathbf{k} = t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

or

$$\frac{x - x_o}{v_1} = \frac{y - y_o}{v_2} = \frac{z - z_o}{v_3} = t$$

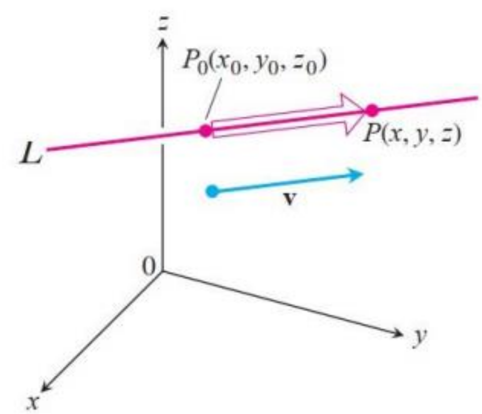


Figure 4.1: A point *P* lies on *L*

Parametric Equations of a Line

The standard parametrization of the line through P_o (x_o, y_o, z_o) parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is $x = x_o + v_1t$, $y = y_o + v_2t$, $z = z_o + v_3t$.

Example 1: Find parametric equations for the line through (-2, 0, 4) parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

Solution: $x = -2 + 2t$, $y = 4t$, $z = 4 - 2t$.

Distance from a Point *S* to a Line Through *P* Parallel to *v* (see Figure 4.2)

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

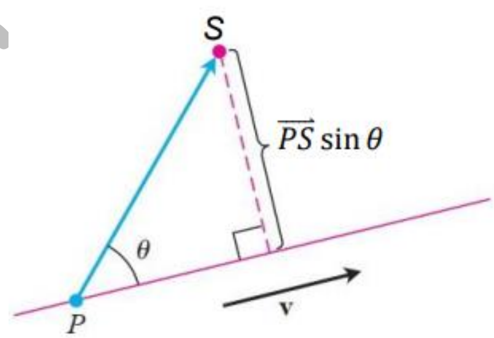


Figure 4.2: The distance from the point to a line

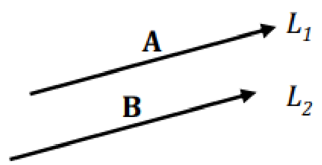
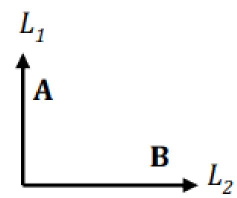
Example 2: Find the distance from the point S(1, 1, 5) to the line

L: $x = 1 + t$, $y = 3 - t$, $z = 2t$

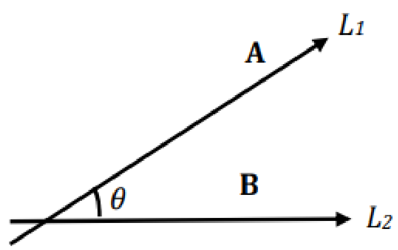
Solution: from the equation for the L, L passes through P (1, 3, 0) and parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. $\overrightarrow{PS} = (1-1)\,\mathbf{i} + (1-3)\,\mathbf{j} + (5-0)\,\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$ and

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$$

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

Parallel Lines: $L_1 \parallel L_2 \Rightarrow \mathbf{A} \times \mathbf{B} = 0.$	**Orthogonal Lines:** $L_1 \perp L_2 \Rightarrow \mathbf{A} \cdot \mathbf{B} = 0.$
	

Angle Between Two Lines:

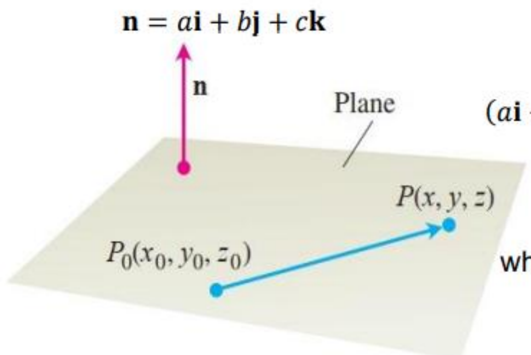


$$\theta = \cos^{-1}\left(\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right)$$

Skew Lines are two lines that do **not intersect** and are **not parallel**.

