Chapter 6

Some Continuous Probability Distributions

6.1 Continuous Uniform Distribution

One of the simplest continuous distributions in all of statistics is the **continuous uniform distribution**. This distribution is characterized by a density function that is "flat," and thus the probability is uniform in a closed interval, say $[A, B]$. Although applications of the continuous uniform distribution are not as abundant as those for other distributions discussed in this chapter, it is appropriate for the novice to begin this introduction to continuous distributions with the uniform distribution.

Uniform Distribution

The density function of the continuous uniform random variable X on the interval $[A, B]$ is

$$f(x; A, B) = \begin{cases} \frac{1}{B-A}, & A \leq x \leq B, \\ 0, & \text{elsewhere.} \end{cases}$$

The density function forms a rectangle with base $B-A$ and **constant height** $\frac{1}{B-A}$. As a result, the uniform distribution is often called the **rectangular distribution**. Note, however, that the interval may not always be closed: $[A, B]$. It can be (A, B) as well. The density function for a uniform random variable on the interval $[1, 3]$ is shown in Figure 6.1.

Probabilities are simple to calculate for the uniform distribution because of the simple nature of the density function. However, note that the application of this distribution is based on the assumption that the probability of falling in an interval of fixed length within $[A, B]$ is constant.

Example 6.1: Suppose that a large conference room at a certain company can be reserved for no more than 4 hours. Both long and short conferences occur quite often. In fact, it can be assumed that the length X of a conference has a uniform distribution on the interval $[0, 4]$.

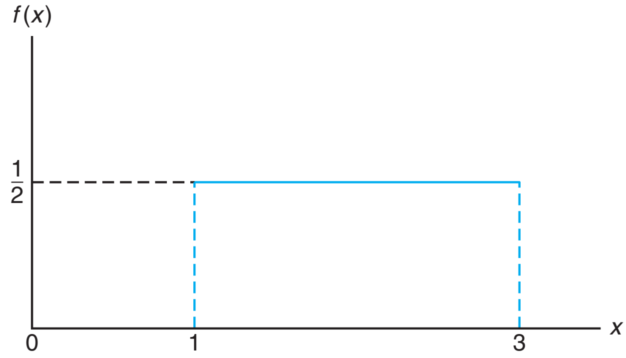

Figure 6.1: The density function for a random variable on the interval $[1, 3]$.

(a) What is the probability density function?

(b) What is the probability that any given conference lasts at least 3 hours?

Solution: (a) The appropriate density function for the uniformly distributed random variable X in this situation is

$$f(x) = \begin{cases} \frac{1}{4}, & 0 \le x \le 4, \\ 0, & \text{elsewhere.} \end{cases}$$

(b) $P[X \ge 3] = \int_3^4 \frac{1}{4}\, dx = \frac{1}{4}$.

Theorem 6.1: The mean and variance of the uniform distribution are

$$\mu = \frac{A+B}{2} \text{ and } \sigma^2 = \frac{(B-A)^2}{12}.$$

The proofs of the theorems are left to the reader. See Exercise 6.1 on page 185.

6.2 Normal Distribution

The most important continuous probability distribution in the entire field of statistics is the **normal distribution**. Its graph, called the **normal curve**, is the bell-shaped curve of Figure 6.2, which approximately describes many phenomena that occur in nature, industry, and research. For example, physical measurements in areas such as meteorological experiments, rainfall studies, and measurements of manufactured parts are often more than adequately explained with a normal distribution. In addition, errors in scientific measurements are extremely well approximated by a normal distribution. In 1733, Abraham DeMoivre developed the mathematical equation of the normal curve. It provided a basis from which much of the theory of inductive statistics is founded. The normal distribution is often referred to as the **Gaussian distribution**, in honor of Karl Friedrich Gauss

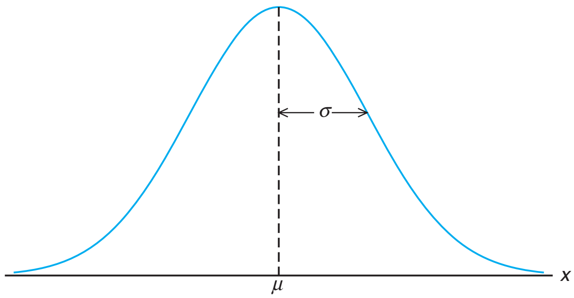

Figure 6.2: The normal curve.

(1777–1855), who also derived its equation from a study of errors in repeated measurements of the same quantity.

A continuous random variable X having the bell-shaped distribution of Figure 6.2 is called a **normal random variable**. The mathematical equation for the probability distribution of the normal variable depends on the two parameters μ and σ, its mean and standard deviation, respectively. Hence, we denote the values of the density of X by $n(x; \mu, \sigma)$.

Normal Distribution

The density of the normal random variable X, with mean μ and variance σ^2, is

$$n(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \qquad -\infty < x < \infty,$$

where $\pi = 3.14159\ldots$ and $e = 2.71828\ldots$.

Once μ and σ are specified, the normal curve is completely determined. For example, if $\mu = 50$ and $\sigma = 5$, then the ordinates $n(x; 50, 5)$ can be computed for various values of x and the curve drawn. In Figure 6.3, we have sketched two normal curves having the same standard deviation but different means. The two curves are identical in form but are centered at different positions along the horizontal axis.

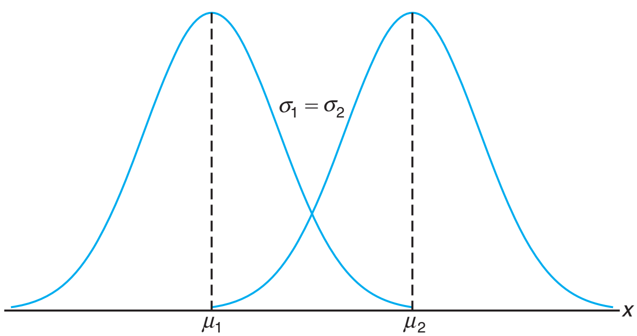

Figure 6.3: Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$.

