3. ELASTIC BEHAVIOUR

This section introduces some fundamental approaches to describe composite materials. These methods can be used to estimate the elastic moduli of concrete if the elastic moduli of the concrete phases and their volume fractions are known.

Special care should be taken when estimating the elastic modulus for different types of concrete (i.e., high-strength concrete, light-weight aggregate concrete, and mass concrete). For instance, the maximum size aggregate and the volume fractions of aggregate and cement paste in mass concrete are quite different from structural concrete, therefore, predicting the elastic modulus using the ACI equation would not be reliable. Assume you are the designer of a large concrete dam and you want to perform a preliminary thermal stress analysis. Unfortunately, in most cases, experimental results are not available at this stage of the project, yet an estimation of the elastic modulus of concrete is crucial for predicting thermal stresses. To solve this problem you must obtain an estimate of the elastic properties using a composite materials formulation that incorporates the elastic moduli and the volume fractions of the cement paste and aggregate.

3.1 Composite Models for Predicting Concrete Elastic Modulus

3.1.1 Two phase models

The two simplest models used to simulate a composite material are shown in Figure (13-1*a* and *b*). In the first model the phases are arranged in a parallel configuration, imposing a condition of *uniform strain*. This arrangement is often referred to as the *Voigt model*. In the second model, the phases are arranged in a series configuration imposing a condition of *uniform stress*; this geometry is known as the *Reuss model*.

We solve the Voigt model by using a simple strength of materials approach. For a first approximation, lateral deformations are neglected. The following equations are obtained:

Equilibrium equation $\sigma A = \sigma_1 A_1 + \sigma_2 A_2$ (3-1)

Compatibility equation $e = e_1 = e_2$ (3-2)

Constitutive relationship $\sigma = E\epsilon$ (3-3)

Substituting Eq. (3-3) into (3-1) we obtain: $E\epsilon A = E_1\epsilon_1 A_1 + E_2\epsilon_2 A_2$ (3-4)

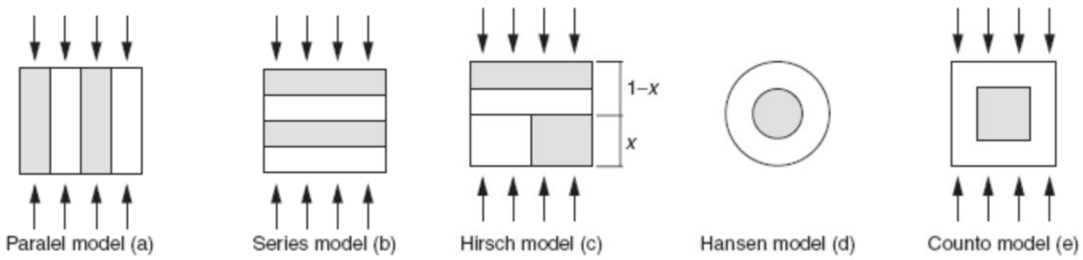

Figure 3.1: Traditional two-phase models for concrete.

Using the compatibility Eq. (3-1), Eq. (3-4) reduces to

$$EA = E_1A_1 + E_2A_2$$

For composite materials it is more convenient to deal with volume than area, therefore, for unit length:

$$EV=E_1V_1+E_2V_2$$

Or

$$E=E_1c_1+E_2c_2 \tag{3.5}$$

where $c_i = V_i/V$ is the volume fraction of the *i*th phase. Using the same approach to solve for the series (*Reuss*) model:

$$\frac{1}{E} = \frac{c_1}{E_1} + \frac{c_2}{E_2} \tag{3.6}$$

To obtain further insight into these models, let us re-derive the parallel and series models to include lateral deformations. Because the structural models shown in Figure (3-1) do not allow the introduction of Poisson's ratio (v), let us consider a homogeneous body of volume V and bulk modulus K subjected to a uniform hydrostatic compression P. The total stored strain energy W is given by

$$W = \frac{P^2V}{2K} \tag{3.7}$$

Or

$$W = \frac{\epsilon^2KV}{2} \tag{3.8}$$

where

$$\epsilon = \frac{dV}{V} = -\frac{P}{K} \tag{3.9}$$

is the volumetric strain.

