CHAPTER 9

Gauss Elimination

This chapter deals with simultaneous linear algebraic equations that can be represented generally as

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdot \qquad\qquad \cdot \qquad\qquad & \\
\cdot \qquad\qquad \cdot \qquad\qquad & \\
\cdot \qquad\qquad \cdot \qquad\qquad & \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
\tag{9.1}
$$

where the a's are constant coefficients and the b's are constants.

The technique described in this chapter is called *Gauss elimination* because it involves combining equations to eliminate unknowns. Although it is one of the earliest methods for solving simultaneous equations, it remains among the most important algorithms in use today and is the basis for linear equation solving on many popular software packages.

9.1 SOLVING SMALL NUMBERS OF EQUATIONS

Before proceeding to the computer methods, we will describe several methods that are appropriate for solving small ($n \leq 3$) sets of simultaneous equations and that do not require a computer. These are the graphical method, Cramer's rule, and the elimination of unknowns.

9.1.1 The Graphical Method

A graphical solution is obtainable for two equations by plotting them on Cartesian coordinates with one axis corresponding to x_1 and the other to x_2. Because we are dealing with linear systems, each equation is a straight line. This can be easily illustrated for the general equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 &= b_1 \\
a_{21}x_1 + a_{22}x_2 &= b_2
\end{aligned}
$$

Both equations can be solved for x_2:

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}}$$

$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$

Thus, the equations are now in the form of straight lines; that is, $x_2 = \text{(slope) } x_1 + \text{intercept}$. These lines can be graphed on Cartesian coordinates with x_2 as the ordinate and x_1 as the abscissa. The values of x_1 and x_2 at the intersection of the lines represent the solution.

EXAMPLE 9.1

The Graphical Method for Two Equations

Problem Statement. Use the graphical method to solve

$$3x_1 + 2x_2 = 18 \qquad \text{(E9.1.1)}$$

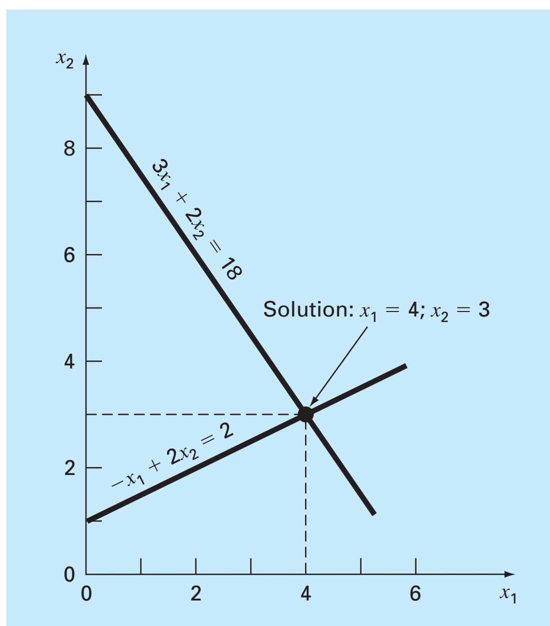
$$-x_1 + 2x_2 = 2 \qquad \text{(E9.1.2)}$$

Solution. Let x_1 be the abscissa. Solve Eq. (E9.1.1) for x_2:

$$x_2 = -\frac{3}{2}x_1 + 9$$

which, when plotted on Fig. 9.1, is a straight line with an intercept of 9 and a slope of $-3/2$.

FIGURE 9.1
Graphical solution of a set of two simultaneous linear algebraic equations. The intersection of the lines represents the solution.



Equation (E9.1.2) can also be solved for x_2:

$$x_2 = \frac{1}{2}x_1 + 1$$

which is also plotted on Fig. 9.1. The solution is the intersection of the two lines at $x_1 = 4$ and $x_2 = 3$. This result can be checked by substituting these values into the original equations to yield