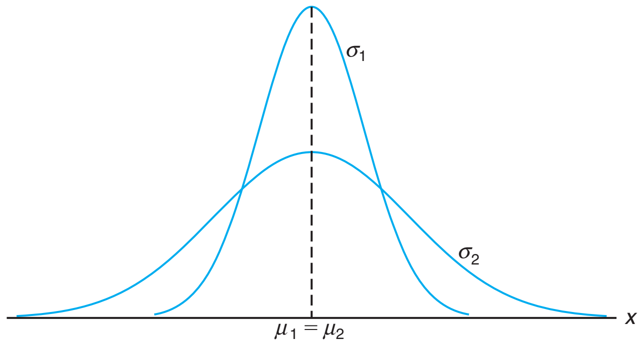

Figure 6.4: Normal curves with $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$.

In Figure 6.4, we have sketched two normal curves with the same mean but different standard deviations. This time we see that the two curves are centered at exactly the same position on the horizontal axis, but the curve with the larger standard deviation is lower and spreads out farther. Remember that the area under a probability curve must be equal to 1, and therefore the more variable the set of observations, the lower and wider the corresponding curve will be.

Figure 6.5 shows two normal curves having different means and different standard deviations. Clearly, they are centered at different positions on the horizontal axis and their shapes reflect the two different values of σ.

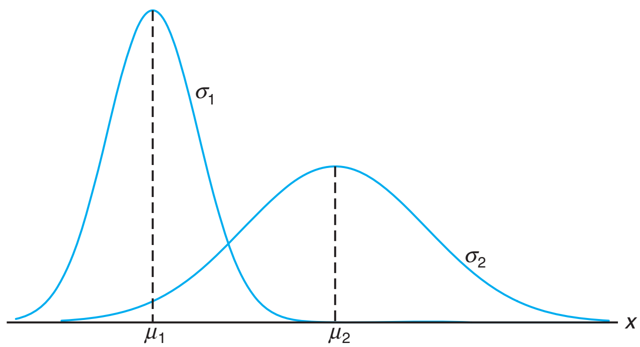

Figure 6.5: Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$.

Based on inspection of Figures 6.2 through 6.5 and examination of the first and second derivatives of $n(x; \mu, \sigma)$, we list the following properties of the normal curve:

1. The mode, which is the point on the horizontal axis where the curve is a maximum, occurs at $x = \mu$.
2. The curve is symmetric about a vertical axis through the mean μ.
3. The curve has its points of inflection at $x = \mu \pm \sigma$; it is concave downward if $\mu - \sigma < X < \mu + \sigma$ and is concave upward otherwise.

4. The normal curve approaches the horizontal axis asymptotically as we proceed in either direction away from the mean.
5. The total area under the curve and above the horizontal axis is equal to 1.

Theorem 6.2: The mean and variance of $n(x; \mu, \sigma)$ are μ and σ^2, respectively. Hence, the standard deviation is σ.

***Proof*:** To evaluate the mean, we first calculate

$$E(X-\mu) = \int_{-\infty}^{\infty} \frac{x-\mu}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Setting $z = (x-\mu)/\sigma$ and $dx = \sigma\ dz$, we obtain

$$E(X-\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz = 0,$$

since the integrand above is an odd function of z. Using Theorem 4.5 on page 128, we conclude that

$$E(X) = \mu.$$

The variance of the normal distribution is given by

$$E[(X-\mu)^2] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}[(x-\mu)/\sigma]^2}\ dx.$$

Again setting $z = (x-\mu)/\sigma$ and $dx = \sigma\ dz$, we obtain

$$E[(X-\mu)^2] = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}}\ dz.$$

Integrating by parts with $u = z$ and $dv = ze^{-z^2/2}\ dz$ so that $du = dz$ and $v = -e^{-z^2/2}$, we find that

$$E[(X-\mu)^2] = \frac{\sigma^2}{\sqrt{2\pi}} \left(-ze^{-z^2/2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-z^2/2}\ dz \right) = \sigma^2(0+1) = \sigma^2.$$

Many random variables have probability distributions that can be described adequately by the normal curve once μ and σ^2 are specified. In this chapter, we shall assume that these two parameters are known, perhaps from previous investigations. Later, we shall make statistical inferences when μ and σ^2 are unknown and have been estimated from the available experimental data.

We pointed out earlier the role that the normal distribution plays as a reasonable approximation of scientific variables in real-life experiments. There are other applications of the normal distribution that the reader will appreciate as he or she moves on in the book. The normal distribution finds enormous application as a *limiting distribution*. Under certain conditions, the normal distribution provides a good continuous approximation to the binomial and hypergeometric distributions. The case of the approximation to the binomial is covered in Section 6.5. In Chapter 8, the reader will learn about **sampling distributions**. It turns out that the limiting distribution of sample averages is normal. This provides a broad base for statistical inference that proves very valuable to the data analyst interested in estimation and hypothesis testing. Theory in the important areas such as analysis of variance (Chapters 13, 14, and 15) and quality control (Chapter 17) is based on assumptions that make use of the normal distribution.

In Section 6.3, examples demonstrate the use of tables of the normal distribution. Section 6.4 follows with examples of applications of the normal distribution.

