4. VISCOELASTICITY

There are two methods used to study the one-dimensional viscoelastic behavior of concrete: ***a***) the creep test, where the stress is kept constant and the increase in strain over time is recorded, and (***b***) the relaxation test, where the strain is kept constant and the decrease in stress over time is recorded. Experimental results from both creep and relaxation tests are shown in Figure (4.1), where the creep response is a function of the *duration of loading* and the *age of concrete* when the load was applied. The longer the concrete is under load, the greater the deformation, and the greater the age of loading, the lower the deformation. This behaviour classifies concrete as an *aging viscoelastic material*. In fact, most of the mechanical properties of concrete are age-dependent. The mathematical formulation for aging materials is more complex than for non-aging materials; this section presents basic expressions for aging materials.

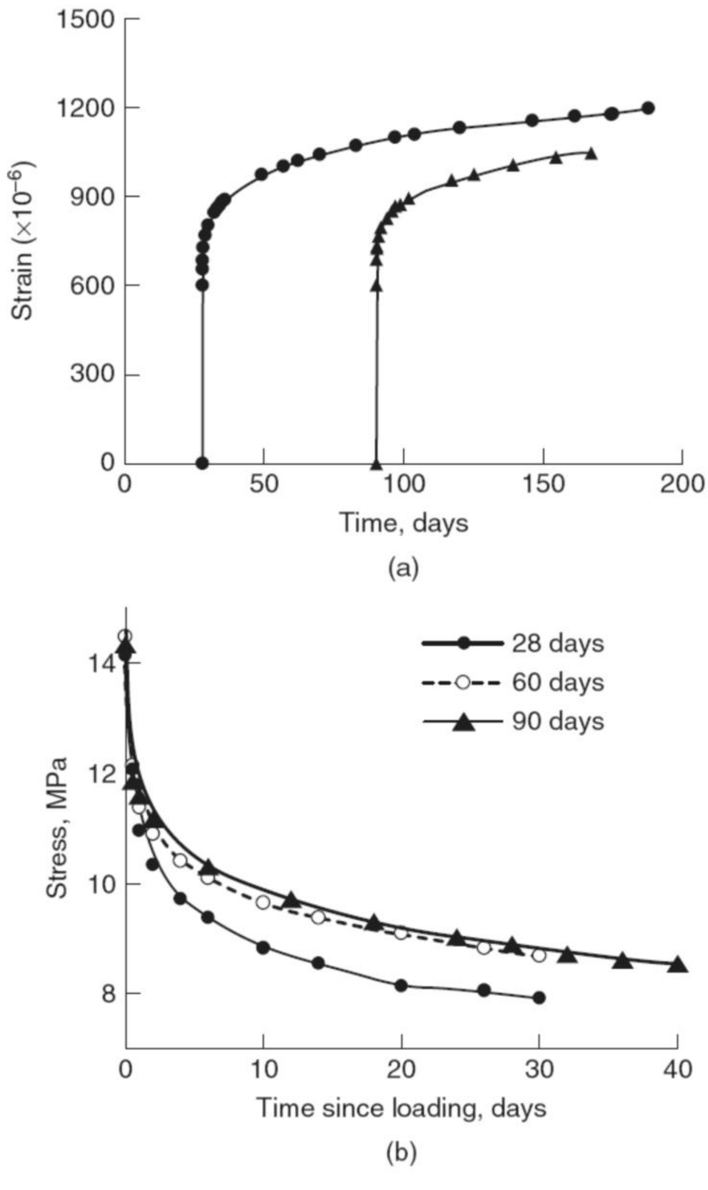


Figure (4.1): (a) Creep test; (b) relaxation test of concrete.

Creep and relaxation experiments are time-consuming, but worthwhile as they yield significant information about the viscoelasticity of the material. Contrary to elastic behavior where two constants are used to describe a homogeneous isotropic elastic material, for viscoelastic behavior an *evolution law* is necessary to describe how the stress or strain changes over time. In this section, *rheological models* are presented that produce such evolution laws, in addition to some practical equations used in design codes. Rheological models will be used to provide some insight into the viscoelastic behavior of concrete, explaining for instance why the rate of stress decrease in the relaxation test is faster than the rate of strain increase in the creep test. Unfortunately, in real concrete structures the state of stress or strain is unlikely to be constant over time. To model more complex loading conditions, the principle of superposition and integral representations are presented. These methods allow to compute the strains if the creep function and stress history are known or to compute the stresses if the relaxation function and strain history are known. If no experimental data are available (i.e., creep or relaxation test results), the recommendations of a code or a model are used: CEB model code 1990, ACI-209, and the Bazant-Panula model.

4.1 Basic Rheological Models

The behavior of viscoelastic materials can be successfully estimated by the creation of rheological models based on two fundamental elements: the linear spring and the linear viscous dashpot. For the linear spring the relationship between stress and strain is given by Hooke's law:

$$\sigma(t) = E\ \varepsilon(t) \tag{4.1}$$

The response of the spring to the stress is immediate. During a creep test, where the stress σ_0 is kept constant, the stain will be σ_0/E, constant over time. Similarly, for a relaxation test, where the strain ε_0 is kept constant the stress will be $\varepsilon_0 E$, constant over time.

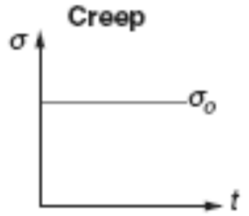
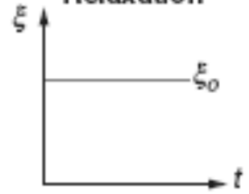
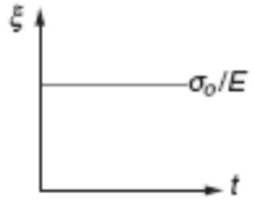
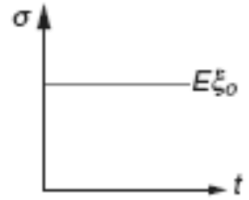
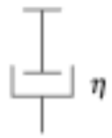
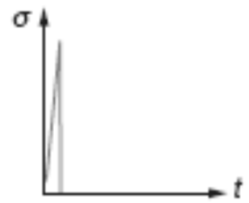
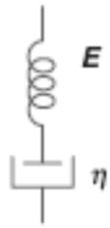
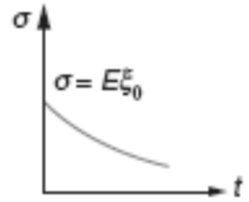
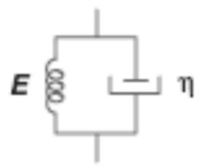
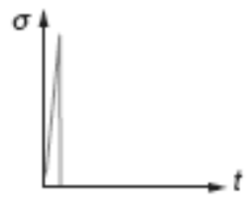
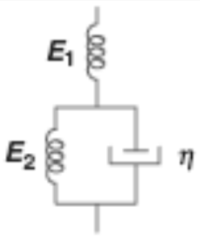
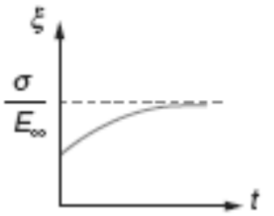
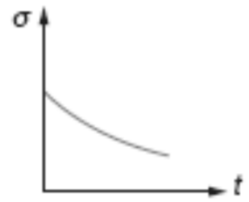
The viscous dashpot can be visualized as a piston displacing a viscous fluid in a cylinder with a perforated bottom. Newton's law of viscosity:

$$\dot{\varepsilon}(t) = \frac{\sigma(t)}{\eta} \tag{4.2}$$

where $\dot{\varepsilon} = \frac{d\varepsilon}{dt} =$ the strain rate

η = the viscosity coefficient

Table 4.1 Simple Rheological Models and their Creep and Relaxation Response

Name	Representation	Creep	Relaxation
			
(a) Spring			
(b) Dashpoot			
(c) Maxwell			
(d) Kelvin			
(e) Standard Solid			

4.1.1 Maxwell model

The *Maxwell model* comprises a linear spring and a linear viscous dashpot connected in series, as shown in Table 4.1*c*. The following equations apply:

Equilibrium equation $$\sigma_E(t) = \sigma_\eta(t) = \sigma(t) \tag{4.3}$$

Compatibility equation $$\varepsilon(t) = \varepsilon_E(t) + \varepsilon_\eta(t) \tag{4.4}$$