The parallel model assumes that each phase undergoes the same strain in the two-phase composite.

$$W = W_1 + W_2 = \frac{\epsilon^2K_1V_1}{2} + \frac{\epsilon^2K_2V_2}{2} \tag{3.10}$$

Equating the strain energy in the composite, Equation (3.8), to the equivalent homogenous medium, Equation (3.10), leads to the following expression for the effective bulk modulus:

$$K = c_1K_1 + c_2K_2 \tag{3.11}$$

A similar expression can be obtained for the effective shear modulus G. The effective modulus of elasticity can be calculated from Equation (3.11) in combination with the following relations from the theory of elasticity (or fluid mechanics):

$$E = \frac{9KG}{3K+G} = 2G(1+v) = 3K(1-2v) \tag{3.12}$$

Using Equations (3.11) and (3.12), the effective modulus of elasticity for the parallel model can be given by

$$E = c_1E_1 + c_2E_2 + \frac{27c_1c_2(G_1K_2 - G_2K_1)^2}{(3K_v + G_v)(3K_1 + G_1)(3K_2 + G_2)} \tag{3.13}$$

where K_v and G_v refer to the values obtained using the Voigt model. For special case where the two phases have the same Poisson's ratio, Equation (3.13) reduces to Equation (3.5), $E = c_1E_1 + c_2E_2$, which was obtained neglecting deformations.

The series model assumes that the stress state in each phase will be a uniform hydrostatic compression of magnitude *P*. The total store energy for the composite is given by

$$W = W_1 + W_2 = \frac{P^2V_1}{2K_1} + \frac{P^2V_2}{2K_2} = \frac{P^2}{2}\left(\frac{V_1}{K_1} + \frac{V_2}{K_2}\right) \tag{3.14}$$

The effective bulk modulus can be obtained by equations (3.7) and (3.14):

$$\frac{1}{K} = \frac{c_1}{K_1} + \frac{c_2}{K_2} \tag{3.15}$$

Using the relationships for elastic modulus given by Equations (3.12), Equations (3.15) can be rewritten as

$$\frac{1}{E} = \frac{c_1}{E_1} + \frac{c_2}{E_2} \tag{3.16}$$

Note that Equation (3.16) is the same as Equation (3.6), which was obtained when we neglected lateral deformations.

Neither the Voigt nor the Reuss models are precise, except in the special case where the moduli of the two materials are equal. This is because the equal-stress assumption satisfies the stress equations of equilibrium, but, in general, gives rise to displacements that are discontinuous at the interface between the two phases. Similarly, the equal-strain assumption leads to an admissible strain field, but the resulting stresses are discontinuous.

Energy considerations from the theory of elasticity, showed that the parallel and series assumptions lead to upper and lower bounds on *G* and *K*. This result is significant because, given the elastic moduli of the phases and their volume fractions, it allows the determination of the maximum and minimum allowable value for the concrete elastic moduli. If the maximum and minimum values are close, the problem is solved from an engineering point of view. When hard inclusions are dispersed in a softer matrix (*concrete for instance*), the maximum and minimum values are far apart, as shown in Figure (3.2). Therefore, it is recommended to establish stricter upper and lower bounds, such as the Hashin-Shtrikman bounds to be discussed later after a brief review of more elaborate models.

Hirsch (1962) proposed a *model* (Figure 3.1*c*) that relates the modulus of elasticity of concrete to the moduli of the two phases (aggregate and matrix), their volume fractions, and an empirical constant, *x*.

$$\frac{1}{E_c} = (1-x)\left(\frac{c}{E_a} + \frac{1-c}{E_m}\right) + x\left(\frac{1}{cE_a + (1-c)E_m}\right) \quad (3.17)$$

where

$$c = \frac{V_a}{V_c}$$

For practical application, the value 0.5 for *x* is often recommended, which gives the arithmetic average of the parallel and series moduli.

