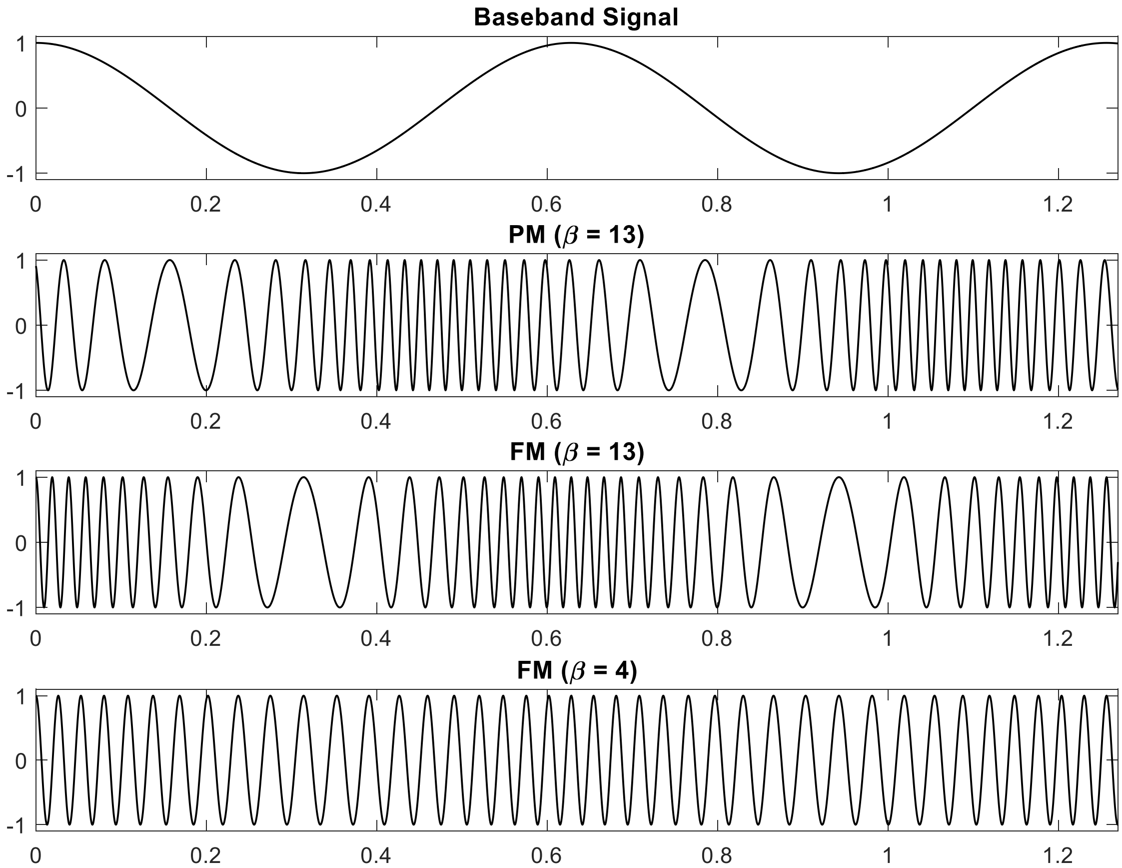
3.6 FREQUENCY MODULATION

We saw in AM that the information contents of $m(t)$ is transmitted through changing the amplitude of the carrier in proportion to the amplitude of $m(t)$. In the angle modulation, the information content is transmitted through changing the frequency of the carrier instead. The instantaneous frequency of the carrier is determined by the value of the baseband signal.

In this modulation type, the carrier frequency varies in a proportion to the amplitude of the message signal $m(t)$.



3.6.1 Definitions

Let us consider the carrier function as: $y(t) = A_c \cos\theta(t)$, where $\theta(t)$ is the angle.

Since the instantaneous frequency $\omega_i(t)$ is the slop of $\theta(t)$, or $\omega_i(t) = \frac{d\theta(t)}{dt}$, then

$$\theta(t) = \int_0^t \omega_i(\varepsilon) d\varepsilon$$

In the case of the Phase Modulation (PM), the angle varies linearly with $m(t)$ as:

$\theta(t) = \omega_c t + k_P m(t)$

where: k_P = modulation sensitivity = constant, and ω_c = the center frequency at $m(t) = 0$.