An Equation for a Plane in Space

A plane in space is determined by knowing a point on the plane and the vector normal (perpendicular) to the plane (see Figure 4.3).



$$\mathbf{n}\cdot\overrightarrow{P_oP} = 0$$

$$(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot[(x-x_o)\mathbf{i}+(y-y_o)\mathbf{j}+(z-z_o)\mathbf{k}] = 0$$

$$a(x-x_o)+b(y-y_o)+c(z-z_o) = 0$$

$$ax+by+cz = d,$$

where

$$d = ax_o+by_o+cz_o$$

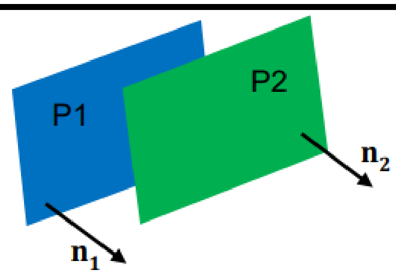
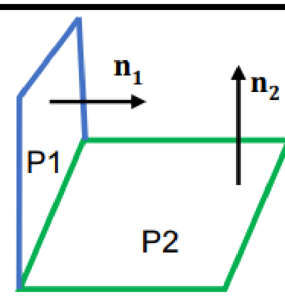
Figure 4.3: A plane in space

Example 3: Find an equation for the plane through $\boldsymbol{P_o}$ (-3, 0, 7) perpendicular to

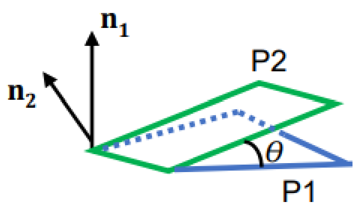
$\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution: the component equation is $5(x-(-3)) + 2(y-0) + (-1)\,(z-7) = 0$

$5x + 2y - z = -22.$

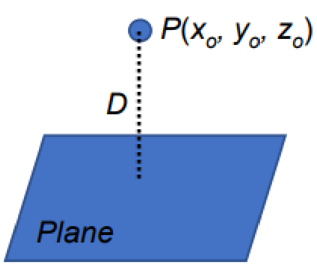
Parallel Planes: $P_1 \parallel P_2 \Rightarrow \mathbf{n_1} \times \mathbf{n_2} = 0.$	**Orthogonal Planes**: $P1 \perp P2 \Rightarrow \mathbf{n_1}\cdot\mathbf{n_2} = 0.$
	

Angle Between Two Planes:



$$\theta = \cos^{-1}\left(\frac{\mathbf{n_1}\cdot\mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|}\right)$$

Distance Between Point and Plane:



$$D = \frac{ax_o + by_o + cz_o - d}{\sqrt{a^2 + b^2 + c^2}}$$

$$D = \begin{cases} + & \text{point } P \text{ lies above} \\ - & \text{point } P \text{ lies below} \\ 0 & \text{point } P \text{ lies on the plane} \end{cases}$$

Or as you can see in the next example:

$$D = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Example 4: Find the distance from ***S*** (1, 1, 3) to the plane $3x + 2y + 6z = 6$.

Solution: First, we find a point in the plane and calculate the length of the vector projection of $\overrightarrow{PS}$ onto a vector n normal to the plane (Figure 4.4). The coefficients in the equation gives

$$\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

We can find interception points from the plane's equation. If we take P to be the y-intercept (0, 3, 0), then

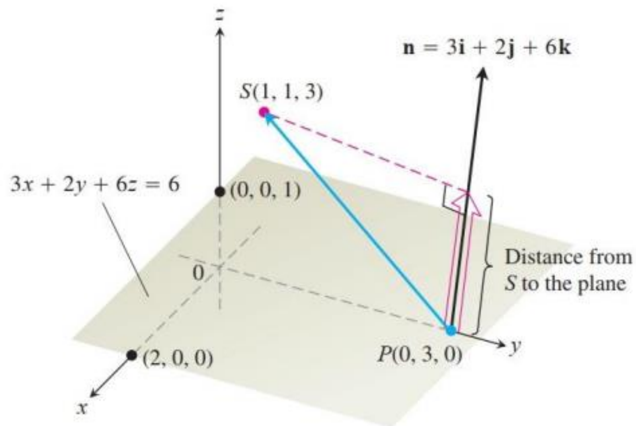


Figure 4.4: The distance from point S to the plane

$$\overrightarrow{PS} = (1-0)\mathbf{i} + (1-3)\mathbf{j} + (3-0)\mathbf{k} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{z}$$

$$|\mathbf{n}| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7$$

The distance from S to the plane is

$$D = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|(\mathbf{i} - 2\mathbf{j} + 3\mathbf{z}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{z}\right)\right| = \left|\frac{3}{7} - \frac{4}{7} + \frac{18}{7}\right| = \frac{17}{7}.$$

Solved Problems:

Prob 1. Find a vector that has a length 15 in the direction of **B** = **i** + 2**j** − **k**.

