3.4 Joint Probability Distributions

Our study of random variables and their probability distributions in the preceding sections is restricted to one-dimensional sample spaces, in that we recorded outcomes of an experiment as values assumed by a single random variable. There will be situations, however, where we may find it desirable to record the simultaneous outcomes of several random variables. For example, we might measure the amount of precipitate P and volume V of gas released from a controlled chemical experiment, giving rise to a two-dimensional sample space consisting of the outcomes (p, v), or we might be interested in the hardness H and tensile strength T of cold-drawn copper, resulting in the outcomes (h, t). In a study to determine the likelihood of success in college based on high school data, we might use a three-dimensional sample space and record for each individual his or her aptitude test score, high school class rank, and grade-point average at the end of freshman year in college.

If X and Y are two discrete random variables, the probability distribution for their simultaneous occurrence can be represented by a function with values $f(x, y)$ for any pair of values (x, y) within the range of the random variables X and Y. It is customary to refer to this function as the **joint probability distribution** of X and Y.

Hence, in the discrete case,

$$f(x, y) = P(X = x, Y = y);$$

that is, the values $f(x, y)$ give the probability that outcomes x and y occur at the same time. For example, if an 18-wheeler is to have its tires serviced and X represents the number of miles these tires have been driven and Y represents the number of tires that need to be replaced, then $f(30000, 5)$ is the probability that the tires are used over 30,000 miles and the truck needs 5 new tires.

Definition 3.8: The function $f(x, y)$ is a **joint probability distribution** or **probability mass function** of the discrete random variables X and Y if

1. $f(x, y) \geq 0$ for all (x, y),
2. $\sum_x \sum_y f(x, y) = 1$,
3. $P(X = x, Y = y) = f(x, y)$.

For any region A in the xy plane, $P[(X, Y) \in A] = \sum \sum_A f(x, y)$.

Example 3.14: Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find

(a) the joint probability function $f(x, y)$,

(b) $P[(X, Y) \in A]$, where A is the region $\{(x, y) | x + y \leq 1\}$.

***Solution*:** The possible pairs of values (x, y) are $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, $(0, 2)$, and $(2, 0)$.

(a) Now, $f(0, 1)$, for example, represents the probability that a red and a green pens are selected. The total number of equally likely ways of selecting any 2 pens from the 8 is $\binom{8}{2} = 28$. The number of ways of selecting 1 red from 2 red pens and 1 green from 3 green pens is $\binom{2}{1}\binom{3}{1} = 6$. Hence, $f(0, 1) = 6/28 = 3/14$. Similar calculations yield the probabilities for the other cases, which are presented in Table 3.1. Note that the probabilities sum to 1. In Chapter

5, it will become clear that the joint probability distribution of Table 3.1 can be represented by the formula

$$f(x,y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}},$$

for $x = 0, 1, 2$; $y = 0, 1, 2$; and $0 \leq x + y \leq 2$.

(b) The probability that (X, Y) fall in the region A is

$$P[(X,Y) \in A] = P(X + Y \leq 1) = f(0,0) + f(0,1) + f(1,0)$$
$$= \frac{3}{28} + \frac{3}{14} + \frac{9}{28} = \frac{9}{14}.$$

Table 3.1: Joint Probability Distribution for Example 3.14

	$f(x,y)$	x = 0	1	2	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

When X and Y are continuous random variables, the **joint density function** $f(x, y)$ is a surface lying above the xy plane, and $P[(X, Y) \in A]$, where A is any region in the xy plane, is equal to the volume of the right cylinder bounded by the base A and the surface.

Definition 3.9: The function $f(x, y)$ is a **joint density function** of the continuous random variables X and Y if

1. $f(x, y) \geq 0$, for all (x, y),
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$,
3. $P[(X,Y) \in A] = \int\int_A f(x,y)\, dx\, dy$, for any region A in the xy plane.

Example 3.15: A privately owned business operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportions of the time that the drive-in and the walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$f(x,y) = \begin{cases} \frac{2}{5}(2x + 3y), & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify condition 2 of Definition 3.9.

(b) Find $P[(X, Y) \in A]$, where $A = \{(x, y) \mid 0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}\}$.