The resulting PM wave is:

$$y_{PM}(t) = A_c \cos[\omega_c t + k_P m(t)]$$

The instantaneous frequency in this case is:

$$\omega_i(t) = \frac{d\theta(t)}{dt} = \omega_c + k_P m'(t)$$

In words: the instantaneous frequency varies linearly with the derivative of the modulating signal.

In the case of the Frequency Modulation (FM), the instantaneous frequency $\omega_i(t)$ is varied linearly with the modulating signal, or:

$$\omega_i(t) = \omega_c + k_F m(t)$$

where: k_F = modulation sensitivity = constant. The angle of the carrier becomes:

$$\theta(t) = \int_0^t [\omega_c + k_F m(\varepsilon)]d\varepsilon + \theta_0$$

If we assume $\theta_0 = 0$, the FM signal becomes:

$$y_{FM}(t) = A_c \cos\left(\omega_c t + k_F \int_0^t m(\varepsilon)d\varepsilon\right)$$

In words: the instantaneous frequency varies linearly with the integral of the modulating signal.

FM and PM are closely related to each other; if we know the properties of the one, we can determine those of the other. For this reason, the material on angle modulation hereafter is devoted to FM.

To simplify the analysis of this equation, we consider $m(t) = A_m \cos(\omega_m t)$. So,

$$\omega_i(t) = \omega_c + k_F A_m \cos\omega_m t = \omega_c + \Delta_f \cos\omega_m t$$

Where $\Delta_f = A_m k_F$ =is the peak frequency deviation or the maximum frequency shift away from f_c (Note: $\Delta_f \ll f_C$). The instantaneous angle of the carrier is:

$$\theta(t) = \int_0^t \omega_i(\varepsilon)d\varepsilon = \omega_c t + \frac{\Delta_f}{\omega_m}\sin\omega_m t\,,\ \text{at}\ \theta_0 = 0$$

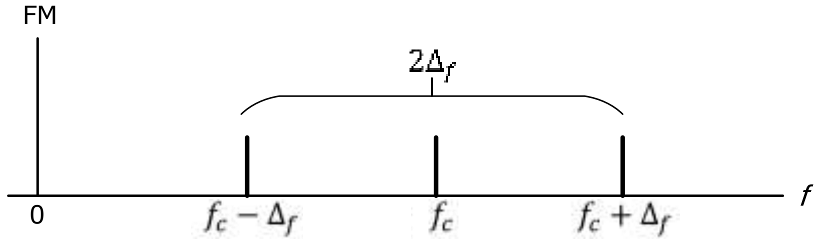
$$\Rightarrow \theta(t) = \omega_c t + \beta\sin\omega_m t$$

Where $\beta = \frac{\Delta_f}{f_m}$ = modulation index of the FM signal

Hence, the formula of the FM waveform will be:

$$y_{FM}(t) = A_c \cos[\omega_c t + \beta \sin(\omega_m t)]$$

Since the peak deviation is given by Δ_f, the carrier swing = $2\Delta_f$.



Note that:

- There is no 'over-modulation' situation with FM signal.
- β can take on any value from 0 to infinity. Its range is not limited as it is for AM.
- As β is increased, the signal becomes more resistant to interfering noise however occupies more bandwidth.

3.6.2 Spectrum of FM Signals

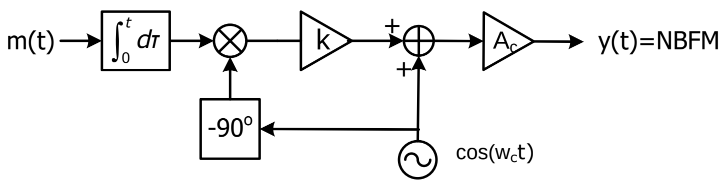
The spectrum of FM signals is rather 'messy' as it has may different sidebands spaced at multiples of f_m from the carrier. As a result, the bandwidth needed to accommodate an FM signal is greater than that for an AM signal having the same modulating frequency, (Except for NBFM).

