1.10 CHANNEL CAPACITY

For any communication system, it is required to send data as fast as possible. But in the presence of noise and distortion along the channel, it is very hard to avoid errors at the reception. The Shannon–Hartley theorem of channel capacity states that: the maximum rate of information transmission $R_{\max}$ over a channel of the bandwidth B and the received signal to ratio SNR is given by:

$$R_{\max} = C = B\log_2\left(1+\frac{S}{N}\right) \quad \text{bps}$$

Where $C =$ channel capacity $=$ the maximum rate at which information can be transmitted across that channel without error; it is measured in bits per second (bps).

let R be the operating information rate.

- If $R \leq C$, it is possible to receive data with small probabilities of error, even with noise.
- If $R > C$, errors can not be avoided regardless of the coding technique used.

If we would like to increase $R_{\max}$ in the above equation, we can increase B and/or the SNR.

- Larger SNR implies working at, for example, higher Tx powers or shorter distances. In some cases, this is not that easy to achieve. But in general, as the SNR increased R would also increase without errors for a given B channel. However, you must note that: when $N \to 0$ then $SNR \to \infty$ and hence $R_{\max} \to \infty$ regardless of B (is it possible?).
- As for increasing B, it requires changing the medium or buying a license for extra bandwidth. In general, as B increased, it can follow faster changes in the information signal, thereby increasing R. Nevertheless, when $B \to \infty$, C does not approach ∞. The noise is assumed to be white: the wider the bandwidth, the more the noise admitted to the system. This means, as B increases, SNR decreases at the same S.

Theoretical Capacity

Suppose that the noise is white with PSD $\eta/2$ (W/Hz), and assume the received signal power is fixed at a value S (W), the channel capacity would be:

$R_{\max} = C = B\log_2\left(1+\frac{S}{\eta B}\right)$, thus, when $B \to \infty$ we get:

$$C = \lim_{B\to\infty}\left\{B\log_2\left(1+\frac{S}{\eta B}\right)\right\} = \lim_{B\to\infty}\left\{\frac{S}{\eta}\log_2\left(1+\frac{S}{\eta B}\right)^{\frac{\eta B}{S}}\right\} \approx \frac{S}{\eta}\log_2 e$$