Solution: (a) The integration of $f(x,y)$ over the whole region is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\,dx\,dy$$
$$= \int_0^1 \left(\frac{2x^2}{5}+\frac{6xy}{5}\right)\Bigg|_{x=0}^{x=1} dy$$
$$= \int_0^1 \left(\frac{2}{5}+\frac{6y}{5}\right) dy = \left(\frac{2y}{5}+\frac{3y^2}{5}\right)\Bigg|_0^1 = \frac{2}{5}+\frac{3}{5} = 1.$$

(b) To calculate the probability, we use

$$P[(X,Y)\in A] = P\left(0<X<\frac{1}{2}, \frac{1}{4}<Y<\frac{1}{2}\right)$$
$$= \int_{1/4}^{1/2}\int_0^{1/2} \frac{2}{5}(2x+3y)\,dx\,dy$$
$$= \int_{1/4}^{1/2} \left(\frac{2x^2}{5}+\frac{6xy}{5}\right)\Bigg|_{x=0}^{x=1/2} dy = \int_{1/4}^{1/2}\left(\frac{1}{10}+\frac{3y}{5}\right) dy$$
$$= \left(\frac{y}{10}+\frac{3y^2}{10}\right)\Bigg|_{1/4}^{1/2}$$
$$= \frac{1}{10}\left[\left(\frac{1}{2}+\frac{3}{4}\right)-\left(\frac{1}{4}+\frac{3}{16}\right)\right] = \frac{13}{160}.$$

Given the joint probability distribution $f(x,y)$ of the discrete random variables X and Y, the probability distribution $g(x)$ of X alone is obtained by summing $f(x,y)$ over the values of Y. Similarly, the probability distribution $h(y)$ of Y alone is obtained by summing $f(x,y)$ over the values of X. We define $g(x)$ and $h(y)$ to be the **marginal distributions** of X and Y, respectively. When X and Y are continuous random variables, summations are replaced by integrals. We can now make the following general definition.

Definition 3.10: The **marginal distributions** of X alone and of Y alone are

$$g(x) = \sum_y f(x,y) \quad \text{and} \quad h(y) = \sum_x f(x,y)$$

for the discrete case, and

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad \text{and} \quad h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

for the continuous case.

The term *marginal* is used here because, in the discrete case, the values of $g(x)$ and $h(y)$ are just the marginal totals of the respective columns and rows when the values of $f(x,y)$ are displayed in a rectangular table.

Example 3.16: Show that the column and row totals of Table 3.1 give the marginal distribution of X alone and of Y alone.

***Solution*:** For the random variable X, we see that

$$g(0) = f(0,0) + f(0,1) + f(0,2) = \frac{3}{28} + \frac{3}{14} + \frac{1}{28} = \frac{5}{14},$$
$$g(1) = f(1,0) + f(1,1) + f(1,2) = \frac{9}{28} + \frac{3}{14} + 0 = \frac{15}{28},$$

and

$$g(2) = f(2,0) + f(2,1) + f(2,2) = \frac{3}{28} + 0 + 0 = \frac{3}{28},$$

which are just the column totals of Table 3.1. In a similar manner we could show that the values of $h(y)$ are given by the row totals. In tabular form, these marginal distributions may be written as follows:

x	0	1	2
$g(x)$	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$

y	0	1	2
$h(y)$	$\frac{15}{28}$	$\frac{3}{7}$	$\frac{1}{28}$

Example 3.17: Find $g(x)$ and $h(y)$ for the joint density function of Example 3.15.

***Solution*:** By definition,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\ dy = \int_0^1 \frac{2}{5}(2x+3y)\ dy = \left(\frac{4xy}{5} + \frac{6y^2}{10}\right)\Bigg|_{y=0}^{y=1} = \frac{4x+3}{5},$$

for $0 \le x \le 1$, and $g(x) = 0$ elsewhere. Similarly,

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\ dx = \int_0^1 \frac{2}{5}(2x+3y)\ dx = \frac{2(1+3y)}{5},$$

for $0 \le y \le 1$, and $h(y) = 0$ elsewhere.

The fact that the marginal distributions $g(x)$ and $h(y)$ are indeed the probability distributions of the individual variables X and Y alone can be verified by showing that the conditions of Definition 3.4 or Definition 3.6 are satisfied. For example, in the continuous case

$$\int_{-\infty}^{\infty} g(x)\ dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ dy\ dx = 1,$$

and

$$\begin{aligned} P(a < X < b) &= P(a < X < b, -\infty < Y < \infty) \\ &= \int_a^b \int_{-\infty}^{\infty} f(x,y)\ dy\ dx = \int_a^b g(x)\ dx. \end{aligned}$$

In Section 3.1, we stated that the value x of the random variable X represents an event that is a subset of the sample space. If we use the definition of conditional probability as stated in Chapter 2,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) > 0,$$

where A and B are now the events defined by $X = x$ and $Y = y$, respectively, then

$$P(Y = y \mid X = x) = \frac{P(X = x, Y = y)}{P(X = x)} = \frac{f(x, y)}{g(x)}, \text{ provided } g(x) > 0,$$

where X and Y are discrete random variables.

It is not difficult to show that the function $f(x, y)/g(x)$, which is strictly a function of y with x fixed, satisfies all the conditions of a probability distribution. This is also true when $f(x, y)$ and $g(x)$ are the joint density and marginal distribution, respectively, of continuous random variables. As a result, it is extremely important that we make use of the special type of distribution of the form $f(x, y)/g(x)$ in order to be able to effectively compute conditional probabilities. This type of distribution is called a **conditional probability distribution**; the formal definition follows.