Solution:

$$|\mathbf{B}| = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{6}\,, \qquad \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}\,,$$

$$\mathbf{v} = 15\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}\right) = \frac{15}{\sqrt{6}}\mathbf{i} + \frac{30}{\sqrt{6}}\mathbf{j} - \frac{15}{\sqrt{6}}\mathbf{k}\,.$$

Prob 2. Find a vector that has a length of 22 in the opposite direction of **A** = 2**i** − 3**j**.

Solution:

$$|\mathbf{A}| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}\,, \qquad \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j}\,,$$

$$\mathbf{v} = -22\left(\frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j}\right) = -\frac{44}{\sqrt{13}}\mathbf{i} + \frac{66}{\sqrt{13}}\mathbf{j}\,.$$

Prob 3. Using vectors, show that the sum of triangle angles is 180°, and the points of the triangle are (1, 1), (4, 3), and (2, 5). Then, find the area of the triangle.

Solution:

$$\mathbf{A} = (2-1)\mathbf{i} + (5-1)\mathbf{j} = \mathbf{i} + 4\mathbf{j}\,, \quad |\mathbf{A}| = \sqrt{1+16} = \sqrt{17}$$

$$\mathbf{B} = (4-1)\mathbf{i} + (3-1)\mathbf{j} = 3\mathbf{i} + 2\mathbf{j}\,, \quad |\mathbf{B}| = \sqrt{9+4} = \sqrt{13}$$

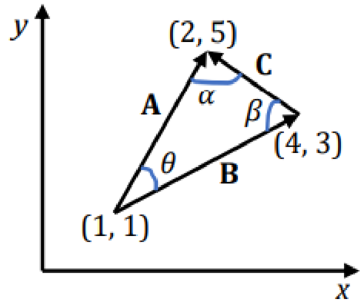
$$\mathbf{C} = (2-4)\mathbf{i} + (5-3)\mathbf{j} = -2\mathbf{i} + 2\mathbf{j}\,, |\mathbf{C}| = \sqrt{4+4} = \sqrt{8}$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right) = \cos^{-1}\left(\frac{3+8}{\sqrt{17}\,\sqrt{13}}\right) = \cos^{-1}\left(\frac{11}{\sqrt{221}}\right) = 42.27^\circ$$

$$\beta = \cos^{-1}\left(\frac{-\mathbf{B}\cdot\mathbf{C}}{|\mathbf{B}||\mathbf{C}|}\right) = \cos^{-1}\left(\frac{6-4}{\sqrt{13}\,\sqrt{8}}\right) = \cos^{-1}\left(\frac{2}{\sqrt{104}}\right) = 78.69^\circ$$

$$\alpha = \cos^{-1}\left(\frac{-\mathbf{A}\cdot-\mathbf{C}}{|\mathbf{A}||\mathbf{C}|}\right) = \cos^{-1}\left(\frac{-2+8}{\sqrt{17}\,\sqrt{8}}\right) = \cos^{-1}\left(\frac{6}{\sqrt{136}}\right) = 59.03^\circ$$