6.3 Areas under the Normal Curve

The curve of any continuous probability distribution or density function is constructed so that the area under the curve bounded by the two ordinates $x = x_1$ and $x = x_2$ equals the probability that the random variable X assumes a value between $x = x_1$ and $x = x_2$. Thus, for the normal curve in Figure 6.6,

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} n(x;\mu,\sigma)\,dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx$$

is represented by the area of the shaded region.

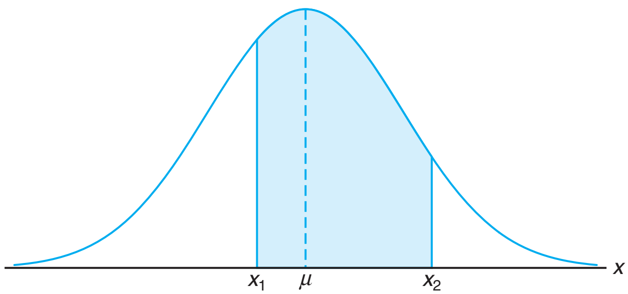

Figure 6.6: $P(x_1 < X < x_2)$ = area of the shaded region.

In Figures 6.3, 6.4, and 6.5 we saw how the normal curve is dependent on the mean and the standard deviation of the distribution under investigation. The area under the curve between any two ordinates must then also depend on the values μ and σ. This is evident in Figure 6.7, where we have shaded regions corresponding to $P(x_1 < X < x_2)$ for two curves with different means and variances. $P(x_1 < X < x_2)$, where X is the random variable describing distribution A, is indicated by the shaded area below the curve of A. If X is the random variable describing distribution B, then $P(x_1 < X < x_2)$ is given by the entire shaded region. Obviously, the two shaded regions are different in size; therefore, the probability associated with each distribution will be different for the two given values of X.

There are many types of statistical software that can be used in calculating areas under the normal curve. The difficulty encountered in solving integrals of normal density functions necessitates the tabulation of normal curve areas for quick reference. However, it would be a hopeless task to attempt to set up separate tables for every conceivable value of μ and σ. Fortunately, we are able to transform all the observations of any normal random variable X into a new set of observations

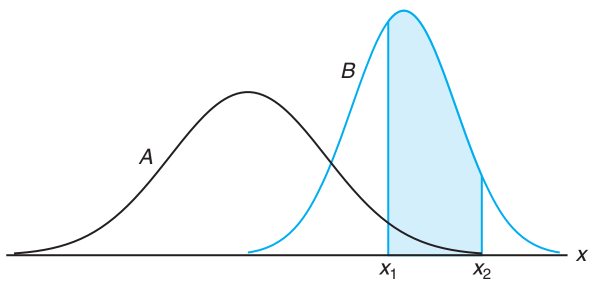

Figure 6.7: $P(x_1 < X < x_2)$ for different normal curves.

of a normal random variable Z with mean 0 and variance 1. This can be done by means of the transformation

$$Z = \frac{X - \mu}{\sigma}.$$

Whenever X assumes a value x, the corresponding value of Z is given by $z = (x - \mu)/\sigma$. Therefore, if X falls between the values $x = x_1$ and $x = x_2$, the random variable Z will fall between the corresponding values $z_1 = (x_1 - \mu)/\sigma$ and $z_2 = (x_2 - \mu)/\sigma$. Consequently, we may write

$$\begin{aligned} P(x_1 < X < x_2) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx = \frac{1}{\sqrt{2\pi}}\int_{z_1}^{z_2} e^{-\frac{1}{2}z^2}\, dz \\ &= \int_{z_1}^{z_2} n(z; 0, 1)\, dz = P(z_1 < Z < z_2), \end{aligned}$$

where Z is seen to be a normal random variable with mean 0 and variance 1.

Definition 6.1: The distribution of a normal random variable with mean 0 and variance 1 is called a **standard normal distribution**.

The original and transformed distributions are illustrated in Figure 6.8. Since all the values of X falling between x_1 and x_2 have corresponding z values between z_1 and z_2, the area under the X-curve between the ordinates $x = x_1$ and $x = x_2$ in Figure 6.8 equals the area under the Z-curve between the transformed ordinates $z = z_1$ and $z = z_2$.

We have now reduced the required number of tables of normal-curve areas to one, that of the standard normal distribution. Table A.3 indicates the area under the standard normal curve corresponding to $P(Z < z)$ for values of z ranging from -3.49 to 3.49. To illustrate the use of this table, let us find the probability that Z is less than 1.74. First, we locate a value of z equal to 1.7 in the left column; then we move across the row to the column under 0.04, where we read 0.9591. Therefore, $P(Z < 1.74) = 0.9591$. To find a z value corresponding to a given probability, the process is reversed. For example, the z value leaving an area of 0.2148 under the curve to the left of z is seen to be -0.79.

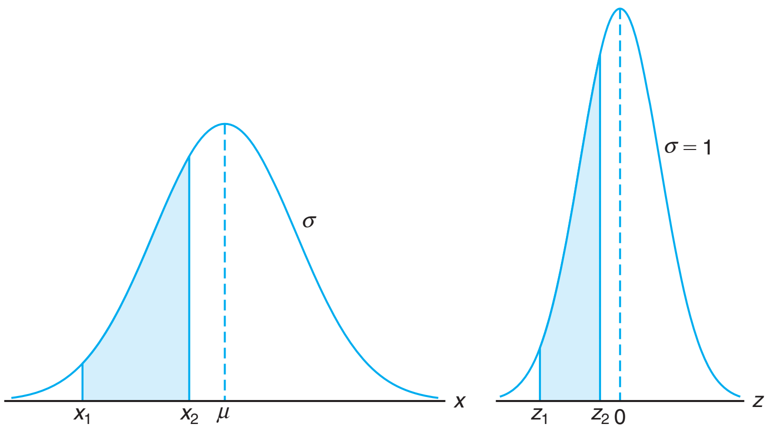

Figure 6.8: The original and transformed normal distributions.