Definition 3.11: Let X and Y be two random variables, discrete or continuous. The **conditional distribution** of the random variable Y given that $X = x$ is

$$f(y|x) = \frac{f(x, y)}{g(x)}, \text{ provided } g(x) > 0.$$

Similarly, the conditional distribution of X given that $Y = y$ is

$$f(x|y) = \frac{f(x, y)}{h(y)}, \text{ provided } h(y) > 0.$$

If we wish to find the probability that the discrete random variable X falls between a and b when it is known that the discrete variable $Y = y$, we evaluate

$$P(a < X < b \mid Y = y) = \sum_{a<x<b} f(x|y),$$

where the summation extends over all values of X between a and b. When X and Y are continuous, we evaluate

$$P(a < X < b \mid Y = y) = \int_a^b f(x|y)\, dx.$$

Example 3.18: Referring to Example 3.14, find the conditional distribution of X, given that $Y = 1$, and use it to determine $P(X = 0 \mid Y = 1)$.

Solution: We need to find $f(x|y)$, where $y = 1$. First, we find that

$$h(1) = \sum_{x=0}^{2} f(x, 1) = \frac{3}{14} + \frac{3}{14} + 0 = \frac{3}{7}.$$

Now

$$f(x|1) = \frac{f(x, 1)}{h(1)} = \left(\frac{7}{3}\right) f(x, 1), \quad x = 0, 1, 2.$$

Therefore,

$$f(0|1) = \left(\frac{7}{3}\right) f(0,1) = \left(\frac{7}{3}\right)\left(\frac{3}{14}\right) = \frac{1}{2},\ f(1|1) = \left(\frac{7}{3}\right) f(1,1) = \left(\frac{7}{3}\right)\left(\frac{3}{14}\right) = \frac{1}{2},$$

$$f(2|1) = \left(\frac{7}{3}\right) f(2,1) = \left(\frac{7}{3}\right)(0) = 0,$$

and the conditional distribution of X, given that $Y = 1$, is

x	0	1	2
$f(x\|1)$	$\frac{1}{2}$	$\frac{1}{2}$	0

Finally,

$$P(X = 0 \mid Y = 1) = f(0|1) = \frac{1}{2}.$$

Therefore, if it is known that 1 of the 2 pen refills selected is red, we have a probability equal to 1/2 that the other refill is not blue.

Example 3.19: The joint density for the random variables (X, Y), where X is the unit temperature change and Y is the proportion of spectrum shift that a certain atomic particle produces, is

$$f(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal densities $g(x)$, $h(y)$, and the conditional density $f(y|x)$.

(b) Find the probability that the spectrum shifts more than half of the total observations, given that the temperature is increased by 0.25 unit.

***Solution*:** (a) By definition,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_x^1 10xy^2\, dy$$
$$= \frac{10}{3}xy^3\Big|_{y=x}^{y=1} = \frac{10}{3}x(1-x^3),\ 0 < x < 1,$$
$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^y 10xy^2\, dx = 5x^2y^2\Big|_{x=0}^{x=y} = 5y^4,\ 0 < y < 1.$$

Now

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{10xy^2}{\frac{10}{3}x(1-x^3)} = \frac{3y^2}{1-x^3},\ 0 < x < y < 1.$$

(b) Therefore,

$$P\left(Y > \frac{1}{2} \,\middle|\, X = 0.25\right) = \int_{1/2}^1 f(y \mid x = 0.25)\, dy = \int_{1/2}^1 \frac{3y^2}{1-0.25^3}\, dy = \frac{8}{9}.$$

Example 3.20: Given the joint density function

$$f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

find $g(x)$, $h(y)$, $f(x|y)$, and evaluate $P(\frac{1}{4} < X < \frac{1}{2} \mid Y = \frac{1}{3})$.

***Solution*:** By definition of the marginal density. for $0 < x < 2$,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\ dy = \int_0^1 \frac{x(1+3y^2)}{4} dy$$
$$= \left(\frac{xy}{4} + \frac{xy^3}{4}\right)\Bigg|_{y=0}^{y=1} = \frac{x}{2},$$

and for $0 < y < 1$,

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\ dx = \int_0^2 \frac{x(1+3y^2)}{4} dx$$
$$= \left(\frac{x^2}{8} + \frac{3x^2y^2}{8}\right)\Bigg|_{x=0}^{x=2} = \frac{1+3y^2}{2}.$$

Therefore, using the conditional density definition, for $0 < x < 2$,

$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{x(1+3y^2)/4}{(1+3y^2)/2} = \frac{x}{2},$$

and

$$P\left(\frac{1}{4} < X < \frac{1}{2} \;\middle|\; Y = \frac{1}{3}\right) = \int_{1/4}^{1/2} \frac{x}{2} dx = \frac{3}{64}.$$

