2.4 Probability of an Event

Perhaps it was humankind's unquenchable thirst for gambling that led to the early development of probability theory. In an effort to increase their winnings, gamblers called upon mathematicians to provide optimum strategies for various games of chance. Some of the mathematicians providing these strategies were Pascal, Leibniz, Fermat, and James Bernoulli. As a result of this development of probability theory, statistical inference, with all its predictions and generalizations, has branched out far beyond games of chance to encompass many other fields associated with chance occurrences, such as politics, business, weather forecasting, and scientific research. For these predictions and generalizations to be reasonably accurate, an understanding of basic probability theory is essential.

What do we mean when we make the statement "John will probably win the tennis match," or "I have a fifty-fifty chance of getting an even number when a die is tossed," or "The university is not likely to win the football game tonight," or "Most of our graduating class will likely be married within 3 years"? In each case, we are expressing an outcome of which we are not certain, but owing to past information or from an understanding of the structure of the experiment, we have some degree of confidence in the validity of the statement.

Throughout the remainder of this chapter, we consider only those experiments for which the sample space contains a finite number of elements. The likelihood of the occurrence of an event resulting from such a statistical experiment is evaluated by means of a set of real numbers, called **weights** or **probabilities**, ranging from 0 to 1. To every point in the sample space we assign a probability such that the sum of all probabilities is 1. If we have reason to believe that a certain sample point is quite likely to occur when the experiment is conducted, the probability assigned should be close to 1. On the other hand, a probability closer to 0 is assigned to a sample point that is not likely to occur. In many experiments, such as tossing a coin or a die, all the sample points have the same chance of occurring and are assigned equal probabilities. For points outside the sample space, that is, for simple events that cannot possibly occur, we assign a probability of 0.

To find the probability of an event A, we sum all the probabilities assigned to the sample points in A. This sum is called the **probability** of A and is denoted by $P(A)$.

Definition 2.9: The **probability** of an event A is the sum of the weights of all sample points in A. Therefore,

$$0 \leq P(A) \leq 1, \quad P(\phi) = 0, \quad \text{and} \quad P(S) = 1.$$

Furthermore, if $A_1, A_2, A_3, \ldots$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots .$$

Example 2.24: A coin is tossed twice. What is the probability that at least 1 head occurs?

***Solution*:** The sample space for this experiment is

$$S = \{HH, HT, TH, TT\}.$$

If the coin is balanced, each of these outcomes is equally likely to occur. Therefore, we assign a probability of ω to each sample point. Then $4\omega = 1$, or $\omega = 1/4$. If A represents the event of at least 1 head occurring, then

$$A = \{HH, HT, TH\} \text{ and } P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

Example 2.25: A die is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.

***Solution*:** The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. We assign a probability of w to each odd number and a probability of $2w$ to each even number. Since the sum of the probabilities must be 1, we have $9w = 1$ or $w = 1/9$. Hence, probabilities of $1/9$ and $2/9$ are assigned to each odd and even number, respectively. Therefore,

$$E = \{1, 2, 3\} \text{ and } P(E) = \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}.$$

Example 2.26: In Example 2.25, let A be the event that an even number turns up and let B be the event that a number divisible by 3 occurs. Find $P(A \cup B)$ and $P(A \cap B)$.

***Solution*:** For the events $A = \{2, 4, 6\}$ and $B = \{3, 6\}$, we have

$$A \cup B = \{2, 3, 4, 6\} \text{ and } A \cap B = \{6\}.$$

By assigning a probability of $1/9$ to each odd number and $2/9$ to each even number, we have

$$P(A \cup B) = \frac{2}{9} + \frac{1}{9} + \frac{2}{9} + \frac{2}{9} = \frac{7}{9} \quad \text{and} \quad P(A \cap B) = \frac{2}{9}.$$

If the sample space for an experiment contains N elements, all of which are equally likely to occur, we assign a probability equal to $1/N$ to each of the N points. The probability of any event A containing n of these N sample points is then the ratio of the number of elements in A to the number of elements in S.

Rule 2.3: If an experiment can result in any one of N different equally likely outcomes, and if exactly n of these outcomes correspond to event A, then the probability of event A is

$$P(A) = \frac{n}{N}.$$

Example 2.27: A statistics class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students. If a person is randomly selected by the instructor to answer a question, find the probability that the student chosen is (a) an industrial engineering major and (b) a civil engineering or an electrical engineering major.

***Solution*:** Denote by I, M, E, and C the students majoring in industrial, mechanical, electrical, and civil engineering, respectively. The total number of students in the class is 53, all of whom are equally likely to be selected.