Example 6.2: Given a standard normal distribution, find the area under the curve that lies

(a) to the right of $z = 1.84$ and

(b) between $z = -1.97$ and $z = 0.86$.

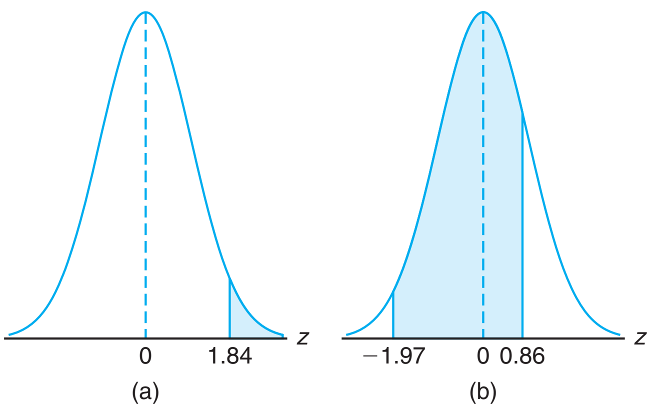

Figure 6.9: Areas for Example 6.2.

***Solution*:** See Figure 6.9 for the specific areas.

(a) The area in Figure 6.9(a) to the right of $z = 1.84$ is equal to 1 minus the area in Table A.3 to the left of $z = 1.84$, namely, $1 - 0.9671 = 0.0329$.

(b) The area in Figure 6.9(b) between $z = -1.97$ and $z = 0.86$ is equal to the area to the left of $z = 0.86$ minus the area to the left of $z = -1.97$. From Table A.3 we find the desired area to be $0.8051 - 0.0244 = 0.7807$.

Example 6.3: Given a standard normal distribution, find the value of k such that

(a) $P(Z > k) = 0.3015$ and

(b) $P(k < Z < -0.18) = 0.4197$.

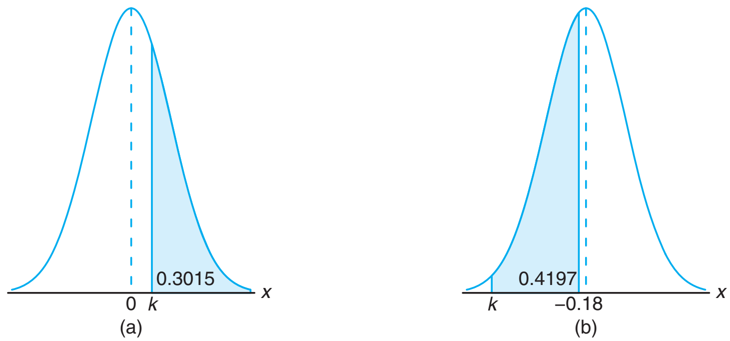

Figure 6.10: Areas for Example 6.3.

***Solution*:** Distributions and the desired areas are shown in Figure 6.10.

(a) In Figure 6.10(a), we see that the k value leaving an area of 0.3015 to the right must then leave an area of 0.6985 to the left. From Table A.3 it follows that $k = 0.52$.

(b) From Table A.3 we note that the total area to the left of -0.18 is equal to 0.4286. In Figure 6.10(b), we see that the area between k and -0.18 is 0.4197, so the area to the left of k must be $0.4286 - 0.4197 = 0.0089$. Hence, from Table A.3, we have $k = -2.37$.

Example 6.4: Given a random variable X having a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X assumes a value between 45 and 62.

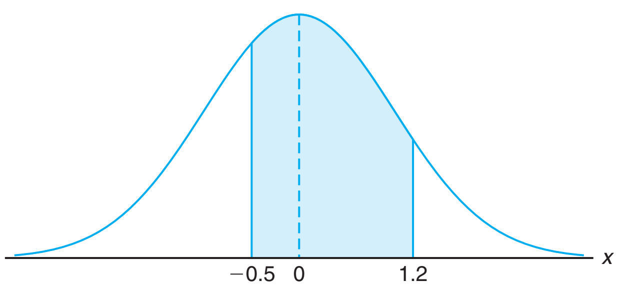

Figure 6.11: Area for Example 6.4.

***Solution*:** The z values corresponding to $x_1 = 45$ and $x_2 = 62$ are

$$z_1 = \frac{45 - 50}{10} = -0.5 \text{ and } z_2 = \frac{62 - 50}{10} = 1.2.$$

Therefore,

$$P(45 < X < 62) = P(-0.5 < Z < 1.2).$$

$P(-0.5 < Z < 1.2)$ is shown by the area of the shaded region in Figure 6.11. This area may be found by subtracting the area to the left of the ordinate $z = -0.5$ from the entire area to the left of $z = 1.2$. Using Table A.3, we have

$$P(45 < X < 62) = P(-0.5 < Z < 1.2) = P(Z < 1.2) - P(Z < -0.5)$$
$$= 0.8849 - 0.3085 = 0.5764.$$

Example 6.5: Given that X has a normal distribution with $\mu = 300$ and $\sigma = 50$, find the probability that X assumes a value greater than 362.