Constitutive relationship (spring) $$\sigma_E(t) = E\varepsilon_E(t) \tag{4.5}$$

(dashpot) $$\sigma_\eta(t) = \eta\dot{\varepsilon}_\eta(t) \tag{4.6}$$

Differentiating Equations (4.4) and (4.5) with respect to time t and using Equations (4.3) and (4.6):

$$\dot{\varepsilon}(t) = \frac{\dot{\sigma}(t)}{E} + \frac{\sigma(t)}{\eta} \tag{4.7}$$

Note that for a rigid spring ($E = \infty$), the model reduces to a Newtonian fluid; likewise, if the dashpot becomes rigid ($\eta = \infty$), the model reduces to a Hookean spring. The response of the Maxwell model to various kinds of time-dependent stress or strain patterns can be determined by solving Equation (4.7). For instance, consider again a creep test, with the initial conditions $\sigma = \sigma_0$ at $t = 0$. Integrating Equation (4.7), we obtain:

$$\varepsilon(t) = \frac{\sigma_0}{E} + \frac{\sigma_0 t}{\eta} \tag{4.8}$$

The model predicts that the strain increases without bounds. This is characteristic of many fluids; therefore, a material described by Equation (4.7) is known as a “Maxwell’’ fluid. When the system is unloaded at time t_1 the elastic strain σ_0/E in the spring recovers instantaneously, while a permanent strain $(\sigma_0/\eta)t_1$ remains in the dashpot.

In a relaxation experiment, where the strain ε_0 is constant, the model predicts:

$$\sigma(t) = E\varepsilon_0 e^{-Et/\eta} \tag{4.9}$$

The ratio $T = \eta/E$ is called the *relaxation time,* and it helps characterize the viscoelastic response of the material. A small relaxation time indicates that the relaxation process will be fast. In the extreme case of a purely viscous fluid, $E = \infty$, Equation (4.9) would indicate an infinitely fast stress relaxation, $T = 0$; while for an elastic spring, $\eta = \infty$, the stress would not relax at all, since $T = \infty$.

4.1.2 Kelvin model

The *Kelvin model* combines a linear spring and a dashpot in parallel as shown in Table 4-1d. The following equations apply:

Equilibrium equation $$\sigma(t) = \sigma_E(t) + \sigma_\eta(t) \tag{4.10}$$

Compatibility equation $$\varepsilon(t) = \varepsilon_E(t) = \varepsilon_\eta(t) \tag{4.11}$$

Constitutive relationship (spring) $$\sigma_E(t) = E\varepsilon_E(t) \tag{4.12}$$

(dashpot) $$\sigma_\eta(t) = \eta\dot{\varepsilon}_\eta(t) \tag{4.13}$$

Resulting in the differential equation $$\sigma(t) = E\varepsilon(t) + \eta\dot{\varepsilon}_\eta(t) \tag{4.14}$$

Note that the model reduces to a Hookean spring if η = 0, and to a Newtonian fluid if E = 0. Equation (4.14) may be used to predict strain if the stress history is given or to predict stress if the strain history is given. For instance, for the creep experiment, integrating Eq. (4.14) with the boundary condition $\sigma = \sigma_0$ at t = 0 yields:

$$\varepsilon(t) = \frac{\sigma_0}{E}\varepsilon_0\left(1 - e^{-Et/\eta}\right) \tag{4.15}$$

In Eq. (4.15), the strain increases at a decreasing rate and has an asymptotic value of σ_0/E, as shown in Table 4-1d. During the creep test the stress is initially carried by the dashpot and, as time goes by, the stress is transferred to the spring. Analogous to the relaxation time, we define the *retardation time* as $\theta = \eta/E$. A small retardation time indicates that the creep process will be fast. In the extreme case of an elastic spring (η = 0), the final strains would be obtained instantaneously since θ = 0.

The Kelvin model requires an infinite stress to produce the instantaneous strain necessary for the relaxation test, which makes it physically impossible to perform.

The Maxwell and Kelvin models have significant limitations in representing the behaviour of most viscoelastic materials. As discussed before, the Maxwell model shows a constant strain rate under constant stress, which may be adequate for fluids, but not for solids. The Kelvin model cannot predict a time-dependent relaxation and does not show a permanent deformation upon unloading.

4.1.3 Standard solid model

A more complex, representative model is the *standard solid model,* where a spring is connected in series with a Kelvin element as shown in Table 4-1e. Assuming ε_1 and ε_2 to be the strain in the spring and Kelvin elements, respectively, the total strain, for the standard solid, is given by

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \tag{4.16}$$

Since the stress in the spring and the Kelvin element is the same, the stress can be determined using Eq. (4-14):

$$\sigma(t) = E_2\varepsilon_2(t) + \eta\,\frac{\partial\varepsilon_2(t)}{\partial t} \tag{4.17}$$

where $\partial/\partial t$ is a differential operator that may be handled as an algebraic entitiy,

$$\sigma(t) = \varepsilon_2(t)\left(E_2 + \eta\,\frac{\partial}{\partial t}\right) \tag{4.18}$$

leading to

$$\varepsilon_2(t) = \frac{\sigma(t)}{\left(E_2+\eta\frac{\partial}{\partial t}\right)} \tag{4.19}$$

Therefore, we can obtain the strain for the standard solid by using Eq. (4.16)

$$\varepsilon(t) = \frac{\sigma(t)}{(E_1)} + \frac{\sigma(t)}{\left(E_2+\eta\frac{\partial}{\partial t}\right)} \tag{4.20}$$

or

$$E_1\varepsilon(t)\left(E_2 + \eta\,\frac{\partial}{\partial t}\right) = E_1\sigma(t) + \sigma(t)\left(E_2 + \eta\,\frac{\partial}{\partial t}\right) \tag{4.21}$$

Which leads to the differential equation

$$\eta E_1\dot{\varepsilon}(t) + E_1E_2\varepsilon(t) = \eta\dot{\sigma}(t) + (E_1 + E_2)\sigma(t) \tag{4.22}$$

Equation (4.22) can be integrated for an arbitrary stress history,

$$\varepsilon(t) = \frac{\sigma(t)}{(E_1)} + \frac{1}{\eta}\int_0^t \sigma(\tau)\, e^{-E_2(t-\tau)/\eta}\, d\tau \tag{4.23}$$

For the particular case of the creep experiment, Equaiton (4.23) reduces to

$$\varepsilon(t) = \frac{\sigma_0}{E_1} + \frac{\sigma_0}{E_2}\left[1 - e^{-E_2 t/\eta}\right] \tag{4.24}$$

Which can be rewritten as

$$\varepsilon(t) = \sigma_0 \left(\frac{E_1+E_2}{E_1 E_2} - \frac{1}{E_2} e^{-E_2 t/\eta} \right) \tag{4.25}$$

Equation (4.25) indicates that the strain is proportional to σ_0, changing from σ_0 / E_1 at $t = 0$ to σ_0 / E_∞ at $t = \infty$. E_∞ is called the asymptotic modulus and is given by

$$E_\infty = \frac{E_1+E_2}{E_1 E_2} \tag{4.26}$$

During the creep test, the elastic modulus of the standard solid model, $E_c(t)$, reduces from the inital value E_1 to its asympototic value E_∞, according to the following law:

$$\frac{1}{E_c(t)} = \frac{\varepsilon(t)}{\sigma_0} = \frac{E_1+E_2}{E_1 E_2} - \frac{1}{E_2} e^{-E_2 t/\eta} \tag{4.27}$$

We now integrate Equation (4.22) for an arbitrary strain histroy

$$\sigma(t) = \varepsilon(t) E_\infty + (E_1 - E_\infty) \int_0^t e^{-(E_1+E_2)(t-\tau)/\eta} \dot{\varepsilon}(\tau) d\tau \tag{4.28}$$

In the particular case of relaxation experiment the stress evolution is given by

$$\sigma(t) = \varepsilon_0 \left[E_\infty + (E_1 - E_\infty) e^{-(E_1+E_2)t/\eta} \right] \tag{4.29}$$

Equation (4.29) indicates that the stress is proportional to ε_0 changing from $E_1\ \varepsilon_0$ at $t = 0$, up to $E_\infty\ \varepsilon_0$ at $t = \infty$. Therefore, during the relaxation test the elastic modulus $E_r(t)$, reduces from the initial value E_1, to its asymptotic value E_∞, according to the following law:

$$E_r(t) = \left[E_\infty + (E_1 - E_\infty) e^{-(E_1+E_2)t/\eta} \right] \tag{4.30}$$

Even though both creep and relaxation may be understood as a decrease in elastic modulus over time from E_1 to its asymptotic value E_∞, Equations (4.25) and (4.28) have different rates of decrease. In a relaxation test, the decrease in the elastic modulus occurs at significantly faster rate than in the creep test. As an example, let us take the following values for concrete: E_1 = 35 GPa, E_2 = 18 GPa, $T(E_2/\eta)$ = 1/300 days. Figure (4.2) illustrates the faster reduction of elastic modulus during relaxation than for the creep test.