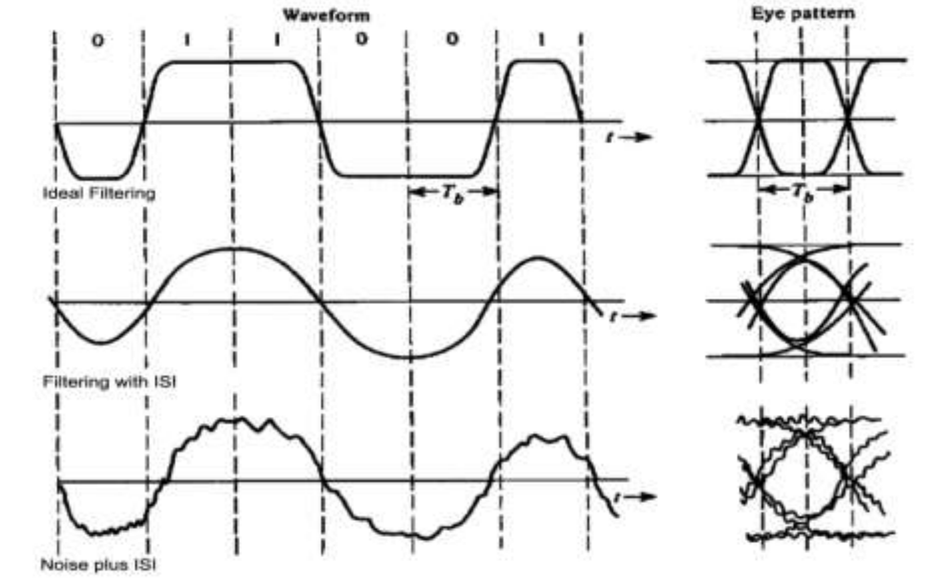
$$\therefore C = 1.44\frac{S}{\eta}$$

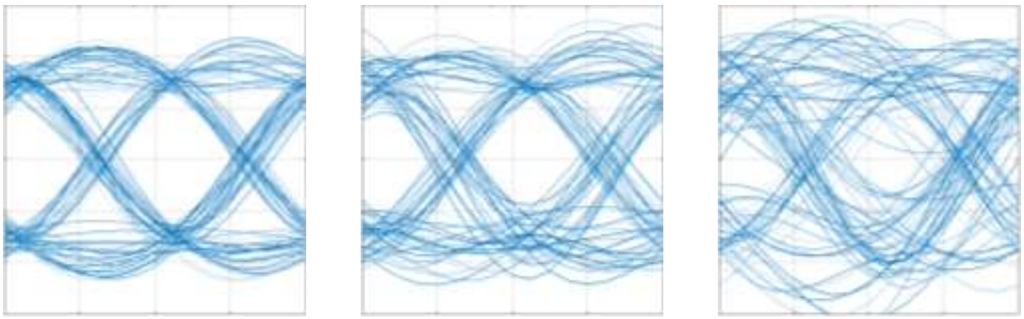
This gives the maximum possible channel capacity as a function of the received signal power and the noise PSD. In actual systems design, the channel capacity might be compared to this value to decide whether a further increase in B is worthwhile.

1.11 INTER-SYMBOL INTERFERENCE (ISI)

Previously, we discussed the crosstalk problem (distortion caused by time dispersion). This results in spreading of time signals and overlapping among adjacent bit waveforms. This overlap is also known as ISI. It is caused not only by channel distortion but also by multi-path effects.

To illustrate the ISI problem, we should first introduce the *eye pattern*.





SNR=20dB SNR=15dB SNR=10dB

To reduce the ISI, we must:

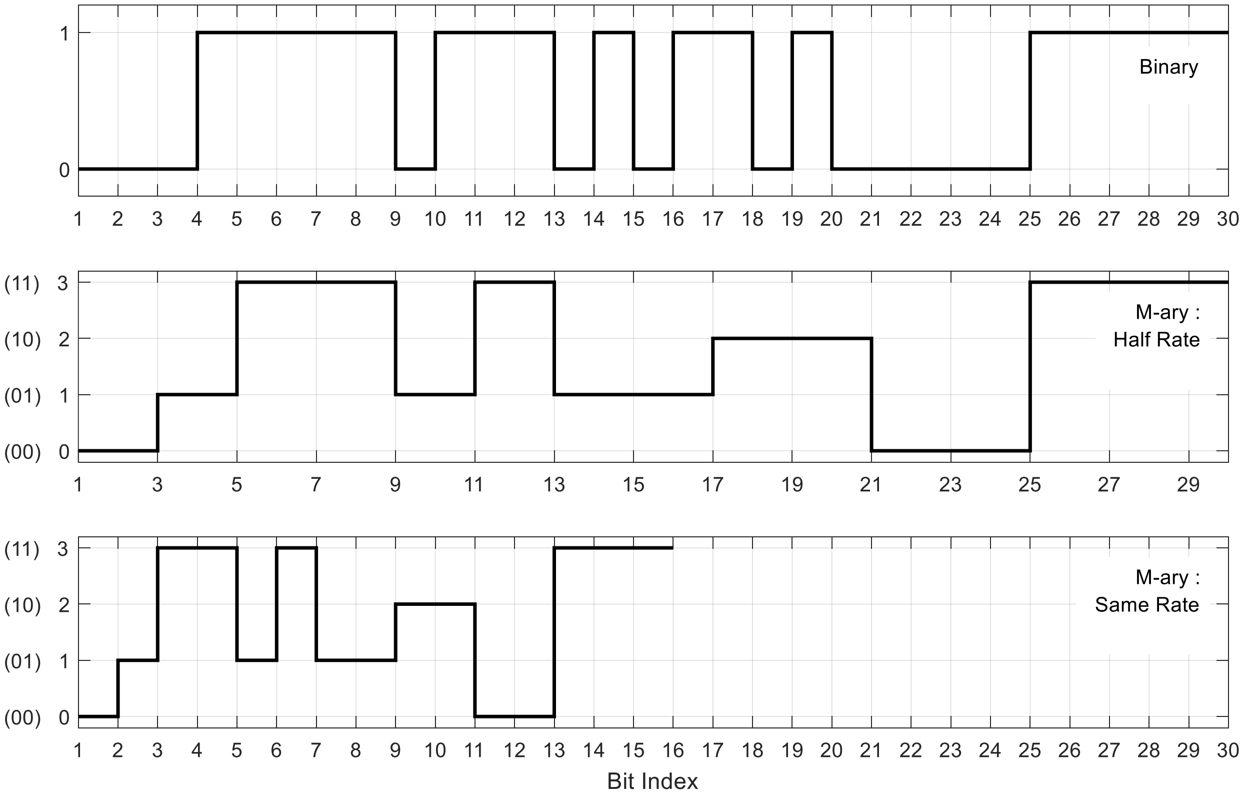
(a) Consider the Shannon criterion for the maximum rate.

(b) Frequency limitation of the transmitted signal to fit B.

(c) Reduce the effects of the multi-path problem *(How?)*.