(a) Since 25 of the 53 students are majoring in industrial engineering, the probability of event I, selecting an industrial engineering major at random, is

$$P(I) = \frac{25}{53}.$$

(b) Since 18 of the 53 students are civil or electrical engineering majors, it follows that

$$P(C \cup E) = \frac{18}{53}.$$

Example 2.28: In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

***Solution*:** The number of ways of being dealt 2 aces from 4 cards is

$$\binom{4}{2} = \frac{4!}{2!\ 2!} = 6,$$

and the number of ways of being dealt 3 jacks from 4 cards is

$$\binom{4}{3} = \frac{4!}{3!\ 1!} = 4.$$

By the multiplication rule (Rule 2.1), there are $n = (6)(4) = 24$ hands with 2 aces and 3 jacks. The total number of 5-card poker hands, all of which are equally likely, is

$$N = \binom{52}{5} = \frac{52!}{5!\ 47!} = 2{,}598{,}960.$$

Therefore, the probability of getting 2 aces and 3 jacks in a 5-card poker hand is

$$P(C) = \frac{24}{2,598,960} = 0.9 \times 10^{-5}.$$

If the outcomes of an experiment are not equally likely to occur, the probabilities must be assigned on the basis of prior knowledge or experimental evidence. For example, if a coin is not balanced, we could estimate the probabilities of heads and tails by tossing the coin a large number of times and recording the outcomes. According to the **relative frequency** definition of probability, the true probabilities would be the fractions of heads and tails that occur in the long run. Another intuitive way of understanding probability is the **indifference** approach. For instance, if you have a die that you believe is balanced, then using this indifference approach, you determine that the probability that each of the six sides will show up after a throw is 1/6.

To find a numerical value that represents adequately the probability of winning at tennis, we must depend on our past performance at the game as well as that of the opponent and, to some extent, our belief in our ability to win. Similarly, to find the probability that a horse will win a race, we must arrive at a probability based on the previous records of all the horses entered in the race as well as the records of the jockeys riding the horses. Intuition would undoubtedly also play a part in determining the size of the bet that we might be willing to wager. The use of intuition, personal beliefs, and other indirect information in arriving at probabilities is referred to as the **subjective** definition of probability.

In most of the applications of probability in this book, the relative frequency interpretation of probability is the operative one. Its foundation is the statistical experiment rather than subjectivity, and it is best viewed as the **limiting relative frequency**. As a result, many applications of probability in science and engineering must be based on experiments that can be repeated. Less objective notions of probability are encountered when we assign probabilities based on prior information and opinions, as in "There is a good chance that the Giants will lose the Super

Bowl." When opinions and prior information differ from individual to individual, subjective probability becomes the relevant resource. In Bayesian statistics (see Chapter 18), a more subjective interpretation of probability will be used, based on an elicitation of prior probability information.

2.5 Additive Rules

Often it is easiest to calculate the probability of some event from known probabilities of other events. This may well be true if the event in question can be represented as the union of two other events or as the complement of some event. Several important laws that frequently simplify the computation of probabilities follow. The first, called the **additive rule**, applies to unions of events.

Theorem 2.7: If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

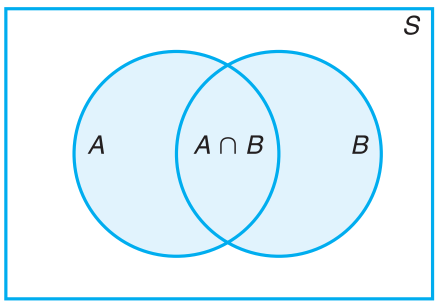

Figure 2.7: Additive rule of probability.

***Proof*:** Consider the Venn diagram in Figure 2.7. The $P(A \cup B)$ is the sum of the probabilities of the sample points in $A \cup B$. Now $P(A) + P(B)$ is the sum of all the probabilities in A plus the sum of all the probabilities in B. Therefore, we have added the probabilities in $(A \cap B)$ twice. Since these probabilities add up to $P(A \cap B)$, we must subtract this probability once to obtain the sum of the probabilities in $A \cup B$.

Corollary 2.1: If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Corollary 2.1 is an immediate result of Theorem 2.7, since if A and B are mutually exclusive, $A \cap B = 0$ and then $P(A \cap B) = P(\phi) = 0$. In general, we can write Corollary 2.2.