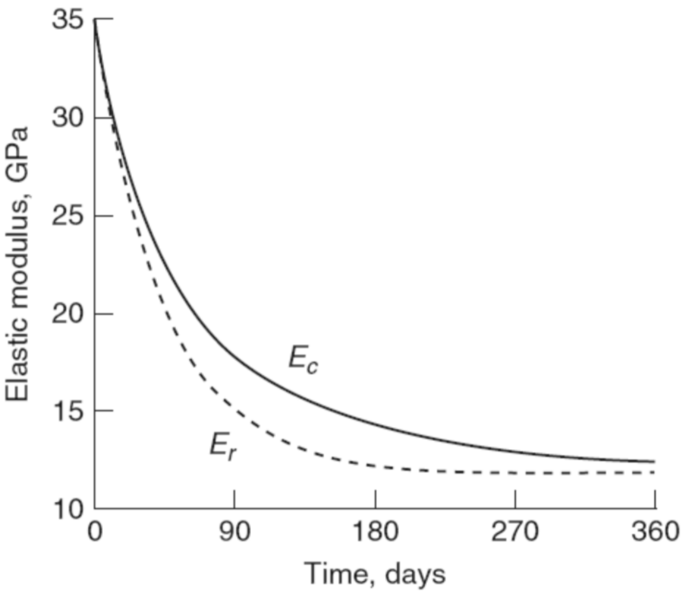


Figure (4.2): Decrease of elastic modulus for relaxation and creep.

Example 4.1

The testing of materials is usually performed either by controlling the stress or strain rate. Study the response of a standard solid model loaded under these conditions. Solve the problem analytically and then expand the discussion for instantaneous, slow, and medium stress and strain rates; assume the following properties for the standard solid: E_1 = 35 GPa, E_2 = 18, GPa, T = 1 min.

(A) Test with a constant stress rate (v): The stress increases linearly with time, according to

$$\sigma(t) = vt \tag{4.31}$$

The strain in the standard solid model is obtained by combining Eqs. (4.23) and (4.31)

$$\varepsilon(t) = \frac{vt}{E_1} + \frac{v}{\eta}\int_0^t \tau\, e^{-E_2(t-\tau)/\eta} d\tau \tag{4.32}$$

which leads to

$$\varepsilon(t) = \frac{vt}{E_\infty} - \frac{v\eta}{E_2^2}\left(1 - e^{-E_2 t/\eta}\right) \qquad (4.33)$$

Figure (4.3) presents the stress (Eq. 4.31) as a function of strain (Eq. 4.33) using the given material properties, showing that the stress-strain diagram is strongly influenced by the rate of loading. Note that the stress-strain diagram may be nonlinear, a common feature for viscoelastic materials where the strain at a given time is influenced by the entire stress history. *This phenomenon will be presented in more detail in the following sections.*

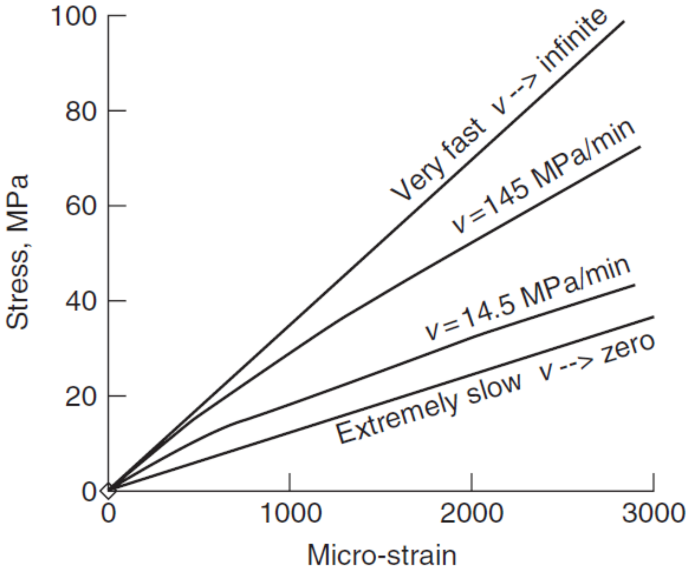


Figure (4.3): Effect of stress rate on the stress-strain diagram.

The stress-strain relationships shown in Figure (4.3) are bounded by very slow and very fast rates. The latter gives the upper bound and physically corresponds to the linear spring (E_1) absorbing all the stress, as the Kelvin element has no time to deform. For very slow rates, the standard solid model responds with the asymptotic modulus E_∞, and physically corresponds to the linear spring E_1 in series with the spring from the Kelvin element E_2 the dashpot does not contribute to the stiffness of the system.

(B) Test with constant strain rate: The strain increases with time, according to

$$\varepsilon(t) = vt \qquad (4.34)$$

The stress in the model is obtained by combining Eqs. (4.28) and (4.34),

$$\sigma(t) = vtE_\infty + (E_1 - E_\infty)\int_0^t ve^{-(E_1+E_2)(t-\tau)/\eta}d\tau \qquad (4.35)$$

which leads to

$$\sigma(t) = vtE_\infty + (E_1 - E_\infty)v\frac{\eta}{(E_1+E_2)}\left(1 - e^{-(E_1+E_2)t/\eta}\right) \tag{4.36}$$

Figure (4.4) shows the stress (Eq. 4.36) in function of strain (Eq. 4.34) with the specified concrete properties.

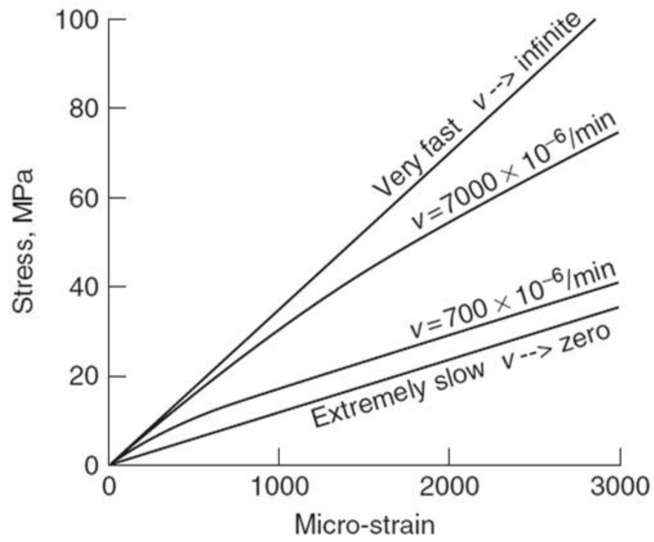


Figure (4.4): Effect of strain rate on the stress-strain diagram.

Example 4.2

Study the response of a viscoelastic material subjected to a cyclic strain ε (t) = ε_0 cos *wt*, where ε_0 is the strain amplitude and *w* the frequency. Write explicit equations for the Maxwell and Kelvin models.