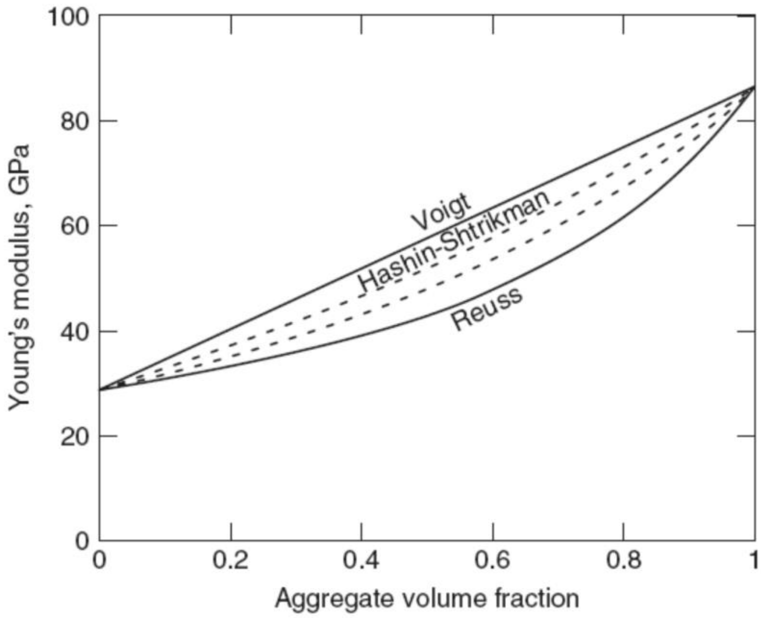

Figure (3.2): Bounds for Young's modulus (elastic moduli of the matrix E_m = 28.7 GPa, K_m = 20.8 GPa and elastic moduli of the aggregate: E_a = 86.7 GPa, Ka = 44 GPa)

Hansen (1965) proposed a model (Figure 3.1d) that consists of spherical aggregate located at the centre of a spherical mass of matrix. This model was based on a general formulation by Hashin (1962), which, for the particular case when the Poisson's ration of both phases is equal to 0.2, yielding

$$E = \left(\frac{c_1E_1 + (1+c_2)E_2}{(1+c_2)E_1 + c_1E_2}\right)E_1 \quad (3.18)$$

where phase-1 corresponds to the matrix and phase-2 to the aggregate.

Counto (1964) considered the case (Figure 3-1e) where an aggregate prism is placed at the centre of a prism of concrete, both with the same ratio of height to area of cross section. By using a simple strength of materials approach, the modulus of elasticity for the concrete can be given by

$$\frac{1}{E} = \frac{1-\sqrt{c_2}}{E_1} + \frac{1}{\left(\frac{1-\sqrt{c_2}}{\sqrt{c_2}}\right)E_1 + E_2} \qquad (3.19)$$

Again, phase 1 corresponds to the matrix and phase 2 to the aggregate.

Example: a typical lean concrete mixture with 75% of aggregate and 25% cement paste by volume, for a given age assuming the elastic modulus of cement and aggregate (quartzite) to be 20 and 45 GPa, respectively, calculate the composite elastic moduli?

Solution:
E using Voigt model = 38.8 GPa
E using Reuss model = 34.3 GPa
No matter how sophisticated or simple a two-phase model may be, any prediction should lie inside these bounds.

E_c using Hirsch model = 36.5 GPa
E_c using Hansen model = 36.4 GPa
E_c using Counto model = 36.2 GPa

So for practical purposes, the three models estimate the same elastic modulus for this particular example. In these examples it has been relatively simple to estimate the elastic modulus. In other cases of estimating elastic moduli may be more problematic.

The models presented so far are limited in computing the effect of voids, cracks, and phase changes (such as water to ice during freezing of the cement paste). Another shortcoming of these methods is that they do not take into account any of the specific geometrical features of the phases or how the pores and aggregate particles interact with one another under various loading conditions. As a general rule for two-component materials, the effect of the shape of the inclusion is more important when the two components have vastly different moduli, but is of minor importance when the two components have roughly equal moduli. *Hence, we can use models that ignore aggregate shape when trying to estimate the moduli of a mixture of cement paste and aggregate.*

3.1.2 Three phase models

Nielsen and Monteiro (1993) studied the limitations of two-phase models for concrete and proposed that concrete be modelled as a three-phase material consisting of aggregate particles, surrounded by a transition zone, embedded in the cement paste matrix.