$\theta + \beta + \alpha = 179.99 \approx 180°$

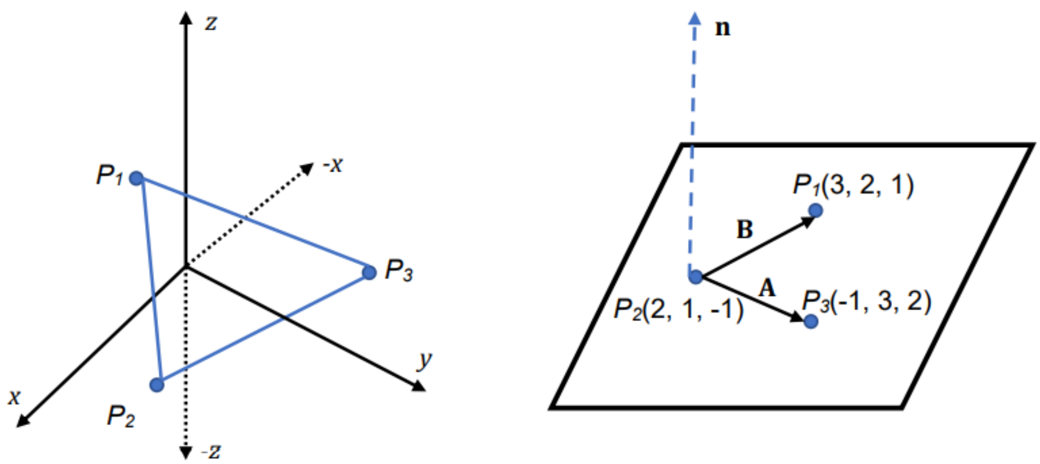


$$\text{Area of traingle} = \frac{1}{2}|\mathbf{A}\times\mathbf{B}| = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ 3 & 2 & 0 \end{vmatrix}$$

$$= \frac{1}{2}|-10\mathbf{k}| = 5 \text{ unit}^2.$$

Prob 4. Find equation of the plane has P1(3, 2, 1), P2(2, 1, -1), and P3(-1, 3, 2).

Solution:



$$\left.\begin{array}{l} \mathbf{A} = -3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \\ \mathbf{B} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} \end{array}\right\} \quad \mathbf{n} = \mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = (4-3)\mathbf{i} - (-6-3)\mathbf{j} + (-3-2)\mathbf{k}$$

$\mathbf{n} = \mathbf{i} + 9\mathbf{j} - 5\mathbf{k}$, point P_2(2, 1, -1), or any point, we find

$$x + 9y - 5z = 1(2) + 9(1) - 5(-1) \xrightarrow{\text{Equ. of plane is}} x + 9y - 5z = 16\,.$$

Prob 5: Given equations of two planes, plane1 ($x + y + z = 1$) and plane2 ($2x - 3y + z = 4$), find: (a) point ∈ plane1; (b) whether the two planes are parallel or not; (c) the intersection point, if they are intersecting; (d) equation of the line of intersection for the two planes.

Solution: (**a**) $y = z = 0, \Rightarrow x = 1, \Rightarrow$ point is (1, 0, 0).

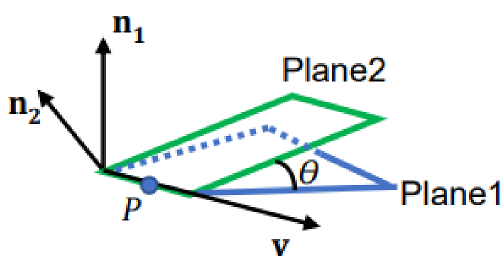
(b) $\left.\begin{matrix} \mathbf{n_1} = \mathbf{i} + \mathbf{j} + \mathbf{k} \\ \mathbf{n_2} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \end{matrix}\right\}$ $\mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -3 & 1 \end{vmatrix} = (1+3)\mathbf{i} \ldots\ldots.. \neq 0$, so they are not parallel.

(c) $z = 0, \Longrightarrow \left.\begin{matrix} x + y = 1 \\ 2x - 3y = 4 \end{matrix}\right\} \xrightarrow{\text{Multiply } 1^{st} \text{ equ. by } 3} \begin{matrix} 3x + 3y = 3 \\ 2x - 3y = 4 \\ \hline 5x = 7 \end{matrix}$

$x = \frac{7}{5}, y = -\frac{2}{5}$, point is $P\left(\frac{7}{5}, -\frac{2}{5}, 0\right)$.

(d) $\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -3 & 1 \end{vmatrix} = (1+3)\mathbf{i} - (1-2)\mathbf{j} + (-3-2)\mathbf{k} = 4\mathbf{i} + \mathbf{j} - 5\mathbf{k}$,

we have $P\left(\frac{7}{5}, -\frac{2}{5}, 0\right)$ and the vector $\mathbf{v}$ $\xrightarrow{\text{Equ. of line is}}$ $\frac{x - \frac{7}{5}}{4} = \frac{y + \frac{2}{5}}{1} = \frac{z}{-5}$.



