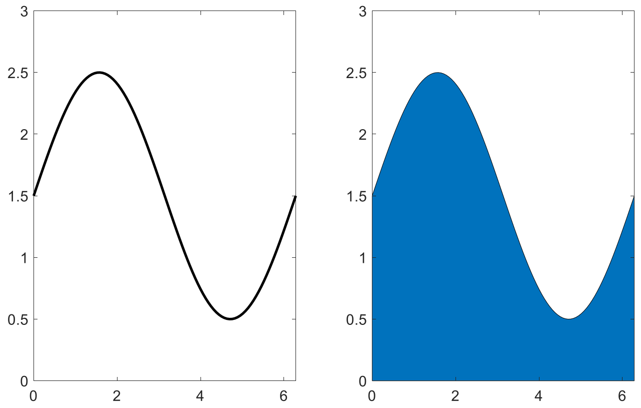
Numerical Integration

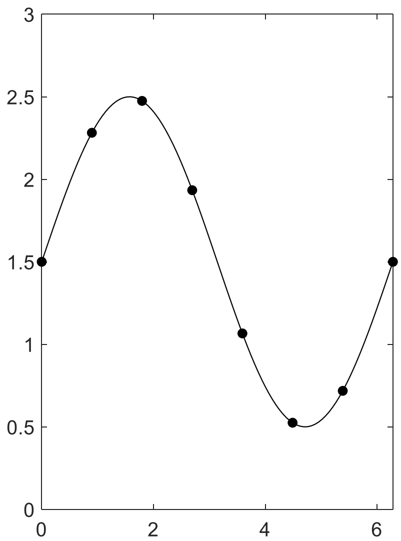
Consider the function: $f(x) = 1.5 + \sin(x)$. The curve of this function is:



The integration of this equation over the indicated period is the area under the curve:

$$I = \int_a^b f(x)dx$$

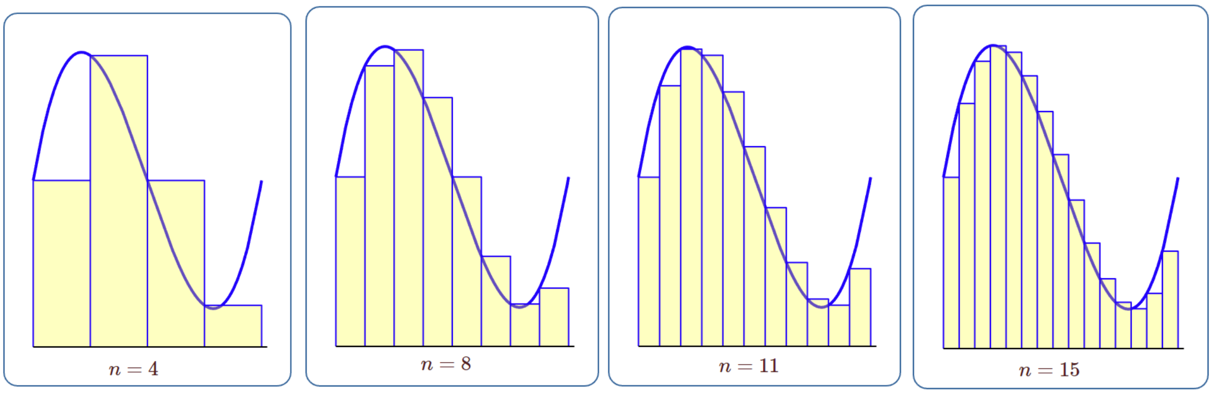
If we sample the function into equally spaced points, we can approximate this integration through several numerical methods, below are the selected ones for this course:



x_i	$f(x_i)$
0	1.5
0.8976	2.2818
1.7952	2.4749
2.6928	1.9339
3.5904	1.0661
4.4880	0.5251
5.3856	0.7182
6.2832	1.5

Rectangular Rule

The simplest way is to add up the values of these samples. This means we divide the area into n rectangles and sum up their areas to compute the overall value. The simplicity of this method comes at the price of errors. However, as n becomes larger the error decreases.