Corollary 2.2: If $A_1,\ A_2,\dots,A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

A collection of events $\{A_1, A_2, \dots, A_n\}$ of a sample space S is called a **partition** of S if $A_1, A_2, \dots, A_n$ are mutually exclusive and $A_1 \cup A_2 \cup \cdots \cup A_n = S$. Thus, we have

Corollary 2.3: If $A_1,\ A_2,\dots,A_n$ is a partition of sample space S, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) = P(S) = 1.$$

As one might expect, Theorem 2.7 extends in an analogous fashion.

Theorem 2.8: For three events A, B, and C,

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C). \end{aligned}$$

Example 2.29: John is going to graduate from an industrial engineering department in a university by the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company A is 0.8, and his probability of getting an offer from company B is 0.6. If he believes that the probability that he will get offers from both companies is 0.5, what is the probability that he will get at least one offer from these two companies?

***Solution*:** Using the additive rule, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.6 - 0.5 = 0.9.$$

Example 2.30: What is the probability of getting a total of 7 or 11 when a pair of fair dice is tossed?

***Solution*:** Let A be the event that 7 occurs and B the event that 11 comes up. Now, a total of 7 occurs for 6 of the 36 sample points, and a total of 11 occurs for only 2 of the sample points. Since all sample points are equally likely, we have $P(A) = 1/6$ and $P(B) = 1/18$. The events A and B are mutually exclusive, since a total of 7 and 11 cannot both occur on the same toss. Therefore,

$$P(A \cup B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}.$$

This result could also have been obtained by counting the total number of points for the event $A \cup B$, namely 8, and writing

$$P(A \cup B) = \frac{n}{N} = \frac{8}{36} = \frac{2}{9}.$$

Theorem 2.7 and its three corollaries should help the reader gain more insight into probability and its interpretation. Corollaries 2.1 and 2.2 suggest the very intuitive result dealing with the probability of occurrence of at least one of a number of events, no two of which can occur simultaneously. The probability that at least one occurs is the sum of the probabilities of occurrence of the individual events. The third corollary simply states that the highest value of a probability (unity) is assigned to the entire sample space S.

Example 2.31: If the probabilities are, respectively, 0.09, 0.15, 0.21, and 0.23 that a person purchasing a new automobile will choose the color green, white, red, or blue, what is the probability that a given buyer will purchase a new automobile that comes in one of those colors?

***Solution*:** Let G, W, R, and B be the events that a buyer selects, respectively, a green, white, red, or blue automobile. Since these four events are mutually exclusive, the probability is

$$\begin{aligned} P(G \cup W \cup R \cup B) &= P(G) + P(W) + P(R) + P(B) \\ &= 0.09 + 0.15 + 0.21 + 0.23 = 0.68. \end{aligned}$$

Often it is more difficult to calculate the probability that an event occurs than it is to calculate the probability that the event does not occur. Should this be the case for some event A, we simply find $P(A')$ first and then, using Theorem 2.7, find $P(A)$ by subtraction.

Theorem 2.9: If A and A' are complementary events, then

$$P(A) + P(A') = 1.$$

***Proof*:** Since $A \cup A' = S$ and the sets A and A' are disjoint,

$$1 = P(S) = P(A \cup A') = P(A) + P(A').$$

Example 2.32: If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

***Solution*:** Let E be the event that at least 5 cars are serviced. Now, $P(E) = 1 - P(E')$, where E' is the event that fewer than 5 cars are serviced. Since

$$P(E') = 0.12 + 0.19 = 0.31,$$

it follows from Theorem 2.9 that

$$P(E) = 1 - 0.31 = 0.69.$$

Example 2.33: Suppose the manufacturer's specifications for the length of a certain type of computer cable are 2000 ± 10 millimeters. In this industry, it is known that small cable is just as likely to be defective (not meeting specifications) as large cable. That is, the probability of randomly producing a cable with length exceeding 2010 millimeters is equal to the probability of producing a cable with length smaller than 1990 millimeters. The probability that the production procedure meets specifications is known to be 0.99.

(a) What is the probability that a cable selected randomly is too large?

(b) What is the probability that a randomly selected cable is larger than 1990 millimeters?

Solution: Let M be the event that a cable meets specifications. Let S and L be the events that the cable is too small and too large, respectively. Then

(a) $P(M) = 0.99$ and $P(S) = P(L) = (1 - 0.99)/2 = 0.005$.

(b) Denoting by X the length of a randomly selected cable, we have

$$P(1990 \leq X \leq 2010) = P(M) = 0.99.$$

Since $P(X \geq 2010) = P(L) = 0.005$,

$$P(X \geq 1990) = P(M) + P(L) = 0.995.$$

This also can be solved by using Theorem 2.9:

$$P(X \geq 1990) + P(X < 1990) = 1.$$

Thus, $P(X \geq 1990) = 1 - P(S) = 1 - 0.005 = 0.995$.