For a linear elastic spring, the stress will be in phase with the cyclic strain that is

$$\sigma(t) = E\varepsilon(t) = E\varepsilon_0 \cos\omega t \tag{4.37}$$

For a Newtonian fluid the stress will lead the strain by π/2:

$$\sigma(t) = \eta\dot{\varepsilon}(t) = -\eta\omega\varepsilon_0 \sin\omega t = \eta\omega\varepsilon_0 \cos(\omega t + \delta) \tag{4.38}$$

where

$$\delta = \frac{\pi}{2} \tag{4.39}$$

For a viscoelastic material the phase difference between stress and strain ranges from 0 to π/2. A convenient way of representing oscillatory strain is by using the expression:

$$e^{i\omega t} = \cos \omega t + i \sin \omega t \tag{4.40}$$

Taking the real part of the expression, the strain equation can be rewritten as

$$\varepsilon(t) = \varepsilon_0 \, e^{i\omega t} \tag{4.41}$$

The stress oscillates with the same frequency *w*, but leads the strain by a phase angle δ where

$$\sigma(t) = \sigma_0 \, e^{i(\omega t + \delta)} \tag{4.42}$$

which can be rewritten as

$$\sigma(t) = \sigma_0 \, e^{i\delta} e^{i\omega t} = \sigma^* e^{i\omega t} \tag{4.43}$$

where σ^* is the complex stress amplitude given by

$$\sigma^* = \sigma_0 \, e^{i\delta} = \sigma_0 \, (\cos \delta + i \sin \delta) \tag{4.44}$$

A complex mosdulus E^* can be defined as

$$E^* = \frac{\sigma^*}{\varepsilon_0} = \frac{\sigma_0 \, (\cos \delta + i \sin \delta)}{\varepsilon_0} = E_1 + iE_2 \tag{4.45}$$

where E_1, the storage modulus, is in phase with the strain, and is given by

$$E_1 = \frac{\sigma_0}{\varepsilon_0} \cos \delta \tag{4.46}$$

E_2, the loss modulus, is the imaginary part, and is given by

$$E_2 = \frac{\sigma_0}{\varepsilon_0} \sin \delta \tag{4.47}$$

and the magnitude of the complex modulus is given by

$$|E^*| = \sqrt{E_1^2 + E_2^2} \tag{4.48}$$

It should be noted that

$$\tan \delta = \frac{E_2}{E_1} \tag{4.49}$$

Represents the mathematical lose per cycle of strain

For the Maxwell model: The constitutive equation of the Maxwell model is given by Equation (4.7)

$$\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon} \tag{4.50}$$

Using Eqs. (4.41) and (4.43) we obtain

$$\sigma_0 e^{i\delta}\left(1 + i\omega\frac{\eta}{E}\right) = i\omega\varepsilon_0\eta \tag{4.51}$$

or

$$\sigma^*\left(1 + i\omega\frac{\eta}{E}\right) = i\omega\varepsilon_0\eta \tag{4.52}$$

Therefore, the complex modulus can be expressed by

$$E^* = \frac{\sigma^*}{\varepsilon_0} = \frac{i\omega\eta}{1+\frac{i\omega\eta}{E}} \tag{4.53}$$

Separating the real and imaginary parts we find

$$E^* = \frac{\eta^2\omega^2/E}{1+\eta^2\omega^2/E^2} + i\,\frac{\eta\omega}{1+\eta^2\omega^2/E^2} \tag{4.54}$$

Hence the magnitude of the complex modulus is given by

$$|E^*| = \eta\omega\left(1 + \frac{\eta^2\omega^2}{E^2}\right)^{-\frac{1}{2}} \tag{4.55}$$

and

$$\tan\delta = \frac{E_2}{E_1} = \frac{E}{\eta\omega} \tag{4.56}$$

Taking the material constants from the previous example, the magnitude of complex modulus can be plotted against the angular frequency, as shown in Figure (4.5). Note that for very high frequencies the dynamic modulus approaches the spring constant E and for the low frequencies the dynamic modulus approaches zero.

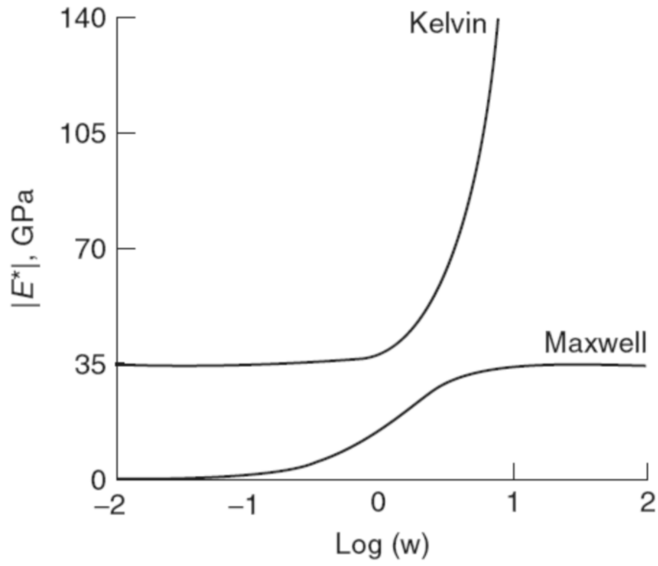


Figure (4.5): Complex elastic modulus in function of frequency.

For the Kelvin model: The constitutive equation for the Kelvin model is given by Eq. (4.14)

$$\sigma(t) = E\varepsilon(t) + \eta\dot{\varepsilon}_\eta(t) \tag{4.57}$$

Using Eqs. (4.42) and (4.57) we obtain

$$\sigma_0\, e^{i\delta} = \varepsilon_0\, (E + i\omega\eta) \tag{4.58}$$

Therefore, the complex modulus is expressed by

$$E^* = \frac{\sigma^*}{\varepsilon_0} = E + i\omega\eta \tag{4.59}$$

And the magnitude of the complex modulus by

$$|E^*| = (E^2 + \eta^2\omega^2)^{\frac{1}{2}} \tag{4.60}$$

The mechanical loss for the model is

$$\tan\delta = \frac{\eta}{E}\omega \tag{4.61}$$

Again, if we take the material constants from the previous example, the results for the Kelvin model can be plotted, as shown in Figure (4.5). Note that for low frequencies the dynamic modulus is given by the spring constant E, while for high frequencies the stiffness increases.

The significantly different responses for the Maxwell and Kelvin models under oscillatory stress points to the advantage of performing such a test to assess which model is most representative for a specific material.

4.2 Generalized Rheological Models

The modelling of viscoelastic behaviour can be improved by combining a large number of springs and dashpots in series or in parallel. By adding many elements, several relaxation times can be obtained, which is characteristic of complex materials such as concrete.

When generalizing the Maxwell model, we must choose to connect the units either in series or in parallel. Let us start by studying the response when the units are connected in series, as shown in Figure (4.6).

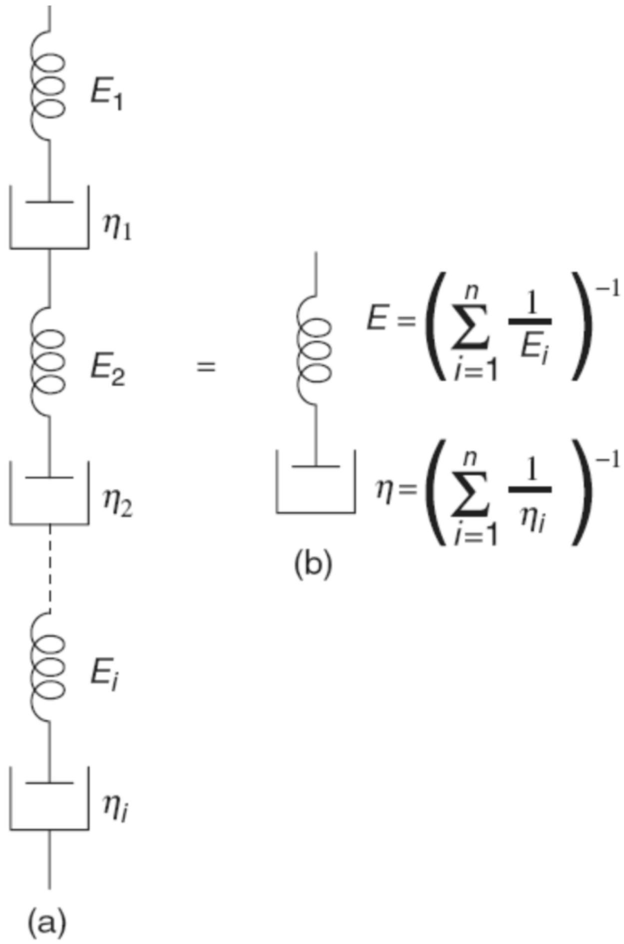


Figure (4.6): Generalized Maxwell model in series.

The constitutive equation has the form:

$$\dot{\varepsilon}(t) = \dot{\sigma}(t) \sum_{i=1}^{n} \frac{1}{E_i} + \sigma(t) \sum_{i=1}^{n} \frac{1}{\eta_i} \qquad (4.62)$$

where *n* is the number of elements. Because the equation is equivalent to Eq. (4.7), the chain of elements is identical to a single Maxwell element, as shown in Figure (4.6b), therefore not much was accomplished by connecting the units in series.