With: $h = \dfrac{b-a}{n}$ The area is $I = h\sum_{i=0}^{n-1} f(x_i)$

Example1: Calculate the following integral using rectangular rule. Use $n = 6$.

$$\int_0^{1.2} \cos x \, dx$$

Solution: here we have $f(x) = \cos(x)$, and $h = \frac{1.2-0}{6} = 0.2$

i	x_i	$f(x_i)$
0	0	1
1	0.2	0.9801
2	0.4	0.9211
3	0.6	0.8253
4	0.8	0.6967
5	1	0.5403
6	1.2	0.3624

$$I = h\sum_{i=0}^{n-1} f(x_i) = 0.2(1 + 0.9801 + \cdots + 0.5403) = 0.9927$$

Example2: repeat **Example1** with $n = 8, 11, 15$.

Answer: $I_8 = 0.9781$. $I_{11} = 0.9659$, $I_{15} = 0.9570$. For very large n, we get $I \approx$ the exact value which is 0.932.

Example3: Calculate the following integral using rectangular rule. Use $n = 8$.

$$\int_{-2}^{3} \{0.5 + \sin(x)\} dx$$

Solution: here we have $f(x) = 0.5 + \sin(x)$, and $h = 0.625$.

i	x_i	$f(x_i)$
0	−2	−0.4093
1	−1.375	−0.4809
2	−0.75	−0.1816
3	−0.125	0.3753
4	0.5	0.9794
5	1.125	1.4023
6	1.75	1.4840
7	2.375	1.1937
8	3	0.6411

$$I = 0.625 \sum_{i=0}^{7} f(x_i) = 2.7268$$

Example4: Calculate the following integral using rectangular rule. Use $n = 6$.

$$\int_{-1}^{3} \frac{e^{-x^2}}{\sqrt{2\pi}} dx$$

Solution: here we can build the table using only:

$$\int_{-1}^{3} e^{-x^2} dx$$

And then divide the result by $\sqrt{2\pi}$. So, $f(x) = e^{-x^2}$, and $h = 0.6667$.

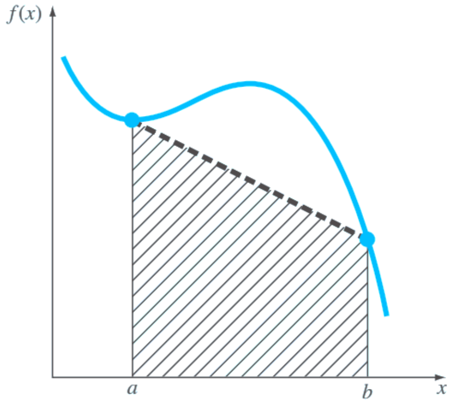
i	x_i	f_i
0	−1	0.3679
1	−0.3333	0.8948
2	0.3333	0.8948
3	1	0.3679
4	1.6667	0.0622
5	2.3333	0.0043
6	3	0.0001