Solution: The normal probability distribution with the desired area shaded is shown in Figure 6.12. To find $P(X > 362)$, we need to evaluate the area under the normal curve to the right of $x = 362$. This can be done by transforming $x = 362$ to the corresponding z value, obtaining the area to the left of z from Table A.3, and then subtracting this area from 1. We find that

$$z = \frac{362 - 300}{50} = 1.24.$$

Hence,

$$P(X > 362) = P(Z > 1.24) = 1 - P(Z < 1.24) = 1 - 0.8925 = 0.1075.$$

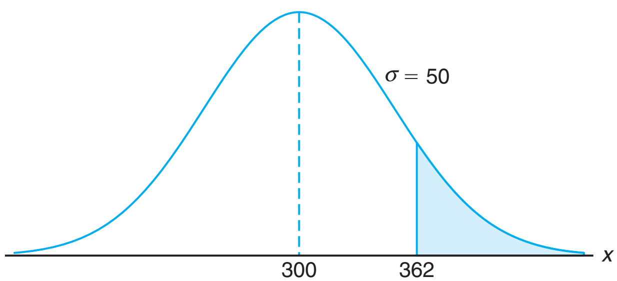

Figure 6.12: Area for Example 6.5.

According to Chebyshev's theorem on page 137, the probability that a random variable assumes a value within 2 standard deviations of the mean is at least 3/4. If the random variable has a normal distribution, the z values corresponding to $x_1 = \mu - 2\sigma$ and $x_2 = \mu + 2\sigma$ are easily computed to be

$$z_1 = \frac{(\mu - 2\sigma) - \mu}{\sigma} = -2 \text{ and } z_2 = \frac{(\mu + 2\sigma) - \mu}{\sigma} = 2.$$

Hence,

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(-2 < Z < 2) = P(Z < 2) - P(Z < -2)$$
$$= 0.9772 - 0.0228 = 0.9544,$$

which is a much stronger statement than that given by Chebyshev's theorem.

Using the Normal Curve in Reverse

Sometimes, we are required to find the value of z corresponding to a specified probability that falls between values listed in Table A.3 (see Example 6.6). For convenience, we shall always choose the z value corresponding to the tabular probability that comes closest to the specified probability.

The preceding two examples were solved by going first from a value of x to a z value and then computing the desired area. In Example 6.6, we reverse the process and begin with a known area or probability, find the z value, and then determine x by rearranging the formula

$$z = \frac{x-\mu}{\sigma} \quad \text{to give} \quad x = \sigma z + \mu.$$

Example 6.6: Given a normal distribution with $\mu = 40$ and $\sigma = 6$, find the value of x that has

(a) 45% of the area to the left and

(b) 14% of the area to the right.

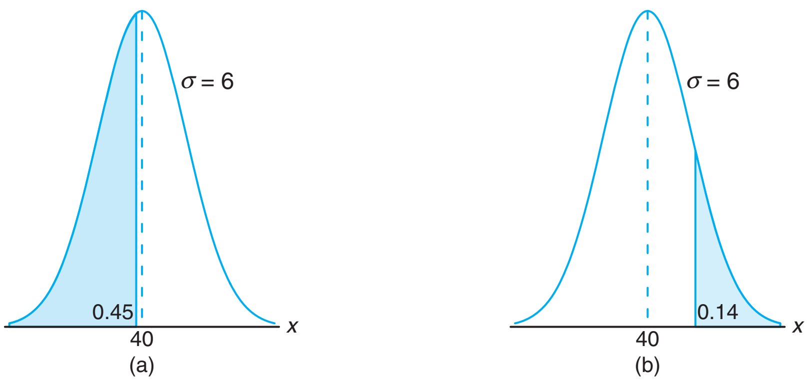

Figure 6.13: Areas for Example 6.6.

Solution: (a) An area of 0.45 to the left of the desired x value is shaded in Figure 6.13(a). We require a z value that leaves an area of 0.45 to the left. From Table A.3 we find $P(Z < -0.13) = 0.45$, so the desired z value is -0.13. Hence,

$$x = (6)(-0.13) + 40 = 39.22.$$

(b) In Figure 6.13(b), we shade an area equal to 0.14 to the right of the desired x value. This time we require a z value that leaves 0.14 of the area to the right and hence an area of 0.86 to the left. Again, from Table A.3, we find $P(Z < 1.08) = 0.86$, so the desired z value is 1.08 and

$$x = (6)(1.08) + 40 = 46.48.$$

6.4 Applications of the Normal Distribution

Some of the many problems for which the normal distribution is applicable are treated in the following examples. The use of the normal curve to approximate binomial probabilities is considered in Section 6.5.

Example 6.7: A certain type of storage battery lasts, on average, 3.0 years with a standard deviation of 0.5 year. Assuming that battery life is normally distributed, find the probability that a given battery will last less than 2.3 years.

Solution: First construct a diagram such as Figure 6.14, showing the given distribution of battery lives and the desired area. To find $P(X < 2.3)$, we need to evaluate the area under the normal curve to the left of 2.3. This is accomplished by finding the area to the left of the corresponding z value. Hence, we find that

$$z = \frac{2.3 - 3}{0.5} = -1.4,$$

and then, using Table A.3, we have

$$P(X < 2.3) = P(Z < -1.4) = 0.0808.$$

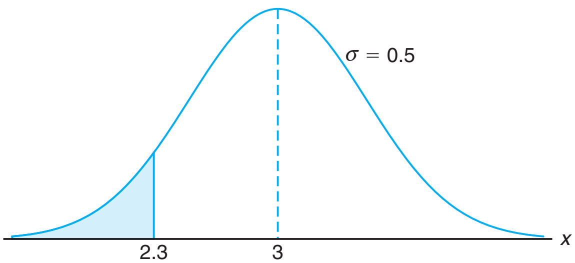

Figure 6.14: Area for Example 6.7.

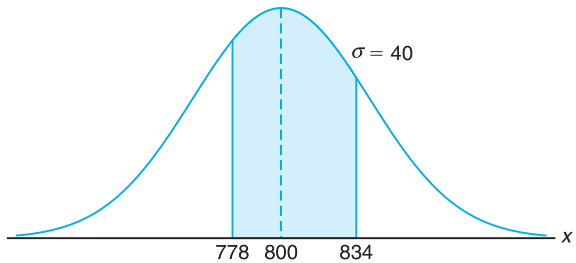

Figure 6.15: Area for Example 6.8.