Hashin and Monteiro (2002) developed a mathematical model based on the following assumption: concrete is a composite consisting of a matrix in which are embedded spherical particles, each of which is surrounded by a concentric spherical shell, which will be called the *interphase*. Matrix, particle, and interphase materials were considered elastic isotropic and the entire composite was assumed to be statistically homogeneous and isotropic. First the authors considered the classical problem of analytical determination of the effective elastic properties of the composite in terms of constituent properties and internal geometry. Then the model was used to determine of interphase properties in terms of particle and matrix properties and effective properties.

Using experimental data, their analysis indicated that the shear and Young's moduli of the interphase are about 50 percent of those of the original bulk cement paste, while the bulk modulus is on the order of 70 percent of the original bulk cement paste.

When studying the effect of pores and cracks, sophisticated models are needed that explicitly consider the shape of the "inclusions." Two of the most accurate models available for estimating the effect of pores and/or cracks on the elastic moduli are the *differential scheme* and the *Mori-Tanaka method*. Discussion on the rationale of these methods is beyond the scope of this course. For two critical idealized pore shapes, namely, the sphere and the "penny-shaped" crack, the results have relatively simple forms which are described below.

If a solid body of modulus E_0 and Poisson's ratio v_0 contains a volume fraction c of spherical pores, its overall moduli E will be as follows:

Differential method: $$E = E_0(1-c)^2 \quad (3.20)$$

Mori-Tanaka method: $$E = E_0(1-c)/(1+\alpha c) \quad (3.21)$$

where $\alpha = 3(1+v_0)(13-15v_0)/6(7-5v_0)$. Parameter α is nearly independent of v_0, and is approximately equal to 1.

3.1.3 Effect of crack

Since a small volume fraction of very thin cracks can cause an appreciable degradation of the moduli, it is not convenient to quantify their concentration using volume fractions. If the body is filled with circular cracks, instead, we use the crack-density parameter Γ, which is defined by $\Gamma = N\alpha^3/V$, where α is the radius of the crack in its plane, and N/V is the number of cracks per unit volume. The effective moduli of a body containing a density Γ of circular cracks are as follows for the two different models

Differential method: $$E = E_0 e^{-16\Gamma/9} \quad (3.22)$$

Mori-Tanaka method: $$E = E_0/(1+\beta\Gamma) \quad (3.23)$$

where $\beta = 16(10-3v_0)(1-v_0^2)/45(2-v_0)$. For typical values of v_0, β is essentially equal to 16/9 = 1.78.

More general treatments of the effect of pores on the elastic moduli assume that the pores are oblate spheroids of a certain aspect ratio. The sphere (aspect ratio = 1) and the crack (aspect ratio = 0) are the two extreme cases. Figure (3-3) shows the elastic moduli as a function of porosity, for various pore aspect ratios, as calculated using the Mori-Tanaka model.

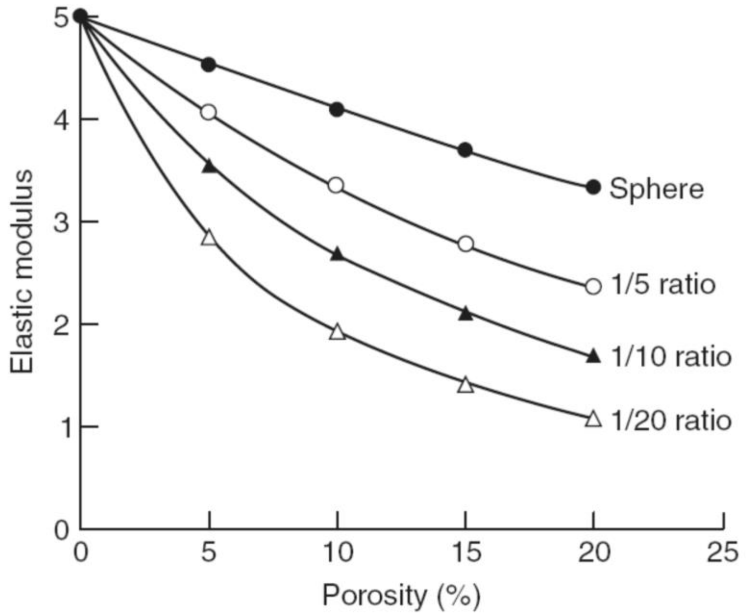

Figure (3.3): Effect of porosity and shape of pores on the elastic modulus of a material.