Statistical Independence

If $f(x|y)$ does not depend on y, as is the case for Example 3.20, then $f(x|y) = g(x)$ and $f(x,y) = g(x)h(y)$. The proof follows by substituting

$$f(x,y) = f(x|y)h(y)$$

into the marginal distribution of X. That is,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\ dy = \int_{-\infty}^{\infty} f(x|y)h(y)\ dy.$$

If $f(x|y)$ does not depend on y, we may write

$$g(x) = f(x|y) \int_{-\infty}^{\infty} h(y)\ dy.$$

Now

$$\int_{-\infty}^{\infty} h(y)\ dy = 1,$$

since $h(y)$ is the probability density function of Y. Therefore,

$$g(x) = f(x|y) \quad \text{and then} \quad f(x,y) = g(x)h(y).$$

It should make sense to the reader that if $f(x|y)$ does not depend on y, then of course the outcome of the random variable Y has no impact on the outcome of the random variable X. In other words, we say that X and Y are independent random variables. We now offer the following formal definition of statistical independence.

Definition 3.12: Let X and Y be two random variables, discrete or continuous, with joint probability distribution $f(x, y)$ and marginal distributions $g(x)$ and $h(y)$, respectively. The random variables X and Y are said to be **statistically independent** if and only if

$$f(x, y) = g(x)h(y)$$

for all (x, y) within their range.

The continuous random variables of Example 3.20 are statistically independent, since the product of the two marginal distributions gives the joint density function. This is obviously not the case, however, for the continuous variables of Example 3.19. Checking for statistical independence of discrete random variables requires a more thorough investigation, since it is possible to have the product of the marginal distributions equal to the joint probability distribution for some but not all combinations of (x, y). If you can find any point (x, y) for which $f(x, y)$ is defined such that $f(x, y) \neq g(x)h(y)$, the discrete variables X and Y are not statistically independent.

Example 3.21: Show that the random variables of Example 3.14 are not statistically independent.

***Proof*:** Let us consider the point $(0, 1)$. From Table 3.1 we find the three probabilities $f(0, 1)$, $g(0)$, and $h(1)$ to be

$$\begin{aligned} f(0,1) &= \frac{3}{14}, \\ g(0) &= \sum_{y=0}^{2} f(0, y) = \frac{3}{28} + \frac{3}{14} + \frac{1}{28} = \frac{5}{14}, \\ h(1) &= \sum_{x=0}^{2} f(x, 1) = \frac{3}{14} + \frac{3}{14} + 0 = \frac{3}{7}. \end{aligned}$$

Clearly,

$$f(0, 1) \neq g(0)h(1),$$

and therefore X and Y are not statistically independent.

All the preceding definitions concerning two random variables can be generalized to the case of n random variables. Let $f(x_1, x_2, \ldots, x_n)$ be the joint probability function of the random variables $X_1, X_2, \ldots, X_n$. The marginal distribution of X_1, for example, is

$$g(x_1) = \sum_{x_2} \cdots \sum_{x_n} f(x_1, x_2, \ldots, x_n)$$

for the discrete case, and

$$g(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n)\, dx_2\, dx_3 \cdots dx_n$$

for the continuous case. We can now obtain **joint marginal distributions** such as $g(x_1, x_2)$, where

$$g(x_1, x_2) = \begin{cases} \sum\limits_{x_3} \cdots \sum\limits_{x_n} f(x_1, x_2, \ldots, x_n) & \text{(discrete case)}, \\ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n)\, dx_3\, dx_4 \cdots dx_n & \text{(continuous case)}. \end{cases}$$

We could consider numerous conditional distributions. For example, the **joint conditional distribution** of X_1, X_2, and X_3, given that $X_4 = x_4, X_5 = x_5, \ldots, X_n = x_n$, is written

$$f(x_1, x_2, x_3 \mid x_4, x_5, \ldots, x_n) = \frac{f(x_1, x_2, \ldots, x_n)}{g(x_4, x_5, \ldots, x_n)},$$

where $g(x_4, x_5, \ldots, x_n)$ is the joint marginal distribution of the random variables $X_4, X_5, \ldots, X_n$.

A generalization of Definition 3.12 leads to the following definition for the mutual statistical independence of the variables $X_1, X_2, \ldots, X_n$.

Definition 3.13: Let $X_1, X_2, \ldots, X_n$ be n random variables, discrete or continuous, with joint probability distribution $f(x_1, x_2, \ldots, x_n)$ and marginal distribution $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$, respectively. The random variables $X_1, X_2, \ldots, X_n$ are said to be mutually **statistically independent** if and only if

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)$$

for all $(x_1, x_2, \ldots, x_n)$ within their range.

