6.3 THE SECANT METHOD

A potential problem in implementing the Newton-Raphson method is the evaluation of the derivative. Although this is not inconvenient for polynomials and many other functions, there are certain functions whose derivatives may be extremely difficult or inconvenient to evaluate. For these cases, the derivative can be approximated by a backward finite divided difference, as in (Fig. 6.7)

$$f'(x_i) \cong \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

This approximation can be substituted into Eq. (6.6) to yield the following iterative equation:

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}} \tag{6.7}$$

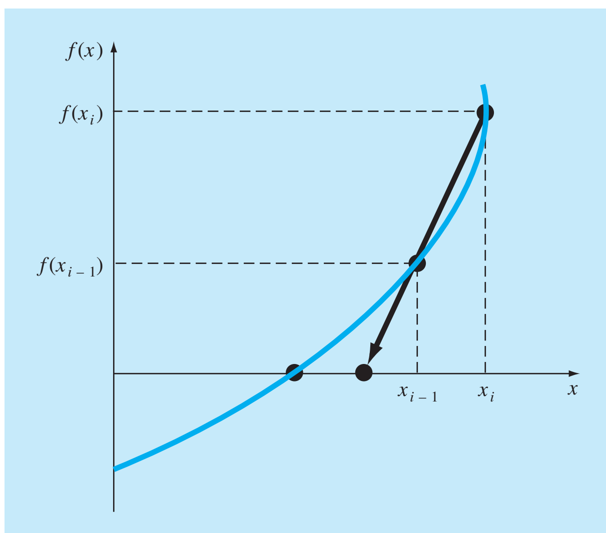


FIGURE 6.7
Graphical depiction of the secant method. This technique is similar to the Newton-Raphson technique (Fig. 6.5) in the sense that an estimate of the root is predicted by extrapolating a tangent of the function to the *x* axis. However, the secant method uses a difference rather than a derivative to estimate the slope.

Equation (6.7) is the formula for the *secant method.* Notice that the approach requires two initial estimates of x. However, because $f(x)$ is not required to change signs between the estimates, it is not classified as a bracketing method.

EXAMPLE 6.6

The Secant Method

Problem Statement. Use the secant method to estimate the root of $f(x) = e^{-x} - x$. Start with initial estimates of $x_{-1} = 0$ and $x_0 = 1.0$.

Solution. Recall that the true root is 0.56714329. . . .

First iteration:

$$x_{-1} = 0 \qquad f(x_{-1}) = 1.00000$$

$$x_0 = 1 \qquad f(x_0) = -0.63212$$

$$x_1 = 1 - \frac{-0.63212(0 - 1)}{1 - (-0.63212)} = 0.61270 \qquad \varepsilon_t = 8.0\%$$

Second iteration:

$$x_0 = 1 \qquad f(x_0) = -0.63212$$
$$x_1 = 0.61270 \qquad f(x_1) = -0.07081$$

(Note that both estimates are now on the same side of the root.)

$$x_2 = 0.61270 - \frac{-0.07081(1 - 0.61270)}{-0.63212 - (-0.07081)} = 0.56384 \qquad \varepsilon_t = 0.58\%$$

Third iteration:

$$x_1 = 0.61270 \qquad f(x_1) = -0.07081$$

$$x_2 = 0.56384 \qquad f(x_2) = 0.00518$$

$$x_3 = 0.56384 - \frac{0.00518(0.61270 - 0.56384)}{-0.07081 - (-0.00518)} = 0.56717 \qquad \varepsilon_t = 0.0048\%$$

6.3.1 The Difference Between the Secant and False-Position Methods

Note the similarity between the secant method and the false-position method. For example, Eqs. (6.7) and (5.7) are identical on a term-by-term basis. Both use two initial estimates to compute an approximation of the slope of the function that is used to project to the x axis for a new estimate of the root. However, a critical difference between the methods is how one of the initial values is replaced by the new estimate. Recall that in the false-position method the latest estimate of the root replaces whichever of the original values yielded a function value with the same sign as $f(x_r)$. Consequently, the two estimates always bracket the root. Therefore, for all practical purposes, the method always converges because the root is kept within the bracket. In contrast, the secant method replaces the values in strict sequence, with the new value x_{i+1} replacing x_i and x_i replacing x_{i-1}. As a result, the two values can sometimes lie on the same side of the root. For certain cases, this can lead to divergence.