Let us now analyze the response when the units are connected in parallel, as shown in Figure (4.7b).

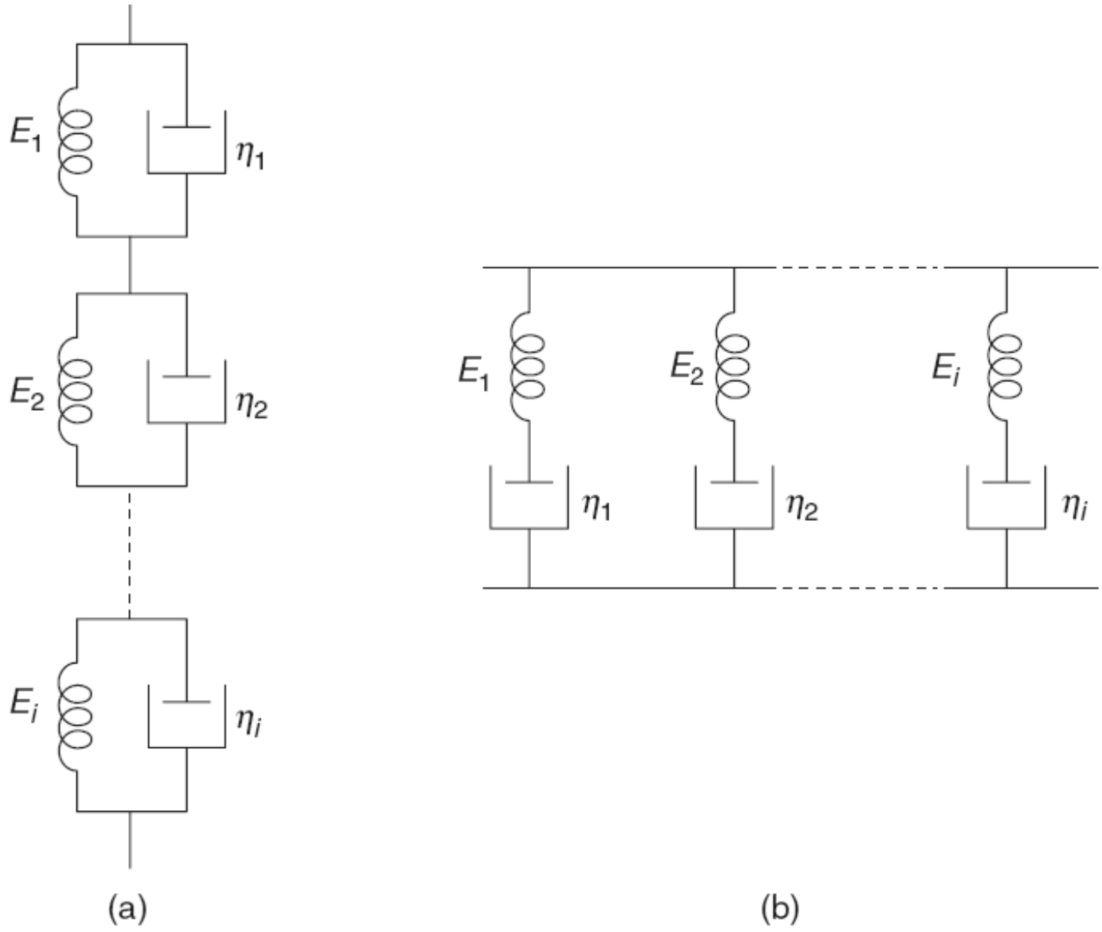


Figure (4.7): (a) Generalized Kelvin model in series and **(b)** generalized Maxwell model in parallel.

The strain in each unit of a generalized Maxwell model in parallel is given by

$$\frac{\partial}{\partial t}\varepsilon_i(t) = \left\{\frac{1}{E_i}\frac{\partial}{\partial t} + \frac{1}{\eta_i}\right\}\sigma_i(t) \qquad (4.63)$$

The stress for the generalized model is given by

$$\sigma(t) = \left\{\sum_{i=1}^{n} \frac{\partial/\partial t}{\frac{1}{E_i}\frac{\partial}{\partial t}+\frac{1}{\eta_i}}\right\}\varepsilon(t) \qquad (4.64)$$

and the relaxation function for the generalized Maxwell model is

$$E(t-\tau) = \sum_{i=1}^{n} E_i \{exp - (t-\tau)/T_i\} \qquad (4.65)$$

indicating that the response of the material depends on a distribution of relaxation times. This formulation is useful in modelling complex viscoelastic materials.

When generalizing the Kelvin model the same question arises: should we connect the units in series or parallel? We start with the units connected in parallel, as shown in Figure (4.8).

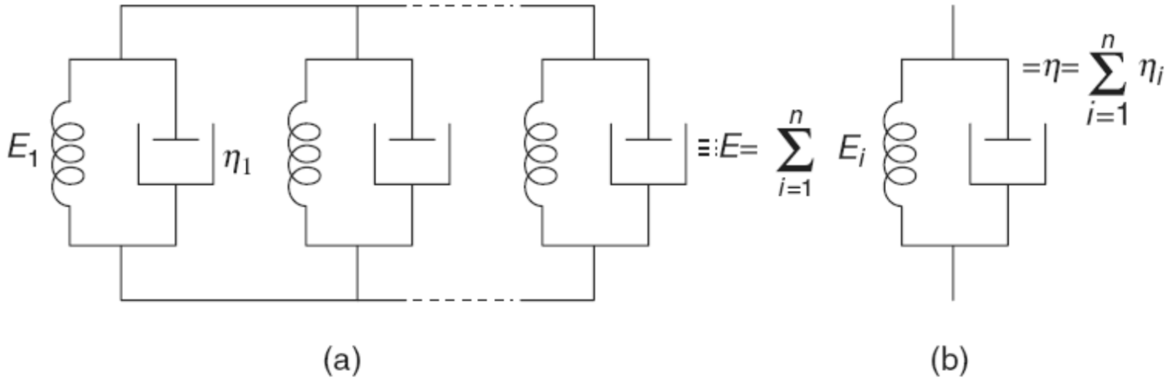


Figure (4.8): Generalized Kelvin model in Parallel

The constitutive equation for the model has the form

$$\sigma(t) = \varepsilon(t)\sum_{i=1}^{n} E_i + \dot{\varepsilon}(t)\sum_{i=1}^{n} \eta_i \qquad (4.66)$$

Which has the same form as a Kelvin element shown in Figure (4.8).

Consider a generalized Kelvin model in series as shown in Figure (4.7a). The stress in each unit is given by

$$\sigma_i(t) = \left\{E_i + \eta_i \frac{\partial}{\partial t}\right\} \varepsilon_i(t) \qquad (4.67)$$

The strain for the generalized model is given by

$$\varepsilon(t) = \sum_{i=1}^{n} \left\{\frac{1}{E_i + \eta_i \frac{\partial}{\partial t}}\right\} \sigma(t) \qquad (4.68)$$

Equations (4.63) and (4.68) are differential equations of the general form

$$\sum_{i=1}^{h} p_i \frac{d^i\sigma}{dt^i} = \sum_{i=1}^{l} q_i \frac{d^i\varepsilon}{dt^i} \qquad (4.69)$$

The specific creep function for the generalized Kelvin model in series is

$$\Phi(t-\tau) = \sum_{i=1}^{n} \frac{1}{E_i}\{1 - exp(-t-\tau)/\theta_i\} \tag{4.70}$$

In order to model the material's response adequately, the spring constants E_i and the dashpot constants η_i should vary over a large range. Sometimes when modelling a fluid or a solid, it is convenient to take some limiting value for the spring or dashpot constant. It should be noted that a Maxwell model with infinite spring constant or a Kelvin model with zero spring constant becomes a dashpot. Conversely a Maxwell model with infinite viscosity or a Kelvin model with zero viscosity results in a spring.

4.3 Time-Variable Rheological Models

Concrete changes its mechanical properties with time due to hydration reaction. In the models presented so far, however, the elastic modulus E and the viscosity coefficient η are constant over time. Consequently they have limited success in modelling the complex response of concrete. To include aging of concrete, we will now study how the differential equations for the basic elements—the spring and dashpot—change when their mechanical properties change with time.