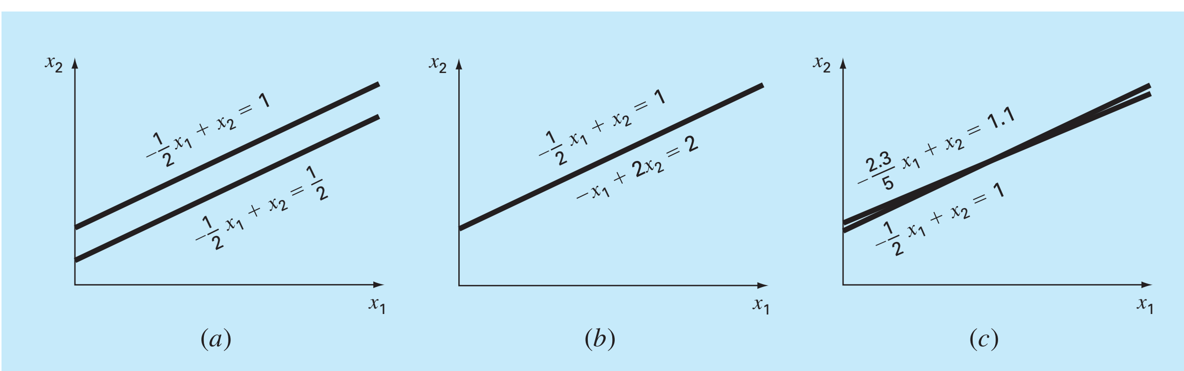
$$3(4) + 2(3) = 18$$
$$-(4) + 2(3) = 2$$

Thus, the results are equivalent to the right-hand sides of the original equations.

For three simultaneous equations, each equation would be represented by a plane in a three-dimensional coordinate system. The point where the three planes intersect would represent the solution. Beyond three equations, graphical methods break down and, consequently, have little practical value for solving simultaneous equations. However, they sometimes prove useful in visualizing properties of the solutions. For example, Fig. 9.2 depicts three cases that can pose problems when solving sets of linear equations. Figure 9.2*a* shows the case where the two equations represent parallel lines. For such situations, there is no solution because the lines never cross. Figure 9.2*b* depicts the case where the two lines are coincident. For such situations there is an infinite number of solutions. Both types of systems are said to be *singular*. In addition, systems that are very close to being singular (Fig. 9.2*c*) can also cause problems. These systems are said to be *ill-conditioned*. Graphically, this corresponds to the fact that it is difficult to identify the exact point at which the lines intersect. Ill-conditioned systems will also pose problems when they are encountered during the numerical solution of linear equations. This is because they will be extremely sensitive to round-off error (recall Sec. 4.2.3).

FIGURE 9.2

Graphical depiction of singular and ill-conditioned systems: (*a*) no solution, (*b*) infinite solutions, and (*c*) ill-conditioned system where the slopes are so close that the point of intersection is difficult to detect visually.



9.2 NAIVE GAUSS ELIMINATION

In the previous section, the elimination of unknowns was used to solve a pair of simultaneous equations. The procedure consisted of two steps:

1. The equations were manipulated to eliminate one of the unknowns from the equations. The result of this *elimination* step was that we had one equation with one unknown.
2. Consequently, this equation could be solved directly and the result *back-substituted* into one of the original equations to solve for the remaining unknown.

This basic approach can be extended to large sets of equations by developing a systematic scheme or algorithm to eliminate unknowns and to back-substitute. *Gauss elimination* is the most basic of these schemes.

This section includes the systematic techniques for forward elimination and back substitution that comprise Gauss elimination. Although these techniques are ideally suited for implementation on computers, some modifications will be required to obtain a reliable algorithm. In particular, the computer program must avoid division by zero. The following method is called *"naive" Gauss elimination* because it does not avoid this problem. Subsequent sections will deal with the additional features required for an effective computer program.

The approach is designed to solve a general set of n equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \tag{9.12a}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \tag{9.12b}$$

$$\vdots \qquad\qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \tag{9.12c}$$

As was the case with the solution of two equations, the technique for n equations consists of two phases: elimination of unknowns and solution through back substitution.

Forward Elimination of Unknowns. The first phase is designed to reduce the set of equations to an upper triangular system (Fig. 9.3). The initial step will be to eliminate the first unknown, x_1, from the second through the nth equations. To do this, multiply Eq. (9.12a) by a_{21}/a_{11} to give

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \cdots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1 \tag{9.13}$$

Now, this equation can be subtracted from Eq. (9.12b) to give

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \cdots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

or

$$a'_{22}x_2 + \cdots + a'_{2n}x_n = b'_2$$

where the prime indicates that the elements have been changed from their original values.