EXAMPLE 6.7 Comparison of Convergence of the Secant and False-Position Techniques

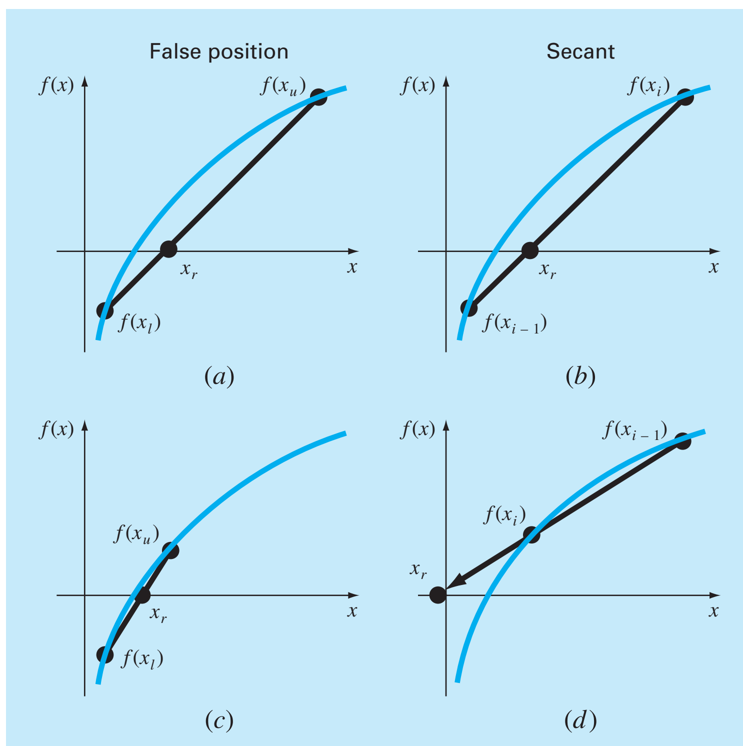
Problem Statement. Use the false-position and secant methods to estimate the root of $f(x) = \ln x$. Start the computation with values of $x_l = x_{i-1} = 0.5$ and $x_u = x_i = 5.0$.

Solution. For the false-position method, the use of Eq. (5.7) and the bracketing criterion for replacing estimates results in the following iterations:

Iteration	x_l	x_u	x_r
1	0.5	5.0	1.8546
2	0.5	1.8546	1.2163
3	0.5	1.2163	1.0585

As can be seen (Fig. 6.8*a* and *c*), the estimates are converging on the true root which is equal to 1.

FIGURE 6.8
Comparison of the false-position and the secant methods. The first iterations (*a*) and (*b*) for both techniques are identical. However, for the second iterations (*c*) and (*d*), the points used differ. As a consequence, the secant method can diverge, as indicated in (*d*).