1.12 MULTI-LEVEL BASEBAND SIGNALING (M-ARY)

Binary shift keying means sending a single bit over the symbol interval T_b (or at the bit rate $1/T_b$ bits/sec). To increase the data transfer rate, it is possible to combine several bits in one symbol. In this technique, we send one symbol per m data bits. So, we the transmitted symbols range between M levels (know that $M = 2^m$). The plots below illustrate the M-ary transmission.



In this example, we set 2 bits per symbol (hence $2^2 = 4$ levels). The information rate is still unchanged, but the symbol rate is halved (as in the second plot). On other words, we can send twice the information rate at the same symbol rate (as in the third plot). Practically, it is difficult to consider the multi-level baseband signaling through noisy and distorting channels. the receiver now must distinguish the incoming symbols according to their levels. Thus the probability of error increases as M becomes larger.

Advantage:

- A higher information transfer rate is possible for a given symbol rate and a channel bandwidth.

Disadvantages:

- M-ary baseband signaling results in reduced noise/interference immunity when it is compared to the binary signaling.

- It involves more complex symbol recovery processing in the receiver.
- It imposes a greater requirement for linearity and/or reduced distortion in the Tx/Rx hardware and in the channel.

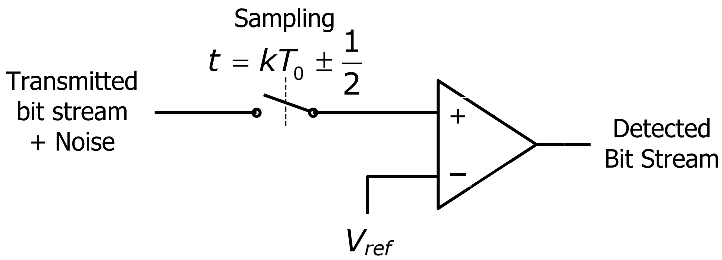
1.13 PROBABILITY OF ERROR AT RECEPTION

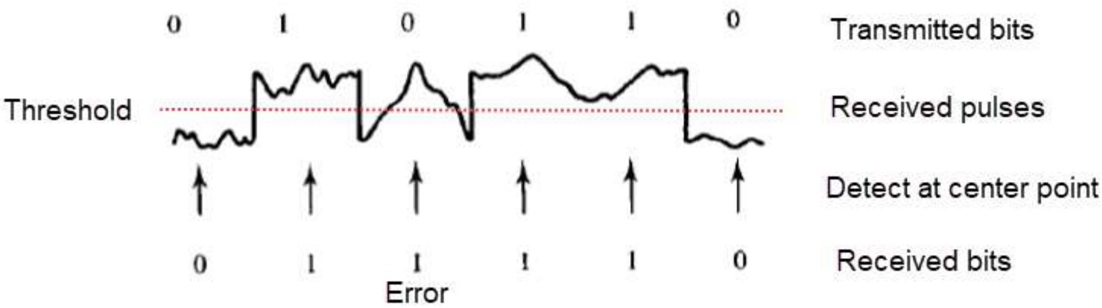
The detection of digital signals involves two processes:

(1) Reduction of each received voltage pulse (i.e. symbol) to a single numerical value, (just like quantization).

(2) Comparison of this value with a reference voltage to determine which symbol was transmitted.

The unipolar binary symbols (0 and 1) is represented by two voltage levels (e.g. 0V and 3V). Intuition tells us that a sensible strategy would be to set the reference, V_{ref}, mid-way between the two voltage levels (i.e. at 1.5V).