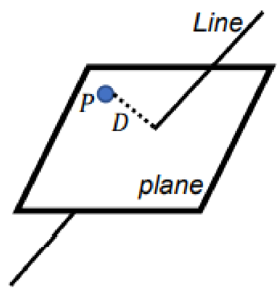
Prob 6: Determine a point ***P*** ∈ plane ($x - 2y + 3z = 0$), then find the distance between that point P and an intersection point of the line (L) with the plane.

$$L: \frac{4 - z}{-3} = \frac{2x}{3} = \frac{1 - \frac{1}{2}y}{4}$$

Solution: $y = z = 0, \Rightarrow x = 0, \Rightarrow \boldsymbol{P}\ (0, 0, 0)$,

$$\frac{4-z}{-3} = \frac{2x}{3} = \frac{1-\frac{1}{2}y}{4} \xrightarrow{\text{rearrange}} \frac{x}{3/2} = \frac{y-2}{-8} = \frac{z-4}{3} = t\,,$$

$$\text{parametric equ.}\quad \left.\begin{matrix} x = \frac{3}{2}t \\ y = 2-8t \\ z = 4+3t \end{matrix}\right\} \xrightarrow{\text{substitute in plane}} \begin{cases} \frac{3}{2}t - 2(2-8t) + 3(4+3t) = 0 \\ \frac{3}{2}t - 4 + 16t + 12 + 9t = 0 \\ \frac{53}{2}t = -8 \Rightarrow t = -\frac{16}{53} \end{cases}$$



$$\text{substitute } t \text{ in parametric equ.}\quad \left.\begin{matrix} x = \frac{3}{2}\left(-\frac{16}{53}\right) = -\frac{24}{53} \\ y = 2-8\left(-\frac{16}{53}\right) = \frac{234}{53} \\ z = 4+3\left(-\frac{16}{53}\right) = \frac{164}{53} \end{matrix}\right\},\ \text{point of intersection } \left(-\frac{24}{53}, \frac{234}{53}, \frac{164}{53}\right)$$

$$D = \sqrt{\left(-\frac{24}{53}-0\right)^2 + \left(\frac{234}{53}-0\right)^2 + \left(\frac{164}{53}-0\right)^2}$$

$$D = \frac{1}{53}\sqrt{(24)^2 + (234)^2 + (164)^2} = 5.41 \text{ unit}$$

Prob7: Find the angle between two planes, plane1 $(2x - 3y + 3z = 1)$ and plane2 $(x - y + \frac{1}{3}z = 0)$. Then, find the distance between $P(1, 1, -2)$ and plane1.

Solution: $\left.\begin{array}{l}\mathbf{n_1} = 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} \\ \mathbf{n_2} = \mathbf{i} - \mathbf{j} + \frac{1}{3}\mathbf{k}\end{array}\right\}, \theta = \cos^{-1}\left(\frac{\mathbf{n_1}\cdot\mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|}\right) = \cos^{-1}\left(\frac{2+3+1}{\sqrt{4+9+9}\sqrt{1+1+\frac{1}{9}}}\right)$

$$= \cos^{-1}\left(\frac{6}{\sqrt{22}\sqrt{\frac{19}{9}}}\right) = 28.3°,$$

$D = \frac{2(1) - 3(1) + 3(-2) - 1}{\sqrt{4+9+9}} = -\frac{8}{\sqrt{22}} = \frac{8}{\sqrt{22}}$, the point lies below the plane.

Prob 8: Check whether these two planes are parallel or not and find the distance between them: plane1 ($x - 2y + 4z = 1$) and plane2 ($3x - 6y + 12z = 5$).

Solution: $\left.\begin{array}{l}\mathbf{n_1} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k} \\ \mathbf{n_2} = 3\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}\end{array}\right\}, \mathbf{n_1}\times\mathbf{n_2} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ 3 & -6 & 12\end{vmatrix}$

$$= (-24+24)\mathbf{i} - (12-12)\mathbf{j} + (-6+6)\mathbf{k} = 0,$$

so, plane1 ∥ plane2,

Then, find point ∈ plane1, $y = z = 0, \Longrightarrow x = 1, P(1,0,0)$,

$D = \frac{3(1) - 6(0) + 12(0) - 5}{\sqrt{9+36+144}} = -\frac{2}{\sqrt{189}} = -0.145 = 0.145$ unit, plane1 lies below plane2.

Prob 9: Find the intersection point of the line that passes through (2, 4, -1), (5, 0, 7) with xz-plane.