Example 3.22: Suppose that the shelf life, in years, of a certain perishable food product packaged in cardboard containers is a random variable whose probability density function is given by

$$f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{elsewhere}. \end{cases}$$

Let X_1, X_2, and X_3 represent the shelf lives for three of these containers selected independently and find $P(X_1 < 2, 1 < X_2 < 3, X_3 > 2)$.

Solution: Since the containers were selected independently, we can assume that the random variables X_1, X_2, and X_3 are statistically independent, having the joint probability density

$$f(x_1, x_2, x_3) = f(x_1) f(x_2) f(x_3) = e^{-x_1} e^{-x_2} e^{-x_3} = e^{-x_1 - x_2 - x_3},$$

for $x_1 > 0$, $x_2 > 0$, $x_3 > 0$, and $f(x_1, x_2, x_3) = 0$ elsewhere. Hence

$$\begin{aligned} P(X_1 < 2, 1 < X_2 < 3, X_3 > 2) &= \int_2^{\infty} \int_1^3 \int_0^2 e^{-x_1 - x_2 - x_3}\, dx_1\, dx_2\, dx_3 \\ &= (1 - e^{-2})(e^{-1} - e^{-3}) e^{-2} = 0.0372. \end{aligned}$$

What Are Important Characteristics of Probability Distributions and Where Do They Come From?

This is an important point in the text to provide the reader with a transition into the next three chapters. We have given illustrations in both examples and exercises of practical scientific and engineering situations in which probability distributions and their properties are used to solve important problems. These probability distributions, either discrete or continuous, were introduced through phrases like "it is known that" or "suppose that" or even in some cases "historical evidence suggests that." These are situations in which the nature of the distribution and even a good estimate of the probability structure can be determined through historical data, data from long-term studies, or even large amounts of planned data. The reader should remember the discussion of the use of histograms in Chapter 1 and from that recall how frequency distributions are estimated from the histograms. However, not all probability functions and probability density functions are derived from large amounts of historical data. There are a substantial number of situations in which the nature of the scientific scenario suggests a distribution type. Indeed, many of these are reflected in exercises in both Chapter 2 and this chapter. When independent repeated observations are binary in nature (e.g., defective or not, survive or not, allergic or not) with value 0 or 1, the distribution covering this situation is called the **binomial distribution** and the probability function is known and will be demonstrated in its generality in Chapter 5. Exercise 3.34 in Section 3.3 and Review Exercise 3.80 are examples, and there are others that the reader should recognize. The scenario of a continuous distribution in time to failure, as in Review Exercise 3.69 or Exercise 3.27 on page 93, often suggests a distribution type called the **exponential distribution**. These types of illustrations are merely two of many so-called standard distributions that are used extensively in real-world problems because the scientific scenario that gives rise to each of them is recognizable and occurs often in practice. Chapters 5 and 6 cover many of these types along with some underlying theory concerning their use.

A second part of this transition to material in future chapters deals with the notion of **population parameters** or **distributional parameters**. Recall in Chapter 1 we discussed the need to use data to provide information about these parameters. We went to some length in discussing the notions of a **mean** and **variance** and provided a vision for the concepts in the context of a population. Indeed, the population mean and variance are easily found from the probability function for the discrete case or probability density function for the continuous case. These parameters and their importance in the solution of many types of real-world problems will provide much of the material in Chapters 8 through 17.

Exercises

3.37 Determine the values of c so that the following functions represent joint probability distributions of the random variables X and Y:

(a) $f(x,y) = cxy$, for $x = 1, 2, 3$; $y = 1, 2, 3$;

(b) $f(x,y) = c|x-y|$, for $x = -2, 0, 2$; $y = -2, 3$.

3.38 If the joint probability distribution of X and Y is given by

$$f(x,y) = \frac{x+y}{30}, \quad \text{for } x = 0, 1, 2, 3;\ y = 0, 1, 2,$$

find

(a) $P(X \leq 2, Y = 1)$;
(b) $P(X > 2, Y \leq 1)$;
(c) $P(X > Y)$;
(d) $P(X + Y = 4)$.

3.39 From a sack of fruit containing 3 oranges, 2 apples, and 3 bananas, a random sample of 4 pieces of fruit is selected. If X is the number of oranges and Y is the number of apples in the sample, find
(a) the joint probability distribution of X and Y;
(b) $P[(X, Y) \in A]$, where A is the region that is given by $\{(x, y) \mid x + y \leq 2\}$.

3.40 A fast-food restaurant operates both a drive-through facility and a walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportions of the time that the drive-through and walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$f(x, y) = \begin{cases} \frac{2}{3}(x + 2y), & 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal density of X.
(b) Find the marginal density of Y.
(c) Find the probability that the drive-through facility is busy less than one-half of the time.