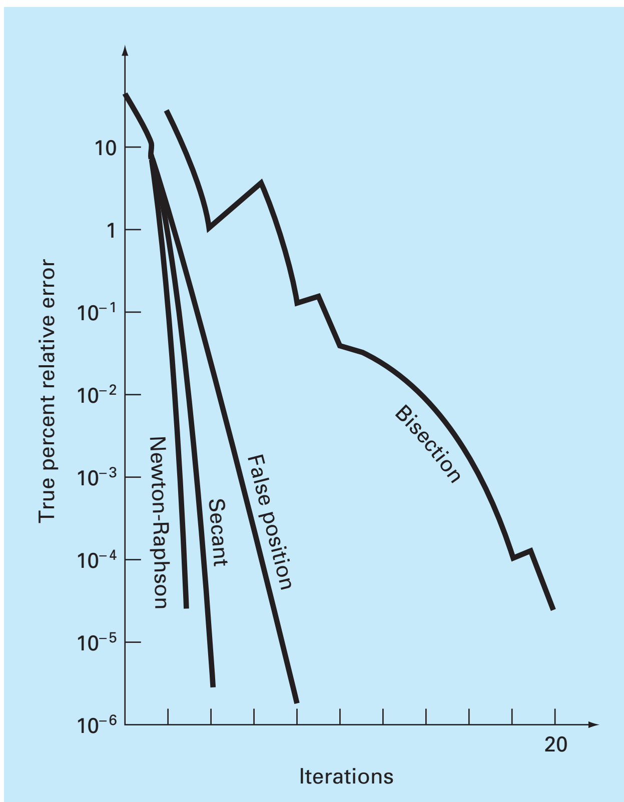
For the secant method, using Eq. (6.7) and the sequential criterion for replacing estimates results in

Iteration	x_{i-1}	x_i	x_{i+1}
1	0.5	5.0	1.8546
2	5.0	1.8546	−0.10438

As in Fig. 6.8*d*, the approach is divergent.

Although the secant method may be divergent, when it converges it usually does so at a quicker rate than the false-position method. For instance, Fig. 6.9 demonstrates the superiority of the secant method in this regard. The inferiority of the false-position method is due to one end staying fixed to maintain the bracketing of the root. This property, which is an advantage in that it prevents divergence, is a shortcoming with regard to the rate of convergence; it makes the finite-difference estimate a less-accurate approximation of the derivative.

FIGURE 6.9
Comparison of the true percent relative errors ε_t for the methods to determine the roots of $f(x) = e^{-x} - x$.



PROBLEMS

6.1 Use simple fixed-point iteration to locate the root of

$$f(x) = \sin(\sqrt{x}) - x$$

Use an initial guess of $x_0 = 0.5$ and iterate until $\varepsilon_a \leq 0.01\%$. Verify that the process is linearly convergent as described in Box 6.1.

6.2 Determine the highest real root of

$$f(x) = 2x^3 - 11.7x^2 + 17.7x - 5$$

(a) Graphically.
(b) Fixed-point iteration method (three iterations, $x_0 = 3$). Note: Make certain that you develop a solution that converges on the root.
(c) Newton-Raphson method (three iterations, $x_0 = 3$).
(d) Secant method (three iterations, $x_{-1} = 3$, $x_0 = 4$).
(e) Modified secant method (three iterations, $x_0 = 3$, $\delta = 0.01$).
Compute the approximate percent relative errors for your solutions.

6.3 Use **(a)** fixed-point iteration and **(b)** the Newton-Raphson method to determine a root of $f(x) = -0.9x^2 + 1.7x + 2.5$ using $x_0 = 5$. Perform the computation until ε_a is less than $\varepsilon_s = 0.01\%$. Also perform an error check of your final answer.

6.4 Determine the real roots of $f(x) = -1 + 5.5x - 4x^2 + 0.5x^3$: **(a)** graphically and **(b)** using the Newton-Raphson method to within $\varepsilon_s = 0.01\%$.

6.5 Employ the Newton-Raphson method to determine a real root for $f(x) = -1 + 5.5x - 4x^2 + 0.5x^3$ using initial guesses of **(a)** 4.52

and **(b)** 4.54. Discuss and use graphical and analytical methods to explain any peculiarities in your results.

6.6 Determine the lowest real root of $f(x) = -12 - 21x + 18x^2 - 2.4x^3$: **(a)** graphically and **(b)** using the secant method to a value of ε_s corresponding to three significant figures.