Consider a linear spring with elastic modulus varying in time. Hooke's law can be expressed in two forms:

$$\sigma(t) = E(t)\varepsilon(t) \tag{4.71}$$

and

$$\dot{\sigma}(t) = E(t)\dot{\varepsilon}(t) \tag{4.72}$$

The equations are not equivalent. Solid mechanics literature defines a body following Eq. (4.71) to be elastic, whereas a body following Eq. (4.72) to be hypoelastic.

A linear viscous dashpot with viscosity coefficient varying in time is expressed unequivocally by

$$\sigma(t) = \eta(t)\dot{\varepsilon}(t) \tag{4.73}$$

If we reconstruct the previous models (Maxwell, Kelvin, standard-solid, generalized) for aging materials such as concrete, the equations for a Maxwell element with a hypoelastic spring or with an elastic spring are given by

$$\dot{\varepsilon}(t) = \frac{\dot{\sigma}(t)}{E(t)} + \frac{\sigma(t)}{\eta(t)} \tag{4.74}$$

$$\dot{\varepsilon}(t) = \frac{\dot{\sigma}(t)}{E(t)} + \left\{\frac{1}{\eta(t)} + \frac{d}{dt}\frac{1}{E(t)}\right\}\sigma(t) \tag{4.75}$$

Note that Eqs. (4.74) and (4.75) may be expressed as

$$\dot{\varepsilon}(t) = q_0(t)\dot{\sigma}(t) + q_1(t)\,\sigma(t) \tag{4.76}$$

where $q_0(t)$, $q_1(t)$ are independent functions of time. Equation (4.76) represents the constitutive law for the Maxwell model with either elastic or hypoelastic spring.

Dischinger (1987) used the aging Maxwell element, Eq. (4.74), to derive the so-called rate-of-creep method. The specific creep function $\Phi(t,\tau)$, that is the strain per unit stress at time t for the stress applied at age τ, is given by

$$\Phi(t,\tau) = \frac{1}{E(\tau)} + \int_{\tau}^{t} \frac{d\acute{\tau}}{\eta(\acute{\tau})} \tag{4.77}$$

The creep coefficient $\Phi(t,\tau)$ representing the ratio between the creep strain and the initial elastic deformation is

$$\Phi(t,\tau) = E(\tau)\int_{\tau}^{t} \frac{d\acute{\tau}}{\eta(\acute{\tau})} \tag{4.78}$$

Equation (4.74) can be expressed in function of the creep coefficient as

$$\frac{\partial \varepsilon}{\partial \varphi} = \frac{1}{E(t)}\frac{\partial \sigma}{\partial \varphi} + \frac{\sigma}{E(\tau)} \tag{4.79}$$

The Dischinger formulation implies that the creep curves are parallel for all ages. Experimental results do not indicate that the assumption is valid, as evident in Figure (4.1a), where the creep curves are not parallel. Usually this method substantially underestimates the creep for stresses applied at ages greater than τ.

A Kelvin element with an elastic spring is described by

$$\sigma(t) = E(t)\varepsilon(t) + \eta(t)\,\dot{\varepsilon}(t) \tag{4.80}$$

and for hypoelastic spring by

$$\dot{\sigma}(t) = [E(t) + \dot{\eta}(t)]\dot{\varepsilon}(t) + \eta(t)\ddot{\varepsilon}(t) \tag{4.81}$$

Equations (4.80) and (4.81) are not equivalent.

Let us now consider the standard solid. Previously for non-aging materials, we solved the model for the Kelvin element in series with a spring. The same differential equation would have been obtained for a Maxwell element in parallel with a spring. For aging materials, the number of combinations for the standard solid greatly increases, as indicated in Table 4.2.

with the notation:

E = elastic spring
H = hypoelastic spring
K_e , K_h = Kelvin element with elastic and hypoelastic spring, respectively
M = Maxwell element (Note Eq. 4.76 satisfies both springs)
−, // = series and parallel configurations, respectively.

Table 4.2

(a) $E - K_e$	(b) $E - K_h$
(c) $H - K_h$	(d) $H - K_e$
(e) $M \,//\, E$	(f) $M \,//\, H$

As an example, let us solve case (a) which was previously analyzed for a non-aging material. In this model we have an elastic spring with an elastic modulus $E_1(t)$ in series with a Kelvin model with an elastic modulus $E_2(t)$ and a dashpot of viscosity $\eta(t)$. Let ε_1 and ε_2 denote the strains of the spring and of the Kelvin element, respectively. Therefore,

$$\varepsilon(t) = \varepsilon_1(t) + \varepsilon_2(t) \qquad \varepsilon_1(t) = \sigma(t)/E_1(t) \tag{4.82}$$

$$E_2(t)\varepsilon_2(t) + \eta(t)\dot{\varepsilon}_2(t) = \sigma(t) \tag{4.83}$$

Eliminating ε_1 and, ε_2 we obtain:

$$E_2(t)\varepsilon(t) + \eta(t)\dot{\varepsilon}(t) = \left(1 + \frac{E_2(t)}{E_1(t)} + \eta\frac{d}{dt}\frac{1}{E_1(t)}\right)\sigma(t) + \frac{\eta(t)}{E_1(t)}\dot{\sigma}(t) \tag{4.84}$$

These aging models can be generalized to obtain the following differential constitutive equation:

$$\frac{d^n}{dt^n} + p_1(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + p_n(t)$$

$$\varepsilon(t) = \left(q_0(t)\frac{d^n}{dt^n} + p_1(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + q_n(t)\right)\sigma(t) \tag{4.85}$$

It should be mentioned that models having two or more Maxwell elements in parallel or Kelvin elements in series will not, in general, lead to a differential equation, but rather to an integro-differential equation.

4.4 Superposition Principle and Integral Representation

In the lifetime of a concrete structure it is unlikely that the load will be kept constant as in a creep test nor will the strain be kept constant, as in a relaxation test.

In order to estimate the strain at a given time from a known stress history further assumptions are necessary. McHenry (1943) made a significant contribution by postulating the following *Principle of Superposition*: “The strains produced in concrete at any time t by a stress increment at any time t_0 are independent of the effects of any stress applied either earlier or later than t_0. The stresses that approach the ultimate strength are excluded.”

Note: Experimental results indicate that the principle of superposition works well for sealed concrete specimens, that are for basic creep. When creep is associated with drying shrinkage other methods should be used.

The principle of superposition may also be formulated as follows “the effect of sum of causes is equal to sum of effects of each of these causes.” Consider $\varepsilon_1(t)$ and $\varepsilon_2(t)$, the strains resulting from the stress history $\sigma_1(t)$ and $\sigma_2(t)$, respectively. For a linear viscoelastic material we simply add the two stress histories

$$\sigma(\tau) = \sigma_1(\tau) + \sigma_2(\tau) \quad (4.86)$$

Using the principal of superposition, the following strain history is obtained:

$$\varepsilon(\tau) = \varepsilon_1(\tau) + \varepsilon_2(\tau) \quad (4.87)$$

Next, by using the principal of superposition and a known creep function, we can determine at any time the strain for a given stress history. For a creep test we may write the strain $\varepsilon(t)$ as a function of the stress σ_0, time t, and age of loading τ,

$$\varepsilon(t) = \Phi(\sigma_0, t, \tau) \quad (4.88)$$

In the linear range Eq. (4.88) may be written as

$$\varepsilon(t) = \sigma_0\, \Phi(t, \tau) \quad (4.89)$$

where $\Phi(t,\tau)$ is the specific creep function.