3.41 A candy company distributes boxes of chocolates with a mixture of creams, toffees, and cordials. Suppose that the weight of each box is 1 kilogram, but the individual weights of the creams, toffees, and cordials vary from box to box. For a randomly selected box, let X and Y represent the weights of the creams and the toffees, respectively, and suppose that the joint density function of these variables is

$$f(x, y) = \begin{cases} 24xy, & 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x + y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the probability that in a given box the cordials account for more than 1/2 of the weight.
(b) Find the marginal density for the weight of the creams.
(c) Find the probability that the weight of the toffees in a box is less than 1/8 of a kilogram if it is known that creams constitute 3/4 of the weight.

3.42 Let X and Y denote the lengths of life, in years, of two components in an electronic system. If the joint density function of these variables is

$$f(x, y) = \begin{cases} e^{-(x+y)}, & x > 0,\ y > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

find $P(0 < X < 1 \mid Y = 2)$.

3.43 Let X denote the reaction time, in seconds, to a certain stimulus and Y denote the temperature (°F) at which a certain reaction starts to take place. Suppose that two random variables X and Y have the joint density

$$f(x, y) = \begin{cases} 4xy, & 0 < x < 1,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find
(a) $P(0 \leq X \leq \frac{1}{2}$ and $\frac{1}{4} \leq Y \leq \frac{1}{2})$;
(b) $P(X < Y)$.

3.44 Each rear tire on an experimental airplane is supposed to be filled to a pressure of 40 pounds per square inch (psi). Let X denote the actual air pressure for the right tire and Y denote the actual air pressure for the left tire. Suppose that X and Y are random variables with the joint density function

$$f(x, y) = \begin{cases} k(x^2 + y^2), & 30 \leq x < 50,\ 30 \leq y < 50, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find k.
(b) Find $P(30 \leq X \leq 40$ and $40 \leq Y < 50)$.
(c) Find the probability that both tires are underfilled.

3.45 Let X denote the diameter of an armored electric cable and Y denote the diameter of the ceramic mold that makes the cable. Both X and Y are scaled so that they range between 0 and 1. Suppose that X and Y have the joint density

$$f(x, y) = \begin{cases} \frac{1}{y}, & 0 < x < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $P(X + Y > 1/2)$.

3.46 Referring to Exercise 3.38, find
(a) the marginal distribution of X;
(b) the marginal distribution of Y.

3.47 The amount of kerosene, in thousands of liters, in a tank at the beginning of any day is a random amount Y from which a random amount X is sold during that day. Suppose that the tank is not resupplied during the day so that $x \leq y$, and assume that the joint density function of these variables is

$$f(x, y) = \begin{cases} 2, & 0 < x \leq y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Determine if X and Y are independent.

(b) Find $P(1/4 < X < 1/2 \mid Y = 3/4)$.

3.48 Referring to Exercise 3.39, find
(a) $f(y|2)$ for all values of y;
(b) $P(Y = 0 \mid X = 2)$.

3.49 Let X denote the number of times a certain numerical control machine will malfunction: 1, 2, or 3 times on any given day. Let Y denote the number of times a technician is called on an emergency call. Their joint probability distribution is given as

$f(x,y)$		$x=1$	$x=2$	$x=3$
	1	0.05	0.05	0.10
y	3	0.05	0.10	0.35
	5	0.00	0.20	0.10

(a) Evaluate the marginal distribution of X.
(b) Evaluate the marginal distribution of Y.
(c) Find $P(Y = 3 \mid X = 2)$.

3.50 Suppose that X and Y have the following joint probability distribution:

$f(x,y)$		$x=2$	$x=4$
	1	0.10	0.15
y	3	0.20	0.30
	5	0.10	0.15

(a) Find the marginal distribution of X.
(b) Find the marginal distribution of Y.

3.51 Three cards are drawn without replacement from the 12 face cards (jacks, queens, and kings) of an ordinary deck of 52 playing cards. Let X be the number of kings selected and Y the number of jacks. Find
(a) the joint probability distribution of X and Y;
(b) $P[(X, Y) \in A]$, where A is the region given by $\{(x, y) \mid x + y \geq 2\}$.

3.52 A coin is tossed twice. Let Z denote the number of heads on the first toss and W the total number of heads on the 2 tosses. If the coin is unbalanced and a head has a 40% chance of occurring, find
(a) the joint probability distribution of W and Z;
(b) the marginal distribution of W;
(c) the marginal distribution of Z;
(d) the probability that at least 1 head occurs.

3.53 Given the joint density function

$$f(x,y) = \begin{cases} \frac{6-x-y}{8}, & 0 < x < 2,\ 2 < y < 4, \\ 0, & \text{elsewhere,} \end{cases}$$

find $P(1 < Y < 3 \mid X = 1)$.