Solution: $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$, with initial point $(2,4,-1), \Rightarrow \frac{x-2}{3} = \frac{y-4}{-4} = \frac{z+1}{8} = t$,

$xz-plane \Rightarrow y = 0, \Rightarrow y = 4 - 4t \overset{y=0}{\Longrightarrow} 4 - 4t = 0 \Rightarrow t = 1$,

$$\left.\begin{array}{l} x = 2 + 3(1) = 5 \\ y = 4 - 4(1) = 0 \\ z = -1 + 8(1) = 7\end{array}\right\} \xrightarrow{\text{the point is}} (5,0,7).$$

Prob 10: Find the equation of the plane through (1, 2, -1) and perpendicular to the line of intersection of these two planes ($2x + y + z = 2$), ($x + 2y + z = 3$).

Solution: $\left.\begin{matrix}\mathbf{n_1} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \\ \mathbf{n_2} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}\end{matrix}\right\}$, $\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & 2 & 1\end{vmatrix} = (1-2)\mathbf{i} - (2-1)\mathbf{j} + (4-1)\mathbf{k}$

$\mathbf{v} = -\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, which is the normal vector to the plane3,

$-x - y + 3z = -1(1) - 1(2) + 3(-1) \Rightarrow -x - y + 3z = -6 \xrightarrow{\text{Equ. of plane}} x + y - 3z = 6.$

H.W (1)

1- (a) $L_1: \quad x=1+4t; \quad y=2+4t; \quad z=1+8t, \quad L_2: \quad x=1+8s; \quad y=2+8s; \quad z=1+16s$

(b) $L_1: \quad \frac{x-2}{1}=\frac{y-3}{-2}=\frac{z-1}{-3}, \quad L_2: \quad \frac{x-3}{1}=\frac{y+4}{3}=\frac{z-2}{-7}$

(c) $L_1: \quad \frac{x+1}{3}=\frac{y-6}{1}=\frac{z-3}{2}, \quad L_2: \quad \frac{x-6}{2}=\frac{y-11}{2}=\frac{z-3}{-1}$

(d) $L_1: \quad x=2t+1; \quad y=3t+2; \quad z=4t+3, \quad L_2: \quad x=s+2; \quad y=2s+4; \quad z=-4s-1$

Determine whether L_1 and L_2 are parallel, intersection, or skew lines.

2- Put the equation of the line in standard form $\frac{x-2}{3}=\frac{-y-5}{-2}=\frac{1-z}{6}$ and convert it into equations.

3- Find the equation of the plane that contains

$L_1: \quad x=1+2t; \quad y=5+t; \quad z=3t, \quad L_2: \quad x=5+4s; \quad y=-2s; \quad z=1+6s$

4- Find the equation of the plane that contains $L: \quad x=1+t; \quad y=2-t; \quad z=4-3t$ and is parallel to the plane $5x+2y+z=1$.

5- Find the equation of the plane passes through the point (1, 5, 1) and perpendicular to

$Plane_1: \quad 2x+y-2z=2, \quad Plane_2: \quad x+3z=4$.

6- Find the point of intersection between the line $L: \quad x=1+2t; \quad y=4t; \quad z=2-3t$ and the plane $x+2y-z+1=0$.

7- $Plane_1: \quad 2x + y - 2z = 2, \quad Plane_2: \quad x + 3z = 4$ find the line intersection between two planes.

8- Find the point of intersection between the line $L: \quad x - 2z; \quad y = 2z$ with the plane $x + 3y - z + 4 = 0$.

9- The line whose equation is given as the line $L: \quad x = 2t - 1; \quad y = 3 + t; \quad z = -t + 4$ intersects the (***xy***) plane at point ***P***. Find the coordinates of ***P***.

10- Find the point of intersection between the line $\frac{x-2}{1} = \frac{y+3}{2} = \frac{z-2}{2}$ and the plane $x + 2y + 2z = 13$.

11- Find the angle between the two planes $2x - 6y - 2z = 7 \quad and \quad 2x + y - 2z = 5$.

12- Find a vector parallel to the line of intersection of the two planes $3x - 6y - 2z = 7 \quad and \quad 2x + y + 2z = 5$.

13- Find the distance (d) between point ***P*** (2, -3, 4) and the plane $x + 2y + 2z = 13$.

14- Find the distance from the point ***P*** (1, 1, 5) to the $L: \quad x = 1 + t; \quad y = 3 - t; \quad z = 2t$.

15- Find the distance from point ***P*** (1, 1, 3) to the plane $3x + 2y + 6z = 6$.

16- Find a vector parallel to the line of intersection of the two planes $3x - 6y - 2z = 15 \quad and \quad 2x + y - 2z = 5$, and find the angle between the two planes.