Exercises

2.49 Find the errors in each of the following statements:

(a) The probabilities that an automobile salesperson will sell 0, 1, 2, or 3 cars on any given day in February are, respectively, 0.19, 0.38, 0.29, and 0.15.

(b) The probability that it will rain tomorrow is 0.40, and the probability that it will not rain tomorrow is 0.52.

(c) The probabilities that a printer will make 0, 1, 2, 3, or 4 or more mistakes in setting a document are, respectively, $0.19, 0.34, -0.25, 0.43$, and 0.29.

(d) On a single draw from a deck of playing cards, the probability of selecting a heart is 1/4, the probability of selecting a black card is 1/2, and the probability of selecting both a heart and a black card is 1/8.

2.50 Assuming that all elements of S in Exercise 2.8 on page 42 are equally likely to occur, find

(a) the probability of event A;

(b) the probability of event C;

(c) the probability of event $A \cap C$.

2.51 A box contains 500 envelopes, of which 75 contain \$100 in cash, 150 contain \$25, and 275 contain \$10. An envelope may be purchased for \$25. What is the sample space for the different amounts of money? Assign probabilities to the sample points and then find the probability that the first envelope purchased contains less than \$100.

2.52 Suppose that in a senior college class of 500 students it is found that 210 smoke, 258 drink alcoholic beverages, 216 eat between meals, 122 smoke and drink alcoholic beverages, 83 eat between meals and drink alcoholic beverages, 97 smoke and eat between meals, and 52 engage in all three of these bad health practices. If a member of this senior class is selected at random, find the probability that the student

(a) smokes but does not drink alcoholic beverages;

(b) eats between meals and drinks alcoholic beverages but does not smoke;

(c) neither smokes nor eats between meals.

2.53 The probability that an American industry will locate in Shanghai, China, is 0.7, the probability that it will locate in Beijing, China, is 0.4, and the probability that it will locate in either Shanghai or Beijing or both is 0.8. What is the probability that the industry will locate

(a) in both cities?

(b) in neither city?

2.54 From past experience, a stockbroker believes that under present economic conditions a customer will invest in tax-free bonds with a probability of 0.6, will invest in mutual funds with a probability of 0.3, and will invest in both tax-free bonds and mutual funds with a probability of 0.15. At this time, find the probability that a customer will invest

(a) in either tax-free bonds or mutual funds;

(b) in neither tax-free bonds nor mutual funds.

2.55 If each coded item in a catalog begins with 3 distinct letters followed by 4 distinct nonzero digits, find the probability of randomly selecting one of these coded items with the first letter a vowel and the last digit even.

2.56 An automobile manufacturer is concerned about a possible recall of its best-selling four-door sedan. If there were a recall, there is a probability of 0.25 of a defect in the brake system, 0.18 of a defect in the transmission, 0.17 of a defect in the fuel system, and 0.40 of a defect in some other area.

(a) What is the probability that the defect is the brakes or the fueling system if the probability of defects in both systems simultaneously is 0.15?

(b) What is the probability that there are no defects in either the brakes or the fueling system?

2.57 If a letter is chosen at random from the English alphabet, find the probability that the letter

(a) is a vowel exclusive of y;

(b) is listed somewhere ahead of the letter j;

(c) is listed somewhere after the letter g.

2.58 A pair of fair dice is tossed. Find the probability of getting

(a) a total of 8;

(b) at most a total of 5.

2.59 In a poker hand consisting of 5 cards, find the probability of holding

(a) 3 aces;

(b) 4 hearts and 1 club.

2.60 If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems, and a dictionary, what is the probability that

(a) the dictionary is selected?

(b) 2 novels and 1 book of poems are selected?

2.61 In a high school graduating class of 100 students, 54 studied mathematics, 69 studied history, and 35 studied both mathematics and history. If one of these students is selected at random, find the probability that

(a) the student took mathematics or history;

(b) the student did not take either of these subjects;

(c) the student took history but not mathematics.

2.62 Dom's Pizza Company uses taste testing and statistical analysis of the data prior to marketing any new product. Consider a study involving three types of crusts (thin, thin with garlic and oregano, and thin with bits of cheese). Dom's is also studying three sauces (standard, a new sauce with more garlic, and a new sauce with fresh basil).

(a) How many combinations of crust and sauce are involved?

(b) What is the probability that a judge will get a plain thin crust with a standard sauce for his first taste test?