3.54 Determine whether the two random variables of Exercise 3.49 are dependent or independent.

3.55 Determine whether the two random variables of Exercise 3.50 are dependent or independent.

3.56 The joint density function of the random variables X and Y is

$$f(x,y) = \begin{cases} 6x, & 0 < x < 1,\ 0 < y < 1 - x, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that X and Y are not independent.
(b) Find $P(X > 0.3 \mid Y = 0.5)$.

3.57 Let X, Y, and Z have the joint probability density function

$$f(x,y,z) = \begin{cases} kxy^2z, & 0 < x, y < 1,\ 0 < z < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find k.
(b) Find $P(X < \frac{1}{4}, Y > \frac{1}{2}, 1 < Z < 2)$.

3.58 Determine whether the two random variables of Exercise 3.43 are dependent or independent.

3.59 Determine whether the two random variables of Exercise 3.44 are dependent or independent.

3.60 The joint probability density function of the random variables X, Y, and Z is

$$f(x,y,z) = \begin{cases} \frac{4xyz^2}{9}, & 0 < x, y < 1,\ 0 < z < 3, \\ 0, & \text{elsewhere.} \end{cases}$$

Find
(a) the joint marginal density function of Y and Z;
(b) the marginal density of Y;
(c) $P(\frac{1}{4} < X < \frac{1}{2},\ Y > \frac{1}{3},\ 1 < Z < 2)$;
(d) $P(0 < X < \frac{1}{2} \mid Y = \frac{1}{4},\ Z = 2)$.

Review Exercises

3.61 A tobacco company produces blends of tobacco, with each blend containing various proportions of Turkish, domestic, and other tobaccos. The proportions of Turkish and domestic in a blend are random variables with joint density function (X = Turkish and Y = domestic)

$$f(x, y) = \begin{cases} 24xy, & 0 \le x, y \le 1,\ x + y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the probability that in a given box the Turkish tobacco accounts for over half the blend.

(b) Find the marginal density function for the proportion of the domestic tobacco.

(c) Find the probability that the proportion of Turkish tobacco is less than 1/8 if it is known that the blend contains 3/4 domestic tobacco.

3.62 An insurance company offers its policyholders a number of different premium payment options. For a randomly selected policyholder, let X be the number of months between successive payments. The cumulative distribution function of X is

$$F(x) = \begin{cases} 0, & \text{if } x < 1, \\ 0.4, & \text{if } 1 \le x < 3, \\ 0.6, & \text{if } 3 \le x < 5, \\ 0.8, & \text{if } 5 \le x < 7, \\ 1.0, & \text{if } x \ge 7. \end{cases}$$

(a) What is the probability mass function of X?

(b) Compute $P(4 < X \le 7)$.

3.63 Two electronic components of a missile system work in harmony for the success of the total system. Let X and Y denote the life in hours of the two components. The joint density of X and Y is

$$f(x, y) = \begin{cases} ye^{-y(1+x)}, & x, y \ge 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Give the marginal density functions for both random variables.

(b) What is the probability that the lives of both components will exceed 2 hours?

3.64 A service facility operates with two service lines. On a randomly selected day, let X be the proportion of time that the first line is in use whereas Y is the proportion of time that the second line is in use. Suppose that the joint probability density function for (X, Y) is

$$f(x, y) = \begin{cases} \frac{3}{2}(x^2 + y^2), & 0 \le x, y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Compute the probability that neither line is busy more than half the time.

(b) Find the probability that the first line is busy more than 75% of the time.

3.65 Let the number of phone calls received by a switchboard during a 5-minute interval be a random variable X with probability function

$$f(x) = \frac{e^{-2}2^x}{x!}, \quad \text{for } x = 0, 1, 2, \ldots.$$

(a) Determine the probability that X equals 0, 1, 2, 3, 4, 5, and 6.

(b) Graph the probability mass function for these values of x.

(c) Determine the cumulative distribution function for these values of X.

3.66 Consider the random variables X and Y with joint density function

$$f(x, y) = \begin{cases} x + y, & 0 \le x, y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal distributions of X and Y.

(b) Find $P(X > 0.5, Y > 0.5)$.

3.67 An industrial process manufactures items that can be classified as either defective or not defective. The probability that an item is defective is 0.1. An experiment is conducted in which 5 items are drawn randomly from the process. Let the random variable X be the number of defectives in this sample of 5. What is the probability mass function of X?

3.68 Consider the following joint probability density function of the random variables X and Y:

$$f(x, y) = \begin{cases} \frac{3x-y}{9}, & 1 < x < 3,\ 1 < y < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal density functions of X and Y.

(b) Are X and Y independent?

(c) Find $P(X > 2)$.

3.69 The life span in hours of an electrical component is a random variable with cumulative distribution function

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{50}}, & x > 0, \\ 0, & \text{eleswhere.} \end{cases}$$

(a) Determine its probability density function.
(b) Determine the probability that the life span of such a component will exceed 70 hours.