The procedure is then repeated for the remaining equations. For instance, Eq. (9.12a) can be multiplied by a_{31}/a_{11} and the result subtracted from the third equation. Repeating

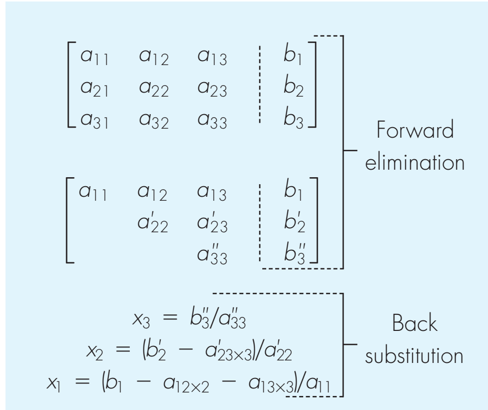


FIGURE 9.3
The two phases of Gauss elimination: forward elimination and back substitution. The primes indicate the number of times that the coefficients and constants have been modified.

the procedure for the remaining equations results in the following modified system:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \tag{9.14a}$$

$$a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n = b'_2 \tag{9.14b}$$

$$a'_{32}x_2 + a'_{33}x_3 + \cdots + a'_{3n}x_n = b'_3 \tag{9.14c}$$

$$\begin{array}{cc} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}$$

$$a'_{n2}x_2 + a'_{n3}x_3 + \cdots + a'_{nn}x_n = b'_n \tag{9.14d}$$

For the foregoing steps, Eq. (9.12*a*) is called the *pivot equation* and a_{11} is called the *pivot coefficient* or *element*. Note that the process of multiplying the first row by a_{21}/a_{11} is equivalent to dividing it by a_{11} and multiplying it by a_{21}. Sometimes the division operation is referred to as normalization. We make this distinction because a zero pivot element can interfere with normalization by causing a division by zero. We will return to this important issue after we complete our description of naive Gauss elimination.

Now repeat the above to eliminate the second unknown from Eq. (9.14*c*) through (9.14*d*). To do this multiply Eq. (9.14*b*) by a'_{32}/a'_{22} and subtract the result from Eq. (9.14*c*). Perform a similar elimination for the remaining equations to yield

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\ a''_{33}x_3 + \cdots + a''_{3n}x_n &= b''_2 \\ \cdot \qquad \cdot & \\ \cdot \qquad \cdot & \\ \cdot \qquad \cdot & \\ a''_{n3}x_3 + \cdots + a''_{nn}x_n &= b''_n \end{aligned}$$

where the double prime indicates that the elements have been modified twice.

The procedure can be continued using the remaining pivot equations. The final manipulation in the sequence is to use the $(n - 1)$th equation to eliminate the x_{n-1} term from the nth equation. At this point, the system will have been transformed to an upper triangular system (recall Box PT3.1):

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \tag{9.15a}$$

$$a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n = b'_2 \tag{9.15b}$$

$$a''_{33}x_3 + \cdots + a''_{3n}x_n = b''_3 \tag{9.15c}$$

$$\vdots \qquad\qquad \vdots$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)} \tag{9.15d}$$

Pseudocode to implement forward elimination is presented in Fig. 9.4*a*. Notice that three nested loops provide a concise representation of the process. The outer loop moves down the matrix from one pivot row to the next. The middle loop moves below the pivot row to each of the subsequent rows where elimination is to take place. Finally, the innermost loop progresses across the columns to eliminate or transform the elements of a particular row.

Back Substitution. Equation (9.15*d*) can now be solved for x_n:

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}} \tag{9.16}$$

This result can be back-substituted into the $(n - 1)$th equation to solve for x_{n-1}. The procedure, which is repeated to evaluate the remaining x's, can be represented by the following formula:

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)}x_j}{a_{ii}^{(i-1)}} \qquad \text{for } i = n - 1, n - 2, \ldots, 1 \tag{9.17}$$

FIGURE 9.4
Pseudocode to perform (*a*) forward elimination and (*b*) back substitution.

(*a*)
```
DOFOR k = 1, n − 1
  DOFOR i = k + 1, n
    factor = a(i,k) / a(k,k)
    DOFOR j = k + 1 to n
      a(i,j) = a(i,j) − factor · a(k,j)
    END DO
    b(i) = b(i) − factor · b(k)
  END DO
END DO
```

(*b*)
```
x(n) = b(n) / a(n,n)
DOFOR i = n − 1, 1, −1
  sum = b(i)
  DOFOR j = i + 1, n
    sum = sum − a(i,j) · x(j)
  END DO
  x(i) = sum / a(i,i)
END DO
```

Pseudocode to implement Eqs. (9.16) and (9.17) is presented in Fig. 9.4*b.* Notice the similarity between this pseudocode and that in Fig. PT3.4 for matrix multiplication. As with Fig. PT3.4, a temporary variable, *sum,* is used to accumulate the summation from Eq. (9.17). This results in a somewhat faster execution time than if the summation were accumulated in b_i. More importantly, it allows efficient improvement in precision if the variable, *sum,* is declared in double precision.