6.7 Locate the first positive root of

$$f(x) = \sin x + \cos(1 + x^2) - 1$$

where x is in radians. Use four iterations of the secant method with initial guesses of **(a)** $x_{i-1} = 1.0$ and $x_i = 3.0$; **(b)** $x_{i-1} = 1.5$ and $x_i = 2.5$, and **(c)** $x_{i-1} = 1.5$ and $x_i = 2.25$ to locate the root. **(d)** Use the graphical method to explain your results.

6.8 Determine the real root of $x^{3.5} = 80$, with the Newton's method to within $\varepsilon_s = 0.1\%$ using an initial guess of $x_0 = 3.5$ and

6.9 Determine the highest real root of $f(x) = x^3 - 6x^2 + 11x - 6.1$:

(a) Graphically.

(b) Using the Newton-Raphson method (three iterations, $x_i = 3.5$).

(c) Using the secant method (three iterations, $x_{i+1} = 2.5$ and $x_i = 3.5$).

6.10 Determine the lowest positive root of $f(x) = 7\sin(x)e^{-x} - 1$:

(a) Graphically.

(b) Using the Newton-Raphson method (three iterations, $x_i = 0.3$).

(c) Using the secant method (five iterations, $x_{i-1} = 0.5$ and $x_i = 0.4$).

6.11 Use the Newton-Raphson method to find the root of

$$f(x) = e^{-0.5x}(4 - x) - 2$$

Employ initial guesses of **(a)** 2, **(b)** 6, and **(c)** 8. Explain your results.

6.12 Given

$$f(x) = -2x^6 - 1.5x^4 + 10x + 2$$

Use a root location technique to determine the maximum of this function. Perform iterations until the approximate relative error falls below 5%. If you use a bracketing method, use initial guesses of $x_l = 0$ and $x_u = 1$. If you use the Newton-Raphson or the modified secant method, use an initial guess of $x_i = 1$. If you use the secant method, use initial guesses of $x_{i-1} = 0$ and $x_i = 1$. Assuming that convergence is not an issue, choose the technique that is best suited to this problem. Justify your choice.

6.13 You must determine the root of the following easily differentiable function,

$$e^{0.5x} = 5 - 5x$$

Pick the best numerical technique, justify your choice and then use that technique to determine the root. Note that it is known that for positive initial guesses, all techniques except fixed-point iteration will eventually converge. Perform iterations until the approximate relative error falls below 2%. If you use a bracketing method, use initial guesses of $x_l = 0$ and $x_u = 2$. If you use the Newton-Raphson or the modified secant method, use an initial guess of $x_i = 0.7$. If you use the secant method, use initial guesses of $x_{i-1} = 0$ and $x_i = 2$.

6.14 Use **(a)** the Newton-Raphson method to determine a root of $f(x) = x^5 - 16.05x^4 + 88.75x^3 - 192.0375x^2 + 116.35x + 31.6875$ using an initial guess of $x = 0.5825$ and $\varepsilon_s = 0.01\%$. Explain your results.

6.16 **(a)** Apply the Newton-Raphson method to the function $f(x) = \tanh(x^2 - 9)$ to evaluate its known real root at $x = 3$. Use an initial guess of $x_0 = 3.2$ and take a minimum of four iterations. **(b)** Did the method exhibit convergence onto its real root? Sketch the plot with the results for each iteration shown.

6.17 The polynomial $f(x) = 0.0074x^4 - 0.284x^3 + 3.355x^2 - 12.183x + 5$ has a real root between 15 and 20. Apply the Newton-Raphson method to this function using an initial guess of $x_0 = 16.15$. Explain your results.

6.18 Use the secant method on the circle function $(x + 1)^2 + (y - 2)^2 = 16$ to find a positive real root. Set your initial guess to $x_i = 3$ and $x_{i-1} = 0.5$. Approach the solution from the first and fourth quadrants. When solving for $f(x)$ in the fourth quadrant, be sure to take the negative value of the square root. Why does your solution diverge?