According to the value of β, we have the following FM formats:

<u>NARROW BAND FM</u>

In the FM systems where β is small (< 0.2), several approximations lead to the following method of FM generation:

$$y(t) = A_c \left[\cos(\omega_c t) + k_F \sin(\omega_c t) \times \int_0^t m(\varepsilon)d\varepsilon\right]$$



We can easily notice in the Bessel chart that the spectrum and BW occupied by this NBFM waveform is $2f_m$ (just like the AM-DSB).

WIDE BAND FM

To determine the spectrum of FM signal with values of $\beta > 0.2$ (which is typical), we may simplify matters by using the complex representation of band-pass signals when $m(t) = A_m \cos \omega_m t$:

$$y(t) = A_c \cos[\omega_c t + \beta \sin \omega_m t]$$

$$= \mathrm{Re}[A_c \exp(j\omega_c t + j\beta \sin(\omega_m t))]$$

$$= \mathrm{Re}[\tilde{y}(t) \exp(j\omega_c t)]$$

where $\tilde{y}(t)$ is the complex envelope of the FM signal $y(t)$, defined by:

$$\tilde{y}(t) = \mathrm{Re}[A_c \exp(j\beta \sin(\omega_m t))]$$

Thus, unlike the original FM signal $y(t)$, the complex envelope $\tilde{y}(t)$ is a periodic function of time with a fundamental frequency equal to the modulation frequency f_m. We may therefore expand $\tilde{y}(t)$ in the form of a complex Fourier series as follows:

$$\tilde{y}(t) = \sum_{n=-\infty}^{\infty} C_n \exp(j\omega_m n t)$$

where the complex Fourier coefficient C_n is defined by

$$C_n = f_m \int_{-\pi/\omega_m}^{\pi/\omega_m} \tilde{y}(t) \exp(-j\omega_m n t)\, dt$$

$$= f_m A_c \int_{-\pi/\omega_m}^{\pi/\omega_m} \exp(j\beta \sin(\omega_m t) - j\omega_m n t)\, dt$$

Define a new variable $x = \omega_m t$,

$$C_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} \exp(j\beta \sin(x) - jnx)\, dx$$

Here the integral on the right-hand side, is recognized as the n^{th} order Bessel function of the first kind and argument β. This function is commonly denoted by the symbol $J_n(\beta)$, as shown by:

$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin(x) - nx)} dx$$

Accordingly, we may reduce C_n Equation to $C_n = A_c J_n(\beta)$.

We substitute to get the complex envelope of the FM signal in terms of the $J_n(\beta)$

$$\tilde{y}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \exp(j\omega_m nt)$$

And

$$y(t) = A_c \operatorname{Re}\left[\sum_{n=-\infty}^{\infty} J_n(\beta) \exp[j2\pi(f_c + nf_m)t]\right]$$

$$\therefore y(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[2\pi(f_c + nf_m)t] \qquad \text{Eq.1}$$

$$= A_c \begin{Bmatrix} J_0(\beta)\cos[2\pi f_c t] \\ +J_1(\beta)\cos[2\pi(f_c + f_m)t] \quad +J_1(\beta)\cos[2\pi(f_c - f_m)t] \\ +J_2(\beta)\cos[2\pi(f_c + 2f_m)t] + J_2(\beta)\cos[2\pi(f_c - 2f_m)t] \\ +J_3(\beta)\cos[2\pi(f_c + 3f_m)t] + J_3(\beta)\cos[2\pi(f_c - 3f_m)t] \\ \dots \end{Bmatrix}$$