EXAMPLE 9.5

Naive Gauss Elimination

Problem Statement. Use Gauss elimination to solve

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \tag{E9.5.1}$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \tag{E9.5.2}$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \tag{E9.5.3}$$

Carry six significant figures during the computation.

Solution. The first part of the procedure is forward elimination. Multiply Eq. (E9.5.1) by (0.1)/3 and subtract the result from Eq. (E9.5.2) to give

$$7.00333x_2 - 0.293333x_3 = -19.5617$$

Then multiply Eq. (E9.5.1) by (0.3)/3 and subtract it from Eq. (E9.5.3) to eliminate x_1. After these operations, the set of equations is

$$3x_1 \quad -0.1x_2 \quad -0.2x_3 = 7.85 \tag{E9.5.4}$$

$$7.00333x_2 - 0.293333x_3 = -19.5617 \tag{E9.5.5}$$

$$-0.190000x_2 + 10.0200x_3 = 70.6150 \tag{E9.5.6}$$

To complete the forward elimination, x_2 must be removed from Eq. (E9.5.6). To accomplish this, multiply Eq. (E9.5.5) by $-0.190000/7.00333$ and subtract the result from Eq. (E9.5.6). This eliminates x_2 from the third equation and reduces the system to an upper triangular form, as in

$$3x_1 \quad -0.1x_2 \quad -0.2x_3 = 7.85 \tag{E9.5.7}$$

$$7.00333x_2 - 0.293333x_3 = -19.5617 \tag{E9.5.8}$$

$$10.0120x_3 = 70.0843 \tag{E9.5.9}$$

We can now solve these equations by back substitution. First, Eq. (E9.5.9) can be solved for

$$x_3 = \frac{70.0843}{10.0120} = 7.0000 \tag{E9.5.10}$$

This result can be back-substituted into Eq. (E9.5.8):

$$7.00333x_2 - 0.293333(7.0000) = -19.5617$$

which can be solved for

$$x_2 = \frac{-19.5617 + 0.293333(7.0000)}{7.00333} = -2.50000 \tag{E9.5.11}$$

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & b_1 \\ a_{21} & a_{22} & a_{23} & \vdots & b_2 \\ a_{31} & a_{32} & a_{33} & \vdots & b_3 \end{bmatrix}
$$

$$\downarrow$$

$$
\begin{bmatrix} 1 & 0 & 0 & \vdots & b_1^{(n)} \\ 0 & 1 & 0 & \vdots & b_2^{(n)} \\ 0 & 0 & 1 & \vdots & b_3^{(n)} \end{bmatrix}
$$

$$\downarrow$$

$$
\begin{aligned} x_1 \qquad\qquad\qquad &= b_1^{(n)} \\ x_2 \qquad\quad &= b_2^{(n)} \\ x_3 &= b_3^{(n)} \end{aligned}
$$

FIGURE 9.9
Graphical depiction of the Gauss-Jordan method. Compare with Fig. 9.3 to elucidate the differences between this technique and Gauss elimination. The superscript (n) means that the elements of the right-hand-side vector have been modified n times (for this case, $n = 3$).

9.7 GAUSS-JORDAN

The Gauss-Jordan method is a variation of Gauss elimination. The major difference is that when an unknown is eliminated in the Gauss-Jordan method, it is eliminated from all other equations rather than just the subsequent ones. In addition, all rows are normalized by dividing them by their pivot elements. Thus, the elimination step results in an identity matrix rather than a triangular matrix (Fig. 9.9). Consequently, it is not necessary to employ back substitution to obtain the solution. The method is best illustrated by an example.

EXAMPLE 9.12

Gauss-Jordan Method

Problem Statement. Use the Gauss-Jordan technique to solve the same system as in Example 9.5:

$$
\begin{aligned}
3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\
0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\
0.3x_1 - 0.2x_2 + 10x_3 &= 71.4
\end{aligned}
$$

Solution. First, express the coefficients and the right-hand side as an augmented matrix:

$$\begin{bmatrix} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

Then normalize the first row by dividing it by the pivot element, 3, to yield

$$\begin{bmatrix} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

The x_1 term can be eliminated from the second row by subtracting 0.1 times the first row from the second row. Similarly, subtracting 0.3 times the first row from the third row will eliminate the x_1 term from the third row:

$$\begin{bmatrix} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0 & 7.00333 & -0.293333 & -19.5617 \\ 0 & -0.190000 & 10.0200 & 70.6150 \end{bmatrix}$$