Example 6.8: An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a bulb burns between 778 and 834 hours.

Solution: The distribution of light bulb life is illustrated in Figure 6.15. The z values corresponding to $x_1 = 778$ and $x_2 = 834$ are

$$z_1 = \frac{778 - 800}{40} = -0.55 \text{ and } z_2 = \frac{834 - 800}{40} = 0.85.$$

Hence,

$$P(778 < X < 834) = P(-0.55 < Z < 0.85) = P(Z < 0.85) - P(Z < -0.55)$$
$$= 0.8023 - 0.2912 = 0.5111.$$

Example 6.9: In an industrial process, the diameter of a ball bearing is an important measurement. The buyer sets specifications for the diameter to be 3.0 ± 0.01 cm. The implication is that no part falling outside these specifications will be accepted. It is known that in the process the diameter of a ball bearing has a normal distribution with mean $\mu = 3.0$ and standard deviation $\sigma = 0.005$. On average, how many manufactured ball bearings will be scrapped?

Solution: The distribution of diameters is illustrated by Figure 6.16. The values corresponding to the specification limits are $x_1 = 2.99$ and $x_2 = 3.01$. The corresponding z values are

$$z_1 = \frac{2.99 - 3.0}{0.005} = -2.0 \text{ and } z_2 = \frac{3.01 - 3.0}{0.005} = +2.0.$$

Hence,

$$P(2.99 < X < 3.01) = P(-2.0 < Z < 2.0).$$

From Table A.3, $P(Z < -2.0) = 0.0228$. Due to symmetry of the normal distribution, we find that

$$P(Z < -2.0) + P(Z > 2.0) = 2(0.0228) = 0.0456.$$

As a result, it is anticipated that, on average, 4.56% of manufactured ball bearings will be scrapped.

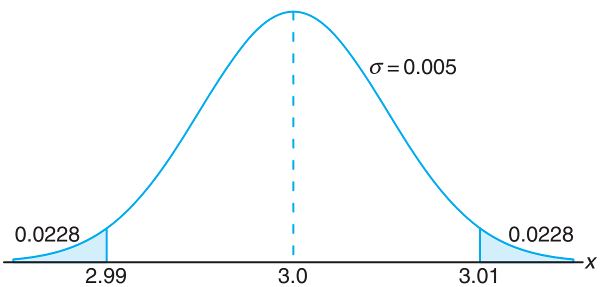

Figure 6.16: Area for Example 6.9.

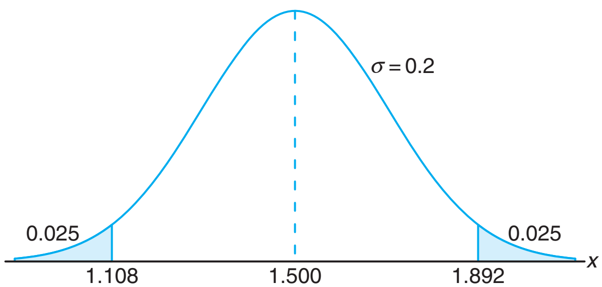

Figure 6.17: Specifications for Example 6.10.

Example 6.10: Gauges are used to reject all components for which a certain dimension is not within the specification $1.50 \pm d$. It is known that this measurement is normally distributed with mean 1.50 and standard deviation 0.2. Determine the value d such that the specifications "cover" 95% of the measurements.

Solution: From Table A.3 we know that

$$P(-1.96 < Z < 1.96) = 0.95.$$

Therefore,

$$1.96 = \frac{(1.50 + d) - 1.50}{0.2},$$

from which we obtain

$$d = (0.2)(1.96) = 0.392.$$

An illustration of the specifications is shown in Figure 6.17.

Example 6.11: A certain machine makes electrical resistors having a mean resistance of 40 ohms and a standard deviation of 2 ohms. Assuming that the resistance follows a normal distribution and can be measured to any degree of accuracy, what percentage of resistors will have a resistance exceeding 43 ohms?

***Solution*:** A percentage is found by multiplying the relative frequency by 100%. Since the relative frequency for an interval is equal to the probability of a value falling in the interval, we must find the area to the right of $x = 43$ in Figure 6.18. This can be done by transforming $x = 43$ to the corresponding z value, obtaining the area to the left of z from Table A.3, and then subtracting this area from 1. We find

$$z = \frac{43 - 40}{2} = 1.5.$$

Therefore,

$$P(X > 43) = P(Z > 1.5) = 1 - P(Z < 1.5) = 1 - 0.9332 = 0.0668.$$

Hence, 6.68% of the resistors will have a resistance exceeding 43 ohms.

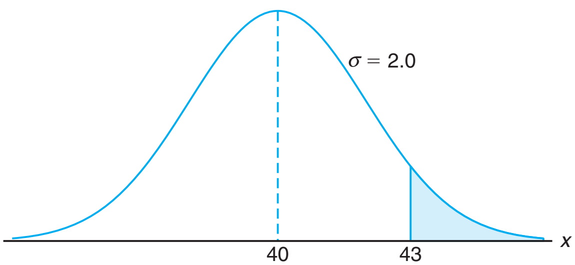

Figure 6.18: Area for Example 6.11.

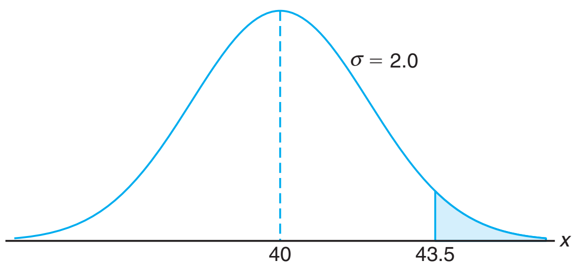

Figure 6.19: Area for Example 6.12.