3.2 **Hashin-Shtrikman (H-S) Bounds**

Although the Voigt and Reuss models produce an upper and lower bounds for the elastic moduli, as shown in Figure (3-2), the bounds are often far apart, in which case they are of little use for certain specific cases. For instance, if we assume a volume fraction of 0.6 in Figure (3-2), the upper and lower bounds are 63.9 and 47.9 MPa, respectively. The spread is large and, therefore, of limited use for engineering applications. Fortunately, Hashin and Shtrikman (H-S) developed more stringent bounds for a composite material, which in a statistical sense, is both isotropic and homogeneous. The H-S bounds were derived using variational principles of the linear theory of elasticity for multiphase materials of arbitrary phase geometry. For two-phase composites the expressions take the form:

$$K_{low} = K_1 + \frac{c_2}{\frac{1}{K_2 - K_1} + \frac{3c_1}{3K_1 + 4G_1}} \qquad K_{up} = K_2 + \frac{c_1}{\frac{1}{K_1 - K_2} + \frac{3c_2}{3K_2 + 4G_2}} \qquad (3.24)$$

$$G_{low} = G_1 + \frac{c_2}{\frac{1}{G_2 - G_1} + \frac{6(K_1 + 2G_1)c_1}{5G_1(3K_1 + 4G_1)}} \qquad G_{up} = G_2 + \frac{c_1}{\frac{1}{G_1 - G_2} + \frac{6(K_2 + 2G_2)c_2}{5G_2(3K_2 + 4G_2)}} \qquad (3.25)$$

where K and G are the bulk and shear moduli, respectively. Here $K_2 > K_1$; $G_2 > G_1$. K_{up} and G_{up} refer to the upper bounds and K_{low} and G_{low} to the lower bounds.

Figure (3-2) shows that the H-S bounds are inside the Voigt-Reuss bounds. Using the previous example for a volume fraction of 0.6, the H-S bounds give 58.4 and 54.0 MPa. The range is significantly narrower than that obtained using the Voigt-Reuss bounds.

3.3 **Transport Properties**

This section has concentrated on various methods for estimating elastic modulus; however, other important properties can also be predicted using the theorems of composite materials. Consider the following relationships that have the same mathematical structure:

Electrical conduction: $j = \sigma E$

Thermal conduction: $Q = -\kappa \nabla T$

Dielectric displacement: $D = \varepsilon E$

Magnetic induction: $B = \mu H$

Diffusion: $Q = -D \nabla c$

For each of these five transport relationships, the flux vector is related to the driving force vector by a second-order physical property tensor, that is, a 3 × 3 matrix (σ, k, ε, μ, D). For isotropic materials, the electrical conductivity σ, the thermal conductivity k, the dielectric constant ε, the magnetic susceptibility μ, and the diffusion constant D reduce to a single constant.

It should be noted that the elastic moduli is a fourth order tensor and, even for isotropic materials, contains two independent constants. Any model that can predict, say, diffusion constant D from the individual phases properties, will also be able to predict σ, k, ε, and μ.

Hashin and Shtrikman derived the following bounds for transport constants. For thermal conductivity ($k_2 > k_1$), in the three-dimensional case we have for the upper bound:

$$\kappa_u = \kappa_2 + \frac{c_1}{\dfrac{1}{\kappa_1 - \kappa_2} + \dfrac{c_2}{3\kappa_2}}$$

and for the lower bound:

$$\kappa_1 = \kappa_1 + \frac{c_2}{\dfrac{1}{\kappa_2 - \kappa_1} + \dfrac{c_1}{3\kappa_1}}$$

The number 3 in the denominator should be replaced by 2 and 1 for two-dimensional or one-dimensional cases, respectively. Similar equations apply for the other transport constants.