Next, normalize the second row by dividing it by 7.00333:

$$\begin{bmatrix} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & -0.190000 & 10.0200 & 70.6150 \end{bmatrix}$$

Reduction of the x_2 terms from the first and third equations gives

$$\begin{bmatrix} 1 & 0 & -0.0680629 & 2.52356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 10.01200 & 70.0843 \end{bmatrix}$$

The third row is then normalized by dividing it by 10.0120:

$$\begin{bmatrix} 1 & 0 & -0.0680629 & 2.52356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 1 & 7.0000 \end{bmatrix}$$

Finally, the x_3 terms can be reduced from the first and the second equations to give

$$\begin{bmatrix} 1 & 0 & 0 & 3.0000 \\ 0 & 1 & 0 & -2.5000 \\ 0 & 0 & 1 & 7.0000 \end{bmatrix}$$

Thus, as depicted in Fig. 9.9, the coefficient matrix has been transformed to the identity matrix, and the solution is obtained in the right-hand-side vector. Notice that no back substitution was required to obtain the solution.

PROBLEMS

9.8 Given the equations

$$10x_1 + 2x_2 - x_3 = 27$$
$$-3x_1 - 6x_2 + 2x_3 = -61.5$$
$$x_1 + x_2 + 5x_3 = -21.5$$

(a) Solve by naive Gauss elimination. Show all steps of the computation.
(b) Substitute your results into the original equations to check your answers.

9.9 Use Gauss elimination to solve:

$$8x_1 + 2x_2 - 2x_3 = -2$$
$$10x_1 + 2x_2 + 4x_3 = 4$$
$$12x_1 + 2x_2 + 2x_3 = 6$$

Employ partial pivoting and check your answers by substituting them into the original equations.

9.10 Given the system of equations

$$-3x_2 + 7x_3 = 2$$
$$x_1 + 2x_2 - x_3 = 3$$
$$5x_1 - 2x_2 = 2$$

(c) Use Gauss elimination with partial pivoting to solve for the x's.
(d) Substitute your results back into the original equations to check your solution.

9.11 Given the equations

$$2x_1 - 6x_2 - x_3 = -38$$
$$-3x_1 - x_2 + 7x_3 = -34$$
$$-8x_1 + x_2 - 2x_3 = -20$$

(a) Solve by Gauss elimination with partial pivoting. Show all steps of the computation.
(b) Substitute your results into the original equations to check your answers.

9.12 Use Gauss-Jordan elimination to solve:

$$2x_1 + x_2 - x_3 = 1$$
$$5x_1 + 2x_2 + 2x_3 = -4$$
$$3x_1 + x_2 + x_3 = 5$$

Do not employ pivoting. Check your answers by substituting them into the original equations.

9.13 Solve:

$$x_1 + x_2 - x_3 = -3$$
$$6x_1 + 2x_2 + 2x_3 = 2$$
$$-3x_1 + 4x_2 + x_3 = 1$$

with **(a)** naive Gauss elimination, **(b)** Gauss elimination with partial pivoting, and **(c)** Gauss-Jordan without partial pivoting.

2.3 The Inverse of a Matrix and Matrix Pathology

Division by a matrix is not defined but the equivalent is obtained from the *inverse* of the matrix. If the product of two square matrices, $A * B$, equals the identity matrix, I, B is said to be the inverse of A (and A is the inverse of B). The usual notation for the inverse of matrix A is A^{-1}. We have said that matrices do not commute on multiplication but inverses are an exception: $A * A^{-1} = A^{-1} * A$.

One way to find the inverse of matrix A is to employ the minors of its determinant but this is not efficient. The better way is to use an elimination method. We augment the A matrix with the identity matrix of the same size and solve. The solution is A^{-1}. This is equivalent to solving the system with n right-hand sides, each column being one of the n unit basis vectors in turn. Here is an example:

EXAMPLE 2.2 Given matrix A, find its inverse. First use the Gauss–Jordan method with exact arithmetic.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Augment A with the identity matrix and then reduce:

$$\begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -5 & -3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(1)} \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & 0 & 1 & -3 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & -1 & 0 & 1 & \frac{2}{5} & -\frac{6}{5} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} \end{bmatrix}.$$

We confirm the fact that we have found the inverse by multiplication:

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{5} & -\frac{1}{5} \\ -1 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$