Example 6.12: Find the percentage of resistances exceeding 43 ohms for Example 6.11 if resistance is measured to the nearest ohm.

***Solution*:** This problem differs from that in Example 6.11 in that we now assign a measurement of 43 ohms to all resistors whose resistances are greater than 42.5 and less than 43.5. We are actually approximating a discrete distribution by means of a continuous normal distribution. The required area is the region shaded to the right of 43.5 in Figure 6.19. We now find that

$$z = \frac{43.5 - 40}{2} = 1.75.$$

Hence,

$$P(X > 43.5) = P(Z > 1.75) = 1 - P(Z < 1.75) = 1 - 0.9599 = 0.0401.$$

Therefore, 4.01% of the resistances exceed 43 ohms when measured to the nearest ohm. The difference 6.68% − 4.01% = 2.67% between this answer and that of Example 6.11 represents all those resistance values greater than 43 and less than 43.5 that are now being recorded as 43 ohms.

Example 6.13: The average grade for an exam is 74, and the standard deviation is 7. If 12% of the class is given As, and the grades are curved to follow a normal distribution, what is the lowest possible A and the highest possible B?

***Solution*:** In this example, we begin with a known area of probability, find the z value, and then determine x from the formula $x = \sigma z + \mu$. An area of 0.12, corresponding to the fraction of students receiving As, is shaded in Figure 6.20. We require a z value that leaves 0.12 of the area to the right and, hence, an area of 0.88 to the left. From Table A.3, $P(Z < 1.18)$ has the closest value to 0.88, so the desired z value is 1.18. Hence,

$$x = (7)(1.18) + 74 = 82.26.$$

Therefore, the lowest A is 83 and the highest B is 82.

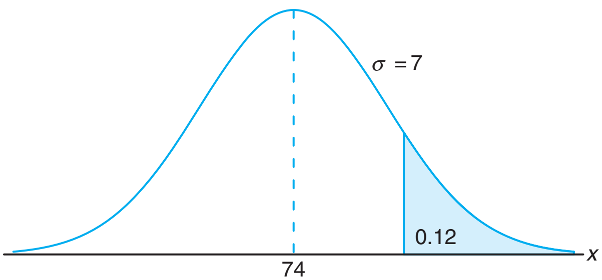

Figure 6.20: Area for Example 6.13.

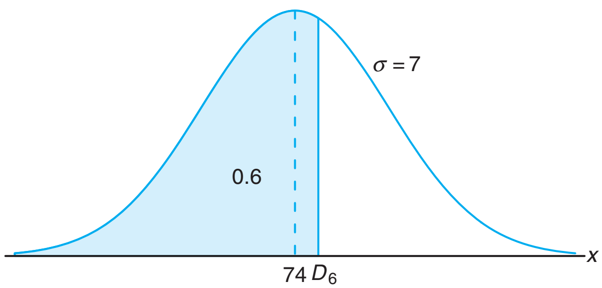

Figure 6.21: Area for Example 6.14.

Example 6.14: Refer to Example 6.13 and find the sixth decile.

***Solution*:** The sixth decile, written D_6, is the x value that leaves 60% of the area to the left, as shown in Figure 6.21. From Table A.3 we find $P(Z < 0.25) \approx 0.6$, so the desired z value is 0.25. Now $x = (7)(0.25) + 74 = 75.75$. Hence, $D_6 = 75.75$. That is, 60% of the grades are 75 or less.

Exercises

6.1 Given a continuous uniform distribution, show that

(a) $\mu = \frac{A+B}{2}$ and

(b) $\sigma^2 = \frac{(B-A)^2}{12}$.

6.2 Suppose X follows a continuous uniform distribution from 1 to 5. Determine the conditional probability $P(X > 2.5 \mid X \leq 4)$.

6.3 The daily amount of coffee, in liters, dispensed by a machine located in an airport lobby is a random variable X having a continuous uniform distribution with $A = 7$ and $B = 10$. Find the probability that on a given day the amount of coffee dispensed by this machine will be

(a) at most 8.8 liters;

(b) more than 7.4 liters but less than 9.5 liters;

(c) at least 8.5 liters.

6.4 A bus arrives every 10 minutes at a bus stop. It is assumed that the waiting time for a particular individual is a random variable with a continuous uniform distribution.

(a) What is the probability that the individual waits more than 7 minutes?
(b) What is the probability that the individual waits between 2 and 7 minutes?

6.5 Given a standard normal distribution, find the area under the curve that lies
(a) to the left of $z = -1.39$;
(b) to the right of $z = 1.96$;
(c) between $z = -2.16$ and $z = -0.65$;
(d) to the left of $z = 1.43$;
(e) to the right of $z = -0.89$;
(f) between $z = -0.48$ and $z = 1.74$.

6.6 Find the value of z if the area under a standard normal curve
(a) to the right of z is 0.3622;
(b) to the left of z is 0.1131;
(c) between 0 and z, with $z > 0$, is 0.4838;
(d) between $-z$ and z, with $z > 0$, is 0.9500.

6.7 Given a standard normal distribution, find the value of k such that
(a) $P(Z > k) = 0.2946$;
(b) $P(Z < k) = 0.0427$;
(c) $P(-0.93 < Z < k) = 0.7235$.

6.8 Given a normal distribution with $\mu = 30$ and $\sigma = 6$, find
(a) the normal curve area to the right of $x = 17$;
(b) the normal curve area to the left of $x = 22$;
(c) the normal curve area between $x = 32$ and $x = 41$;
(d) the value of x that has 80% of the normal curve area to the left;
(e) the two values of x that contain the middle 75% of the normal curve area.