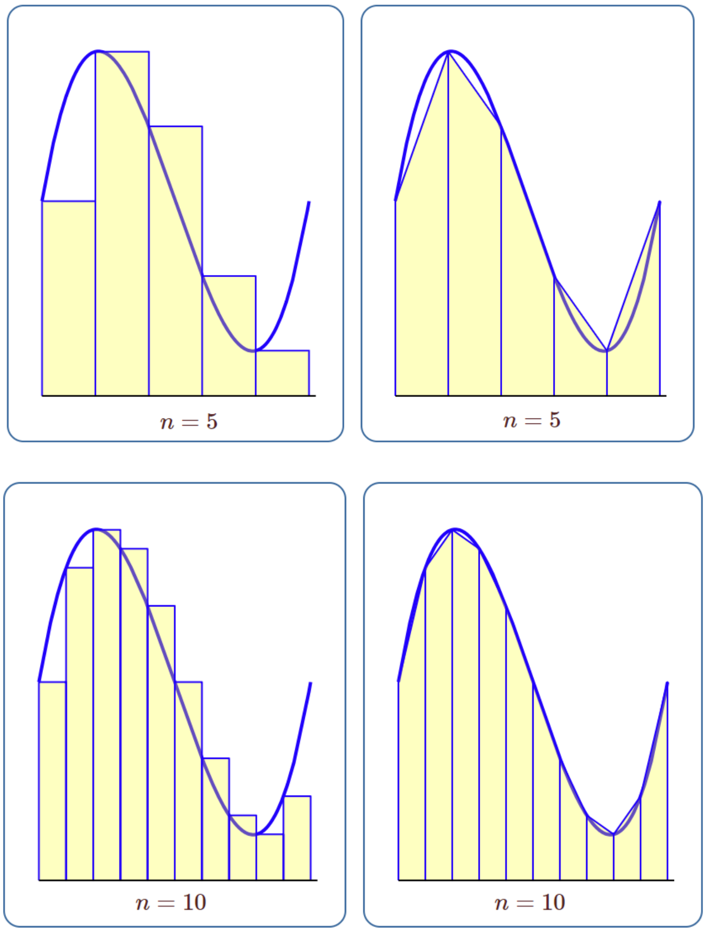
$$I = \frac{0.6667}{\sqrt{2\pi}} \sum_{i=0}^{5} f_i = 0.6894$$

We can write $f(x_i)$ as f_i for simplicity.

The Trapezoidal Rule

Instead of a simple rectangle, the slice here is a Trapezoid. This provides a closer approximation to the actual function.





With: $h = \frac{b-a}{n}$ The area is $I = \frac{h}{2}\{f_0 + f_n\} + h\sum_{i=1}^{n-1} f_i$

Example5: repeat **Example1** using Trapezoidal rule.

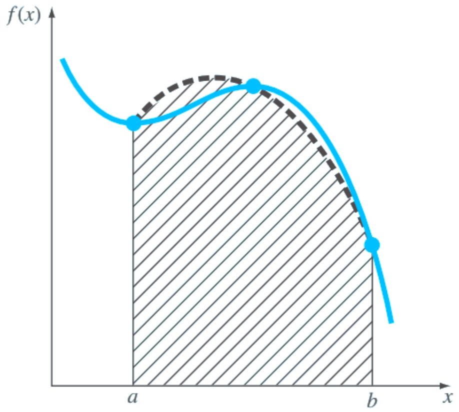
Solution: for the same table, we get

$$I = \frac{0.2}{2}(1 + 0.3624) + 0.2(0.9801 + \cdots + 0.5403) = 0.9289$$

Which is closer to the actual value 0.9320 at the same n. Even for larger n, $I_8 = 0.9303$. $I_{11} = 0.9312$, $I_{15} = 0.9315$.

Simpson's 1/3 Rule

This method obtains a more accurate estimate of an integral by using higher-order polynomials to connect the points.



With: $h = \frac{b-a}{n}$ where n is **even** $I = \frac{h}{3}\left[f_0 + f_n + 4\sum_{i=1,3,5}^{n-1} f_i + 2\sum_{i=2,4,6}^{n-2} f_i\right]$

Example6: repeat **Example1** using Simpson's 1/3 Rule.

Solution:

$$I = \frac{0.2}{3}[1 + 0.3624 + 4(0.9801 + 0.8253 + 0.5403) + 2(0.9211 + 0.6967)] = 0.9321$$

The table below shows the comparison between the actual value from the studied rules:

	Computed	ε_t %
Actual Value	0.9320	
Simpson's 1/3 Rule	0.9321	0.011
The Trapezoidal Rule	0.9289	0.333
Rectangular Rule	0.9927	6.513