2.63 According to *Consumer Digest* (July/August 1996), the probable location of personal computers (PC) in the home is as follows:

Adult bedroom:	0.03
Child bedroom:	0.15
Other bedroom:	0.14
Office or den:	0.40
Other rooms:	0.28

(a) What is the probability that a PC is in a bedroom?

(b) What is the probability that it is not in a bedroom?

(c) Suppose a household is selected at random from households with a PC; in what room would you expect to find a PC?

2.64 Interest centers around the life of an electronic component. Suppose it is known that the probability that the component survives for more than 6000 hours is 0.42. Suppose also that the probability that the component survives *no longer than* 4000 hours is 0.04.

(a) What is the probability that the life of the component is less than or equal to 6000 hours?

(b) What is the probability that the life is greater than 4000 hours?

2.65 Consider the situation of Exercise 2.64. Let A be the event that the component fails a particular test and B be the event that the component displays strain but does not actually fail. Event A occurs with probability 0.20, and event B occurs with probability 0.35.

(a) What is the probability that the component does not fail the test?

(b) What is the probability that the component works perfectly well (i.e., neither displays strain nor fails the test)?

(c) What is the probability that the component either fails or shows strain in the test?

2.66 Factory workers are constantly encouraged to practice zero tolerance when it comes to accidents in factories. Accidents can occur because the working environment or conditions themselves are unsafe. On the other hand, accidents can occur due to carelessness or so-called human error. In addition, the worker's shift, 7:00 A.M.–3:00 P.M. (day shift), 3:00 P.M.–11:00 P.M. (evening shift), or 11:00 P.M.–7:00 A.M. (graveyard shift), may be a factor. During the last year, 300 accidents have occurred. The percentages of the accidents for the condition combinations are as follows:

Shift	Unsafe Conditions	Human Error
Day	5%	32%
Evening	6%	25%
Graveyard	2%	30%

If an accident report is selected randomly from the 300 reports,

(a) what is the probability that the accident occurred on the graveyard shift?

(b) what is the probability that the accident occurred due to human error?

(c) what is the probability that the accident occurred due to unsafe conditions?

(d) what is the probability that the accident occurred on either the evening or the graveyard shift?

2.67 Consider the situation of Example 2.32 on page 58.

(a) What is the probability that no more than 4 cars will be serviced by the mechanic?

(b) What is the probability that he will service fewer than 8 cars?

(c) What is the probability that he will service either 3 or 4 cars?

2.68 Interest centers around the nature of an oven purchased at a particular department store. It can be either a gas or an electric oven. Consider the decisions made by six distinct customers.

(a) Suppose that the probability is 0.40 that at most two of these individuals purchase an electric oven. What is the probability that at least three purchase the electric oven?

(b) Suppose it is known that the probability that all six purchase the electric oven is 0.007 while 0.104 is the probability that all six purchase the gas oven. What is the probability that at least one of each type is purchased?

2.69 It is common in many industrial areas to use a filling machine to fill boxes full of product. This occurs in the food industry as well as other areas in which the product is used in the home, for example, detergent. These machines are not perfect, and indeed they may A, fill to specification, B, underfill, and C, overfill. Generally, the practice of underfilling is that which one hopes to avoid. Let $P(B) = 0.001$ while $P(A) = 0.990$.

(a) Give $P(C)$.

(b) What is the probability that the machine does not underfill?

(c) What is the probability that the machine either overfills or underfills?

2.70 Consider the situation of Exercise 2.69. Suppose 50,000 boxes of detergent are produced per week and suppose also that those underfilled are "sent back," with customers requesting reimbursement of the purchase price. Suppose also that the cost of production is known to be \$4.00 per box while the purchase price is \$4.50 per box.

(a) What is the weekly profit under the condition of no defective boxes?

(b) What is the loss in profit expected due to underfilling?

2.71 As the situation of Exercise 2.69 might suggest, statistical procedures are often used for control of quality (i.e., industrial quality control). At times, the *weight* of a product is an important variable to control. Specifications are given for the weight of a certain packaged product, and a package is rejected if it is either too light or too heavy. Historical data suggest that 0.95 is the probability that the product meets weight specifications whereas 0.002 is the probability that the product is too light. For each single packaged product, the manufacturer invests \$20.00 in production and the purchase price for the consumer is \$25.00.

(a) What is the probability that a package chosen randomly from the production line is too heavy?

(b) For each 10,000 packages sold, what profit is received by the manufacturer if all packages meet weight specification?

(c) Assuming that all defective packages are rejected