6.9 Given the normally distributed variable X with mean 18 and standard deviation 2.5, find
(a) $P(X < 15)$;
(b) the value of k such that $P(X < k) = 0.2236$;
(c) the value of k such that $P(X > k) = 0.1814$;
(d) $P(17 < X < 21)$.

6.10 According to Chebyshev's theorem, the probability that any random variable assumes a value within 3 standard deviations of the mean is at least 8/9. If it is known that the probability distribution of a random variable X is normal with mean μ and variance σ^2, what is the exact value of $P(\mu - 3\sigma < X < \mu + 3\sigma)$?

6.11 A soft-drink machine is regulated so that it discharges an average of 200 milliliters per cup. If the amount of drink is normally distributed with a standard deviation equal to 15 milliliters,
(a) what fraction of the cups will contain more than 224 milliliters?
(b) what is the probability that a cup contains between 191 and 209 milliliters?
(c) how many cups will probably overflow if 230-milliliter cups are used for the next 1000 drinks?
(d) below what value do we get the smallest 25% of the drinks?

6.12 The loaves of rye bread distributed to local stores by a certain bakery have an average length of 30 centimeters and a standard deviation of 2 centimeters. Assuming that the lengths are normally distributed, what percentage of the loaves are
(a) longer than 31.7 centimeters?
(b) between 29.3 and 33.5 centimeters in length?
(c) shorter than 25.5 centimeters?

6.13 A research scientist reports that mice will live an average of 40 months when their diets are sharply restricted and then enriched with vitamins and proteins. Assuming that the lifetimes of such mice are normally distributed with a standard deviation of 6.3 months, find the probability that a given mouse will live
(a) more than 32 months;
(b) less than 28 months;
(c) between 37 and 49 months.

6.14 The finished inside diameter of a piston ring is normally distributed with a mean of 10 centimeters and a standard deviation of 0.03 centimeter.
(a) What proportion of rings will have inside diameters exceeding 10.075 centimeters?
(b) What is the probability that a piston ring will have an inside diameter between 9.97 and 10.03 centimeters?
(c) Below what value of inside diameter will 15% of the piston rings fall?

6.15 A lawyer commutes daily from his suburban home to his midtown office. The average time for a one-way trip is 24 minutes, with a standard deviation of 3.8 minutes. Assume the distribution of trip times to be normally distributed.
(a) What is the probability that a trip will take at least 1/2 hour?
(b) If the office opens at 9:00 A.M. and the lawyer leaves his house at 8:45 A.M. daily, what percentage of the time is he late for work?

(c) If he leaves the house at 8:35 A.M. and coffee is served at the office from 8:50 A.M. until 9:00 A.M., what is the probability that he misses coffee?

(d) Find the length of time above which we find the slowest 15% of the trips.

(e) Find the probability that 2 of the next 3 trips will take at least 1/2 hour.

6.16 In the November 1990 issue of *Chemical Engineering Progress*, a study discussed the percent purity of oxygen from a certain supplier. Assume that the mean was 99.61 with a standard deviation of 0.08. Assume that the distribution of percent purity was approximately normal.

(a) What percentage of the purity values would you expect to be between 99.5 and 99.7?

(b) What purity value would you expect to exceed exactly 5% of the population?

6.17 The average life of a certain type of small motor is 10 years with a standard deviation of 2 years. The manufacturer replaces free all motors that fail while under guarantee. If she is willing to replace only 3% of the motors that fail, how long a guarantee should be offered? Assume that the lifetime of a motor follows a normal distribution.

6.18 The heights of 1000 students are normally distributed with a mean of 174.5 centimeters and a standard deviation of 6.9 centimeters. Assuming that the heights are recorded to the nearest half-centimeter, how many of these students would you expect to have heights

(a) less than 160.0 centimeters?

(b) between 171.5 and 182.0 centimeters inclusive?

(c) equal to 175.0 centimeters?

(d) greater than or equal to 188.0 centimeters?

6.19 A company pays its employees an average wage of \$15.90 an hour with a standard deviation of \$1.50. If the wages are approximately normally distributed and paid to the nearest cent,

(a) what percentage of the workers receive wages between \$13.75 and \$16.22 an hour inclusive?

(b) the highest 5% of the employee hourly wages is greater than what amount?

6.20 The weights of a large number of miniature poodles are approximately normally distributed with a mean of 8 kilograms and a standard deviation of 0.9 kilogram. If measurements are recorded to the nearest tenth of a kilogram, find the fraction of these poodles with weights

(a) over 9.5 kilograms;

(b) of at most 8.6 kilograms;

(c) between 7.3 and 9.1 kilograms inclusive.

6.21 The tensile strength of a certain metal component is normally distributed with a mean of 10,000 kilograms per square centimeter and a standard deviation of 100 kilograms per square centimeter. Measurements are recorded to the nearest 50 kilograms per square centimeter.

(a) What proportion of these components exceed 10,150 kilograms per square centimeter in tensile strength?

(b) If specifications require that all components have tensile strength between 9800 and 10,200 kilograms per square centimeter inclusive, what proportion of pieces would we expect to scrap?

6.22 If a set of observations is normally distributed, what percent of these differ from the mean by

(a) more than 1.3σ?

(b) less than 0.52σ?

6.23 The IQs of 600 applicants to a certain college are approximately normally distributed with a mean of 115 and a standard deviation of 12. If the college requires an IQ of at least 95, how many of these students will be rejected on this basis of IQ, regardless of their other qualifications? Note that IQs are recorded to the nearest integers.