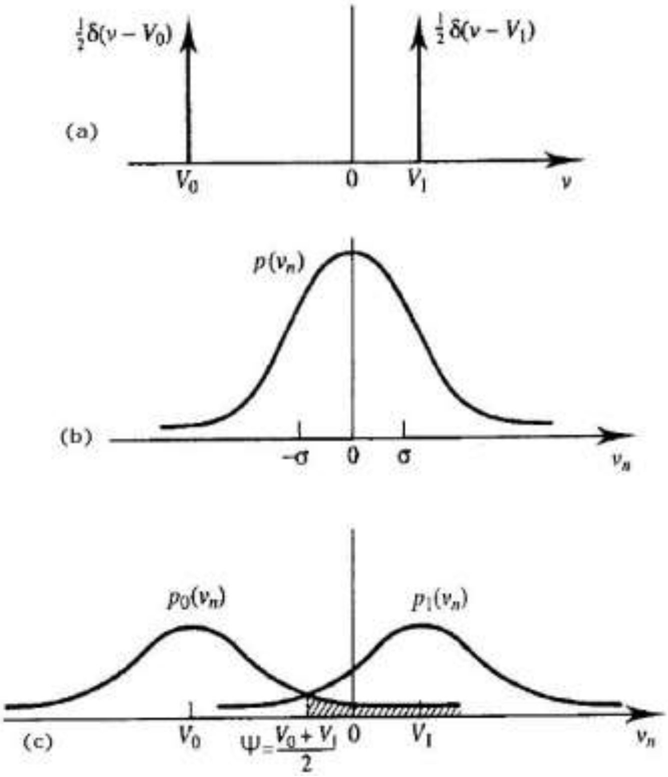




In general, for equiprobable symbols the decision level is set to $\psi = \frac{V_0+V_1}{2}$. For unequal transmission of the symbols, we need to determine the optimum threshold using more advanced mathematics.

Since the noise with a Gaussian PDF♣ is common and analytically tractable, the bit error rate (BER) of a communication system is often modeled assuming Gaussian noise alone.

♣ See Str. Section 8.6.4.



(a) The PDF of a binary information signal which can employ voltage levels V_0 and V_1 only. (b) The PDF of a zero mean Gaussian noise process, $v_n(t)$, with RMS value σ Volts. (c) The PDF of the sum of the signal and the noise.

Let the probability of sending symbol 0 (the voltage level V_0) is:

$$p_0(v_n) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(v_n-V_0)^2}{2\sigma^2}}$$

And the PDF of sending symbol 1 (the voltage level V_1) is:

$$p_1(v_n) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(v_n-V_1)^2}{2\sigma^2}}$$

Now, if the symbol 0 is transmitted, let P_{e1} be the probability of the received signal plus noise that is above the threshold at the decision instant (i.e. seen as voltage level V_1), [shaded area under the curve $p_0(v_n)$ in Figure (c)]. The equation of the probability of error is:

$$P_{e1} = \int_{\psi}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(v_n-V_0)^2}{2\sigma^2}} dv_n$$

Also, if the digital symbol 1 is transmitted, let P_{e0} be the probability that the received signal plus noise is below the threshold at the decision instant (i.e. seen as voltage level V_0). So:

$$P_{e0} = \int_{-\infty}^{\psi} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(v_n-V_1)^2}{2\sigma^2}} dv_n$$

It is clear from the symmetry of this problem that P_{e0} is identical to $P_{e1},$ and for equiprobable symbols $p_0(v_n) = p_1(v_n) = \frac{1}{2}$, the net probability of error P_E will be:

$$P_E = p_0 P_{e0} + p_1 P_{e1} = \frac{1}{2}(P_{e0} + P_{e1}) = \int_{\psi}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(v_n - V_0)^2}{2\sigma^2}} dv_n$$

using $x = \dfrac{v_n - V_0}{\sigma} \quad \Rightarrow P_E = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\frac{\psi}{\sigma}}^{\infty} e^{-x^2} dx$

This integral cannot be evaluated analytically but it can be recast as a complementary error function, which is defined by:

$$\text{Erfc}(z) \approx \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-x^2} dx$$

Thus $\quad P_E = \text{Erfc}\left(\dfrac{\psi}{\sigma}\right)$

The advantage of using Erfc (also called Q-function) in the expression for P_E is that this function has been extensively tabulated. Sometimes is tabulated as erf(z) or erfc(z)♣.

For **unipolar** binary $\left(V_0 = 0, \; V_1 = A \; \rightarrow \psi = \frac{A}{2}\right)$, the average signal power is: $S = \frac{(0^2 + A^2)}{2}$

And the average noise power is $N = \sigma^2$, so that: $\quad P_E = \text{Erfc}\left\{\sqrt{\dfrac{S}{2N}}\right\}$

For **polar** binary $\left(V_0 = \frac{-A}{2}, \; V_1 = \frac{A}{2} \rightarrow \psi = 0\right)$, the average signal power is: $S = \frac{\left[\left(\frac{-A}{2}\right)^2 + \left(\frac{A}{2}\right)^2\right]}{2}$

And the average noise power is $N = \sigma^2$, we get: $\quad P_E = \text{Erfc}\left\{\sqrt{\dfrac{S}{N}}\right\}$

Therefore, the average transmitted power for the unipolar signal must be twice that of the polar binary signal to achieve the same probability of error. For equiprobable signals, the polar binary signaling also has an advantage in the optimum decision threshold. It is simply set at the

♣ See Q-Function and Error Function Complementary in this document and Str. Appendix G.

zero Volt, whereas the receiver for the ON-OFF binary signaling, the threshold must be adjusted to half the amplitude of the received signal.

So, which is the better: the polar or the unipolar signaling in terms of the probability of error? why?