Figure (4.9) shows an arbitrary stress changing with time. Breaking the stress history up into small intervals, we have

$$\sigma(\tau) \cong \sum_{i=0}^{n} \Delta\sigma(\tau_i), \qquad \tau_n = t \quad (4.90)$$

Using Eq. (4.89), the strain history is given by

$$\varepsilon(\tau) \cong \sum_{i=0}^{n} \Delta\sigma(\tau_i)\Phi(t, \tau) \quad (4.91)$$

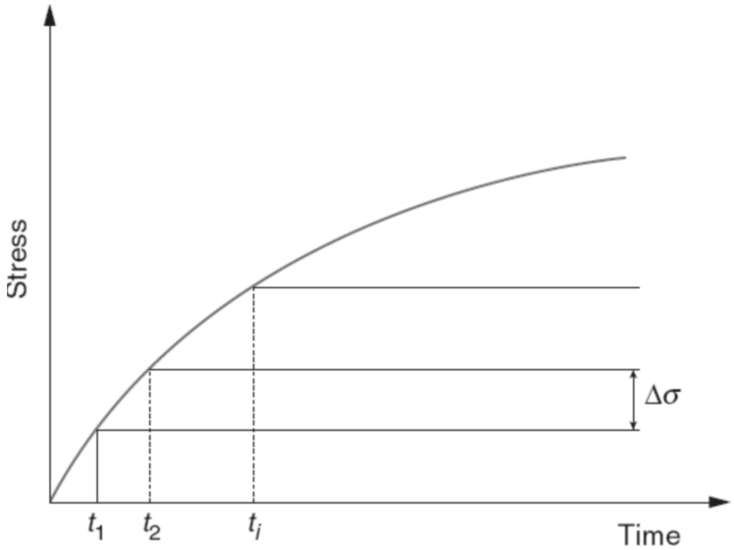


Figure (4.9): Incremental application of load over time.

and in the limit

$$\varepsilon(t) = \int_{\tau_0}^{t} \Phi(t,\tau)\, d\sigma(\tau) \qquad (4.92)$$

Equation (4.92) is often referred to as the hereditary or Volterra integral. It shows that at time *t* the strain *ε(t)* not only depends on the stress *σ(t)* but rather on the whole stress history. Integrating Equation (4.92) by parts we obtain

$$\varepsilon(t) = \frac{\sigma(t)}{E(t)} - \int_{\tau_0}^{t} \sigma(\tau) \frac{\partial \Phi(t,\tau)}{\partial \tau}\, d\tau \qquad (4.93)$$

where $E(t) = 1/\Phi(t,t)$

Our next objective is to compute the stress for a given strain history and relaxation function $E(t,\tau)$. Equations analogous to Eqs. (4.91) and (4.92) can be formulated.

$$\sigma(t) = \int_{\tau_0}^{t} E(t,\tau)\, \dot{\varepsilon}(\tau) d\tau \qquad (4.94)$$

$$\sigma(t) = E(t)\varepsilon(t) - \int_{\tau_0}^{t} \varepsilon(\tau) \frac{\partial E(t,\tau)}{\partial \tau}\, d\tau \qquad (4.95)$$

where $E(t) = E(t,t)$.

4.5 Mathematical Expressions for Creep

As we mentioned before, creep tests are time-consuming and special care needs to be taken to select a creep function that best fits the experimental results. In addition, the relatively short (time-wise) creep experiments, the selected creep function also must predict the long-term deformation. Previously, the curve fitting was done manually; researchers had to use intuition and experience to select simple and well-behaved functions. Today, because curve-fitting can be performed on almost any personal computer, the number and degree of sophistication of the functions has increased significantly. Before presenting some functions for creep of concrete commonly used in structural analysis, we will make the following general statements regarding the specific creep function $\Phi(t,\tau)$. Consider it as a guideline in case you feel the need to introduce a new creep function!

1. For a given age of loading τ, the creep function is a monotonic increasing function of time t;

$$\frac{\partial \Phi(t,\tau)}{\partial t} \geq 0 \tag{4.96}$$

2. However, the rate of creep increment is always negative;

$$\frac{\partial^2 \Phi(t,\tau)}{\partial t^2} \leq 0 \tag{4.97}$$

3. The aging of concrete causes a decrease in creep as the age of loading τ increases. For a given value of load duration (t – τ) due to aging of concrete;

$$\left(\frac{\partial \Phi(t,\tau)}{\partial t\tau}\right)_{(t-\tau)} \leq 0 \tag{4.98}$$

4. Creep has an asymptotic value

$$\text{Lim}_{t\to\infty} \Phi(t,\tau) \leq M \tag{4.99}$$

In many structural models, the function $\Phi(t, \tau)$ is separated into instantaneous and delayed components.

$$\Phi(t,\tau) = \frac{1}{E(\tau)} + C(t,\tau) \tag{4.100}$$

if we take aging of the concrete into account, the specific creep function $C(t, \tau)$ is further separated into:

$$C(t,\tau) = F(\tau) + f(t-\tau) \tag{4.101}$$

By writing $C(t, \tau)$ in this fashion, we indicate that at a given time concrete should recall not only the actions to which it was subjected since time t, given by the function $f(t-\tau)$, but also its own material state at time t, given by the function $F(\tau)$. Therefore, function $F(\tau)$ characterizes the aging of concrete. The following expressions for $F(\tau)$ and $f(t-\tau)$ have been traditionally used for fitting short term experimental data, with the objective of predicting the long-term deformation.

Expressions for $f(t - \tau)$

1. ***Logarithmic expression*:** The U.S. Bureau of Reclamation (1956) proposed using the following logarithmic expression for its projects dealing with mass concrete. When the stress-strength ratio does not exceed 0.40 the following equation is used:

$$f(t-\tau) = a + b\log[1 + (t-\tau)] \qquad (4.102)$$

Constants a and b are easily obtained when the creep data are plotted semi-logarithmically. The equation was originally developed for modelling basic creep of large dams, and the duration of load (t – τ) is measured in days. The expression is unbounded and usually overestimates the later creep.

2. *Power expression*: The general expression is given by

$$f(t-\tau) = a(t-\tau)^m \qquad (4.103)$$

Constants a and m can be easily obtained on a log-log plot, where the power expression gives a straight line. The expression captures the early creep well but overestimates the later creep with unbounded results.

3. *Hyperbolic expression*: Ross(1937) proposed the following hyperbolic expression:

$$f(t-\tau) = \frac{(t-\tau)}{a+b(t-\tau)} \qquad (4.104)$$

This expression provides a limiting value for creep, $1/b$. It usually underestimates early creep but provides good agreement for late creep. ACI code uses this formulation for creep evolution.

4. ***Exponential expression*:** The exponential expression provides a limiting value for creep. In its simplest formulation it is given by

$$f(t-\tau) = a\left(1 - e^{-b(t-\tau)}\right) \qquad (4.105)$$

It does not provide a good fit for experimental values. For numerical analysis more terms are usually incorporated.

Expressions for $F(\tau)$

$F(\tau)$ takes into account the aging of concrete, therefore it should be monotonically decreasing. While expressions for $f(t - \tau)$ have been developed during the last 70 years, expressions for $F(\tau)$ are much more recent. Among the expressions, we cite:

1. Power law:

$$F(\tau) = a + b\tau^{-c} \qquad (4.106)$$

2. Exponential:

$$F(\tau) = a + b\tau^{-c\tau} \qquad (4.107)$$

4.6 Methods for Predicting Creep

When experimental data are not available, the designer relies on a relevant code, which usually represents the consensus among researchers and practitioners. This section presents the 90 CEB-FIP model as well as the recommendations of ACI-209 and the Bazant-Panula model.

The creep function $\Phi(t, t_0)$ that represents the strain at time t for a constant unit stress acting from time t_0 ($t_0 = \tau$ = the age of loading) is given by

$$\Phi(t, t_0) = \frac{\varepsilon(t, t_0)}{\sigma_0} = \frac{E}{E_c(t_0)} + C(t, t_0) \qquad (4.108)$$

In the prediction models two types of creep coefficient exists:

1. The creep coefficient representing the ratio between creep strain at time t and initial strain at time t_0. This definition is used in the ACI and Bazant-Panula models

$$\Phi_0(t, t_0) = \frac{\varepsilon_c(t, t_0)}{\sigma_0 / E_c(t_0)} \qquad (4.109)$$

Therefore Eq. (4.106) may be written as

$$\Phi(t, t_0) = \frac{1}{E_c(t_0)}\left[1 + \Phi(t, t_0)\right] \qquad (4.110)$$

2. The creep coefficient representing the ratio between the creep strain at time t and the initial strain for a stress applied at 28 days.

$$\Phi_{28}(t, t_0) = \frac{\varepsilon_c(t, t_0)}{\sigma_0 / E_{c28}} \qquad (4.111)$$

Therefore Equation (4.108) may be written as

$$\Phi(t, t_0) = \frac{1}{E_c(t_0)} + \frac{\Phi_{28}(t, t_0)}{E_{c28}} \qquad (4.112)$$