3.70 Pairs of pants are being produced by a particular outlet facility. The pants are checked by a group of 10 workers. The workers inspect pairs of pants taken randomly from the production line. Each inspector is assigned a number from 1 through 10. A buyer selects a pair of pants for purchase. Let the random variable X be the inspector number.
(a) Give a reasonable probability mass function for X.
(b) Plot the cumulative distribution function for X.

3.71 The shelf life of a product is a random variable that is related to consumer acceptance. It turns out that the shelf life Y in days of a certain type of bakery product has a density function

$$f(y) = \begin{cases} \frac{1}{2}e^{-y/2}, & 0 \le y < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

What fraction of the loaves of this product stocked today would you expect to be sellable 3 days from now?

3.72 Passenger congestion is a service problem in airports. Trains are installed within the airport to reduce the congestion. With the use of the train, the time X in minutes that it takes to travel from the main terminal to a particular concourse has density function

$$f(x) = \begin{cases} \frac{1}{10}, & 0 \le x \le 10, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that the above is a valid probability density function.
(b) Find the probability that the time it takes a passenger to travel from the main terminal to the concourse will not exceed 7 minutes.

3.73 Impurities in a batch of final product of a chemical process often reflect a serious problem. From considerable plant data gathered, it is known that the proportion Y of impurities in a batch has a density function given by

$$f(y) = \begin{cases} 10(1-y)^9, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify that the above is a valid density function.
(b) A batch is considered not sellable and then not acceptable if the percentage of impurities exceeds 60%. With the current quality of the process, what is the percentage of batches that are not acceptable?

3.74 The time Z in minutes between calls to an electrical supply system has the probability density function

$$f(z) = \begin{cases} \frac{1}{10}e^{-z/10}, & 0 < z < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) What is the probability that there are no calls within a 20-minute time interval?
(b) What is the probability that the first call comes within 10 minutes of opening?

3.75 A chemical system that results from a chemical reaction has two important components among others in a blend. The joint distribution describing the proportions X_1 and X_2 of these two components is given by

$$f(x_1, x_2) = \begin{cases} 2, & 0 < x_1 < x_2 < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Give the marginal distribution of X_1.
(b) Give the marginal distribution of X_2.
(c) What is the probability that component proportions produce the results $X_1 < 0.2$ and $X_2 > 0.5$?
(d) Give the conditional distribution $f_{X_1|X_2}(x_1|x_2)$.

3.76 Consider the situation of Review Exercise 3.75. But suppose the joint distribution of the two proportions is given by

$$f(x_1, x_2) = \begin{cases} 6x_2, & 0 < x_2 < x_1 < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Give the marginal distribution $f_{X_1}(x_1)$ of the proportion X_1 and verify that it is a valid density function.
(b) What is the probability that proportion X_2 is less than 0.5, given that X_1 is 0.7?

3.77 Consider the random variables X and Y that represent the number of vehicles that arrive at two separate street corners during a certain 2-minute period. These street corners are fairly close together so it is important that traffic engineers deal with them jointly if necessary. The joint distribution of X and Y is known to be

$$f(x, y) = \frac{9}{16} \cdot \frac{1}{4^{(x+y)}},$$

for $x = 0, 1, 2, \ldots$ and $y = 0, 1, 2, \ldots$.
(a) Are the two random variables X and Y independent? Explain why or why not.
(b) What is the probability that during the time period in question less than 4 vehicles arrive at the two street corners?

3.78 The behavior of series of components plays a huge role in scientific and engineering reliability problems. The reliability of the entire system is certainly no better than that of the weakest component in the series. In a series system, the components operate independently of each other. In a particular system containing three components, the probabilities of meeting specifications for components 1, 2, and 3, respectively, are 0.95, 0.99, and 0.92. What is the probability that the entire system works?

3.79 Another type of system that is employed in engineering work is a group of parallel components or a parallel system. In this more conservative approach, the probability that the system operates is larger than the probability that any component operates. The system fails only when all components fail. Consider a situation in which there are 4 independent components in a parallel system with probability of operation given by

Component 1: 0.95; Component 2: 0.94;
Component 3: 0.90; Component 4: 0.97.

What is the probability that the system does not fail?

3.80 Consider a system of components in which there are 5 independent components, each of which possesses an operational probability of 0.92. The system does have a redundancy built in such that it does not fail if 3 out of the 5 components are operational. What is the probability that the total system is operational?

3.81 **Project**: Take 5 class periods to observe the shoe color of individuals in class. Assume the shoe color categories are red, white, black, brown, and other. Complete a frequency table for each color category.

(a) Estimate and interpret the meaning of the probability distribution.

(b) What is the estimated probability that in the next class period a randomly selected student will be wearing a red or a white pair of shoes?