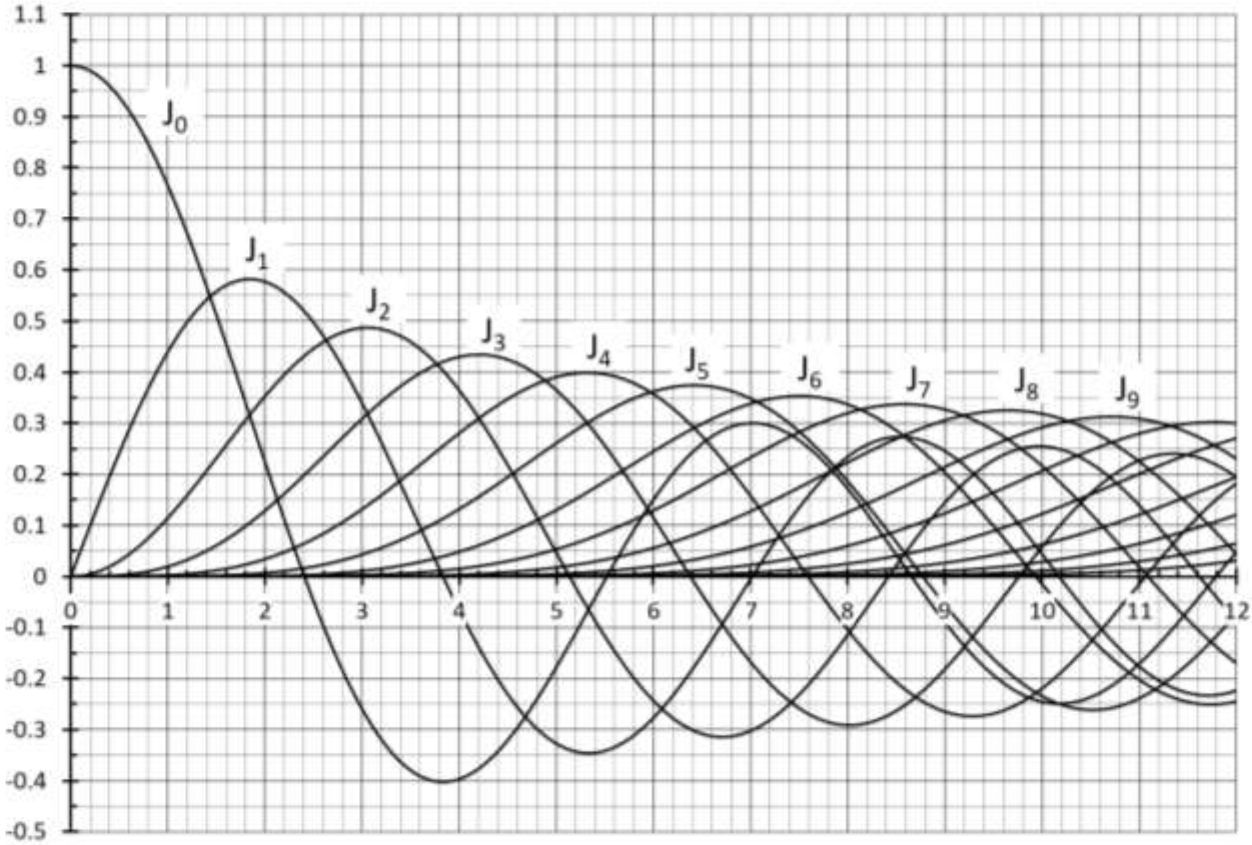
$$= A_c \begin{Bmatrix} J_0(\beta)\cos[2\pi f_c t] \\ +2J_1(\beta)\cos[2\pi(f_c + f_m)t] \\ +2J_2(\beta)\cos[2\pi(f_c + 2f_m)t] \\ +2J_3(\beta)\cos[2\pi(f_c + 3f_m)t] \\ \dots \end{Bmatrix}$$

The discrete spectrum of $y(t)$ is obtained by taking the Fourier transforms of both sides, yields:

$$Y(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)] \qquad \text{Eq.2}$$

Knowing that: $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$ Eq.3

In the figure below, we plot the Bessel function $J_n(\beta)$ versus the modulation index β for different positive integer values of n, (see the properties of Bessel function: Str. Page 291)



Thus, using Equations 2 & 3 and the curves of the above Figure, we may observe:

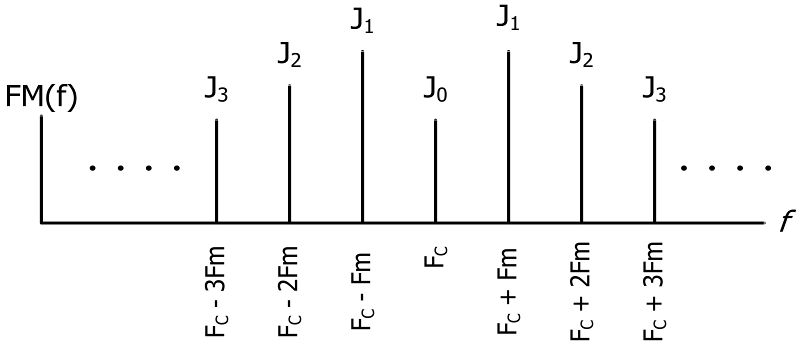
(1) The spectrum of an FM signal contains a carrier component and an infinite set of side frequencies located symmetrically on either side of the carrier at frequency separations of $f_m, 2f_m, 3f_m$ ...

(2) For the special case of $\beta < 0.2$, only the Bessel coefficients $J_0(\beta)$ and $J_1(\beta)$ have significant values, so that the FM signal is effectively composed of a carrier and a single pair of side frequencies at $f_c \pm f_m$. (NBFM)

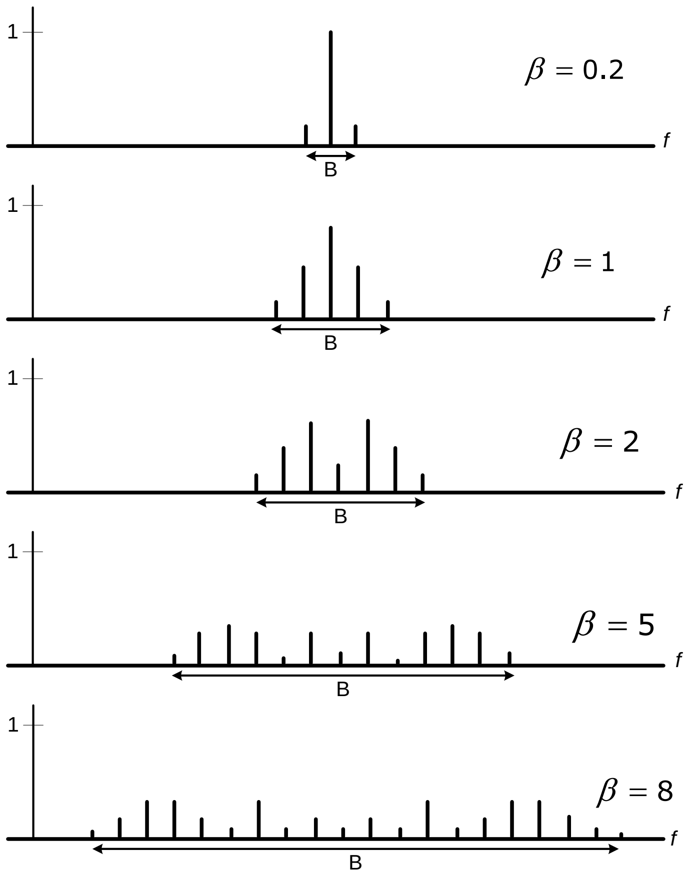
(3) The amplitude of the carrier component varies with β according to $J_0(\beta)$. That is, unlike an AM signal, the amplitude of the carrier component of an FM signal is dependent on β. As the envelope of an FM signal is constant, so that the average power of such a signal is:

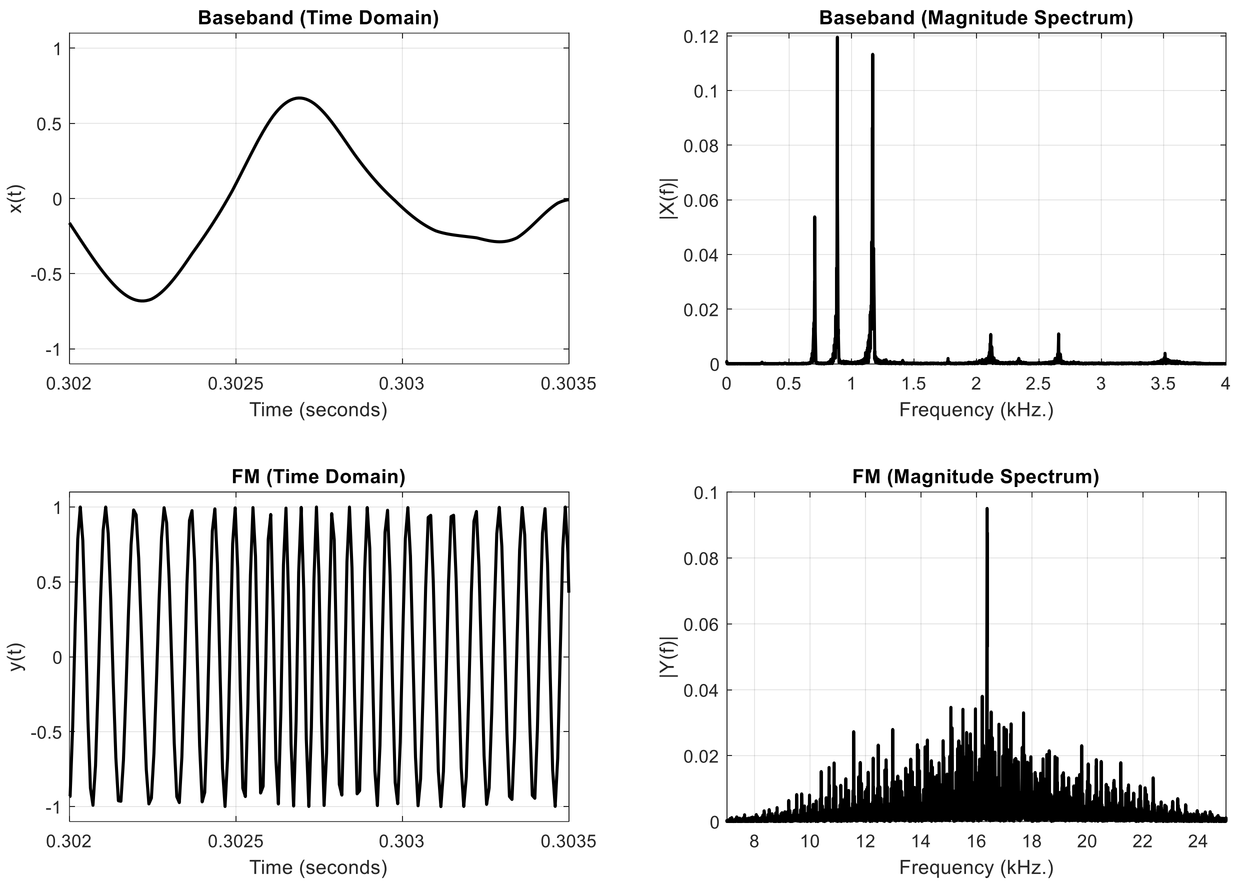
$$P = \overline{y^2(t)} = \frac{A_c^2}{2R} \qquad , \qquad P_{Total} = \frac{A_c^2}{2R} \sum_{n=-\infty}^{\infty} J_n^2(\beta)$$

To sketch the spectrum of an FM signal at a certain β, we must get the values of $J_n(\beta)$ from a plot or a table of Bessel function (see **Error! Reference source not found.**). So, the plot may look like the following depiction, which is plotted using Eq.2 (Note: we assume $A_c = 2\text{V}$)



It is evident that, the frequency modulation of $m(t) = \cos\omega_m t$ has infinite number of sidebands. However, the magnitude of the spectral components of the higher order sidebands becomes negligible




Baseband (Time Domain)
x(t)
Time (seconds)
Baseband (Magnitude Spectrum)
|X(f)|
Frequency (kHz.)
FM (Time Domain)
y(t)
Time (seconds)
FM (Magnitude Spectrum)
|Y(f)|
Frequency (kHz.)