CEB 1990: This method estimates creep and shrinkage for structural concretes in the range of 12 to 80 MPa in the linear domain, that is, for compressive stresses $\sigma_c(t_0)$ not exceeding 0.4 $f_{cm}(t_0)$ at the age of loading t_0. Here the total strain at time t, $\varepsilon_c(t)$ may be subdivided into

$$\varepsilon(t) = \varepsilon_{ci}(t) + \varepsilon_{cc}(t) + \varepsilon_{cs}(t) + \varepsilon_{cT}(t) = \varepsilon_{c\sigma}(t) + \varepsilon_{cn}(t) \quad (4.113)$$

where $\varepsilon_{c\sigma}(t) = \varepsilon_{ci}(t) + \varepsilon_{cc}(t)$

$\varepsilon_{cn}(t) = \varepsilon_{cs}(t) + \varepsilon_{cT}(t)$

$\varepsilon_{ci}(t_0)$ = initial strain at loading

$\varepsilon_{cc}(t)$ = creep strain

$\varepsilon_{cs}(t)$ = shrinkage strain

$\varepsilon_{cT}(t)$ = thermal strain

$\varepsilon_{cs}(t)$ = stress dependent strain

$\varepsilon_{cn}(t)$ = stress independent strain

The creep strain $\varepsilon_{cc}(t, t_0)$ is given by

$$\varepsilon_{cc}(t, t_0) = \frac{\sigma_c(t_0)}{E_c}\varphi(t, t_0) \quad (4.114)$$

where $\varphi(t, t_0)$ = creep coefficient

E_c = 28-day modulus of elasticity

Table 4.3 indicates the parameters necessary to compute the creep coefficient

where t and t_0 = measured in days

t_1 = 1 Day

f_{cm} = 28-day compressive strength, in MPa

f_{cmo} = 10 MPa

RH = percent relative humidity

A_c = cross section of the member

u = perimeter of the member in contact with the atmosphere

Table 4.3

$\varphi(t, t_0) = \phi_0\ \beta_c\ (t - t_0)$	$\varphi_0 = \phi_{RH}\ \beta(f_{cm})\ \beta(t_o)$
$h_0 = \frac{2A_c}{u}$	$\phi_{RH} = 1 + \frac{1 - RH/100}{0.46\ (h_0/100)^{1/3}}$
$\beta(f_{cm}) = \frac{5.3}{\sqrt{f_{cm}/f_{cmo}}}$	$\beta(t_0) = \frac{1}{0.1 + (t_0/t_i)^{0.20}}$
$\beta_c(t - t_0) = \left[\frac{(t - t_0)/t_i}{\beta_H + (t - t_0)/t_i}\right]^{0.3}$	$\beta_H = 150\left[1 + \left(1.2\frac{RH}{100}\right)^{18}\right]\frac{h}{100} + 250 \leq 1500$

The development of creep with time β_c is hyperbolic, therefore giving an asymptotic value of strain as $t \rightarrow \infty$. The effect of type of cement may be considered by modifying the age of loading to, as

$$t_0 = t_{0,T}\left(\frac{9}{2+t_{0,T}^{1/2}}+1\right)^{\alpha} \geq 0.5 \text{ days} \tag{4.115}$$

$$t_{0,T} = \sum_{i=1}^{n} \Delta t_i \exp - \left(\frac{4000}{273+T(\Delta t_i)/T_0} - 13.65\right) \tag{4.116}$$

where α = -1.0 for slow hardening cements, 0.0 for normal or rapid hardening cements, 1.0 for rapid hardening, high-strength cement

$T(\Delta t_i)$ = temperature, in C, during the time period of Δt_i

Δt_i = number of days with temperature T

T_0 = 1°C

4.7 Methods for Predicting Shrinkage

The total shrinkage $\varepsilon_{cs}(t, t_s)$ can be computed from the equations shown in Table 4.4,

Table 4.4

$\varepsilon_{cs}(t,t_s) = \varepsilon_{cso}\,\beta_s(t-t_s)$	$\varepsilon_{cso} = \varepsilon_s(f_{cm})\beta_{\text{RH}}$
$\varepsilon_s(f_{cm}) = [160 + 10\,\beta_{sc}(9 - f_{cm}/f_{cmo})] \times 10^{-6}$	$\beta_s(t-t_s) = \sqrt{\dfrac{(t-t_s)/t_i}{350(h/h_0)^2 + (t-t_s)/t_i}}$

where t = age of concrete (days)

t_s = age of concrete (days) at the beginning of the shrinkage

t_i = 1 day

h_0 = 100 mm

f_{cm} = mean compressive strength of concrete at the age of 28 days [MPa]

f_{cmo} = 10 MPa

β_{sc} = coefficient (4 for slowly hardening cements, 5 for normal or rapid hardening cements, 8 for rapid hardening, high – strength cements)

$\beta_{\text{RH}} = -1.55\,[1-(\text{RH}/100)^3]$ for $40\% \leq \text{RH} \leq 99\%$

$\beta_{\text{RH}} = 0.25$ for $\text{RH} \geq 99\%$

ACI 209: The creep coefficient $\varphi(t, t_0)$ is defined as

$$\varphi = \frac{(t-t_0)^{0.6}}{10+(t-t_0)^{0.6}}\varphi(\infty, t_0) \qquad (4.117)$$

where (t,t_0) = time since application of load
$\varphi(\infty, t_0)$ = ultimate creep coefficient given by

$$\varphi(\infty, t_0) = 2.35k_1k_2k_3k_4k_5k_6 \qquad (4.118)$$

At loading ages greater than 7 days for moist cured concrete and greater than 1-3 days for steam cured concrete

$k_1 = 1.25t_0^{-0.118}$ for moist cured concrete

$k_1 = 1.13t_0^{-0.095}$ for steam cured concrete

Coefficients k_4, k_5, and k_6 are all related to the concrete composition

$k_4 = 0.82 + 0.00264s$ s = slump of concrete (mm)

$k_5 = 0.88 + 0.0024f$ f = ratio of fine to total aggregate by weight in percent

$k_6 = 0.46 + 0.09a$ a = air content(percent) $\geq$1.0.

k_2, the humidity coefficient is given by

$k_2 = 1.27 - 0.006\,RH$ (RH > 40%)

where RH is the relative humidity in percent.

The member thickness coefficient $k3$ can be computed by two methods:

1. *Average-thickness method* for average thickness less than 150 mm:

$k_3 = 1.14 - 0.023h$ for $(t - t_0) < 1$ year

$k_3 = 1.10 - 0.017h$ for $(t - t_0) > 1$ year

where h is the average thickness in mm.

2. Volume-surface ratio method:

$$k_3 = \frac{2}{3}[1 + 1.13\exp(-0.0213V/S)]$$

where V/S is the volume-surface ratio (mm).

Bazant-Panula method: This model separates total creep into basic and drying creep. The basic creep function is given by

$$\Phi_b(t,t_0) = \frac{1}{E_0} + C_0(t,t_0) = \frac{1}{E_0} + \frac{\varphi_1}{E_0}(t_0^{-m} + \alpha)(t - t_0)^n \qquad (4.119)$$

E_0, asymptotic modulus, is a material parameter. It is not an actual elastic modulus for any load duration.

The five materials parameters of Eq. (4.119) are given by the following relations:

$$\frac{1}{E_0} = 0.09 + \frac{0.465}{f_{cm28}} \qquad \alpha = 0.05$$

$$\varphi_1 = 0.3 + 15 f_{cm28}^{-1.2} \qquad m = 0.28 + \frac{1}{f_{cm28}^2}$$

$$n = 0.115 + 0.00013 f_{cm28}^{3/4}$$

where f_{cm28} is in ksi and $1/E_0$ is in 10^{-6} per psi.

The only input is the 28 day mean cylinder strength. In the refined formulation, the material parameters are corrected both to the strength f_{cm28} and to several composition parameters.

Equation (4.119) also computes the static and dynamic modulus. The load duration ($t - t_0$) for the static modulus is approximately 0.1 days, and for the dynamic modulus, 10^{-7} days.

The total creep function is given by

$$\Phi(t,t_0) = \Phi_0(t,t_0) + \frac{\varphi_{ds}(t,t_0,t_s)}{E_0} \qquad (4.120)$$

The drying creep coefficient $\varphi(t, t_0, t_s)$ is given by a series of equations relating concrete composition, thickness of the member, and environmental conditions.