

Runge-Kutta Methods

This chapter is devoted to solving ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y)$$

In Chap. 1, we used a numerical method to solve such an equation for the velocity of the falling parachutist. Recall that the method was of the general form

$$\text{New value} = \text{old value} + \text{slope} \times \text{step size}$$

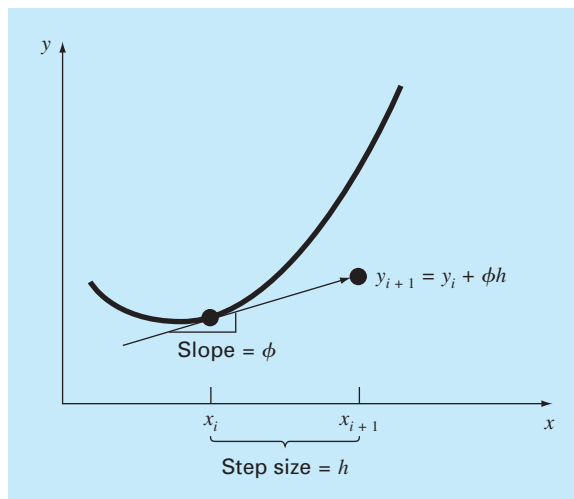
or, in mathematical terms,

$$y_{i+1} = y_i + \phi h \tag{25.1}$$

According to this equation, the slope estimate of ϕ is used to extrapolate from an old value y_i to a new value y_{i+1} over a distance h (Fig. 25.1). This formula can be applied step by step to compute out into the future and, hence, trace out the trajectory of the solution.

FIGURE 25.1

Graphical depiction of a one-step method.



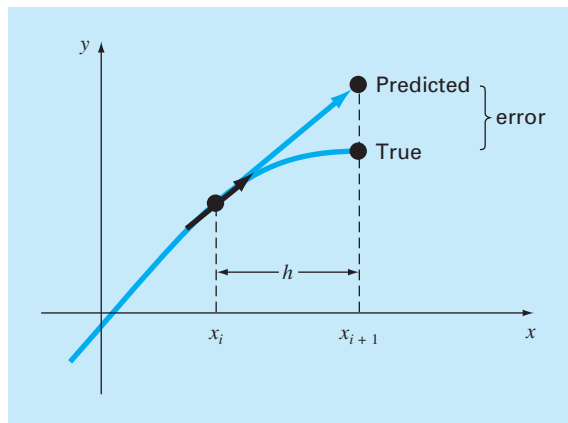


FIGURE 25.2
Euler's method.

All one-step methods can be expressed in this general form, with the only difference being the manner in which the slope is estimated. As in the falling parachutist problem, the simplest approach is to use the differential equation to estimate the slope in the form of the first derivative at x_i . In other words, the slope at the beginning of the interval is taken as an approximation of the average slope over the whole interval. This approach, called *Euler's method*, is discussed in the first part of this chapter. This is followed by other one-step methods that employ alternative slope estimates that result in more accurate predictions. All these techniques are generally called *Runge-Kutta* methods.

25.1 EULER'S METHOD

The first derivative provides a direct estimate of the slope at x_i (Fig. 25.2):

$$\phi = f(x_i, y_i)$$

where $f(x_i, y_i)$ is the differential equation evaluated at x_i and y_i . This estimate can be substituted into Eq. (25.1):

$$y_{i+1} = y_i + f(x_i, y_i)h \quad (25.2)$$

This formula is referred to as *Euler's* (or the *Euler-Cauchy* or the *point-slope*) *method*. A new value of y is predicted using the slope (equal to the first derivative at the original value of x) to extrapolate linearly over the step size h (Fig. 25.2).

EXAMPLE 25.1 Euler's Method

Problem Statement. Use Euler's method to numerically integrate Eq. (PT7.13):

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

from $x = 0$ to $x = 4$ with a step size of 0.5. The initial condition at $x = 0$ is $y = 1$. Recall that the exact solution is given by Eq. (PT7.16):

$$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$$

Solution. Equation (25.2) can be used to implement Euler's method:

$$y(0.5) = y(0) + f(0, 1)0.5$$

where $y(0) = 1$ and the slope estimate at $x = 0$ is

$$f(0, 1) = -2(0)^3 + 12(0)^2 - 20(0) + 8.5 = 8.5$$

Therefore,

$$y(0.5) = 1.0 + 8.5(0.5) = 5.25$$

The true solution at $x = 0.5$ is

$$y = -0.5(0.5)^4 + 4(0.5)^3 - 10(0.5)^2 + 8.5(0.5) + 1 = 3.21875$$

Thus, the error is

$$E_t = \text{true} - \text{approximate} = 3.21875 - 5.25 = -2.03125$$

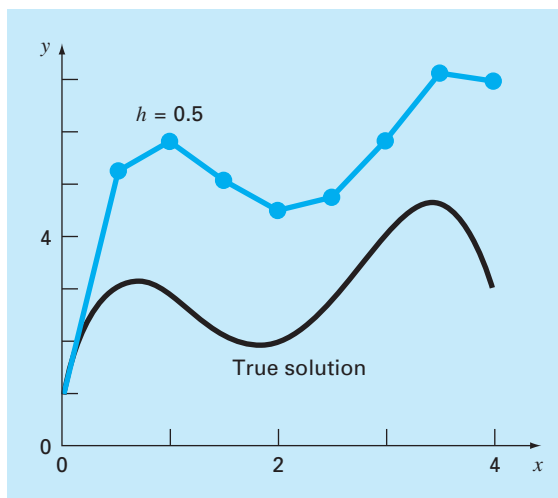
or, expressed as percent relative error, $\varepsilon_t = -63.1\%$. For the second step,

$$\begin{aligned} y(1) &= y(0.5) + f(0.5, 5.25)0.5 \\ &= 5.25 + [-2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5]0.5 \\ &= 5.875 \end{aligned}$$

The true solution at $x = 1.0$ is 3.0, and therefore, the percent relative error is -95.8% . The computation is repeated, and the results are compiled in Table 25.1 and Fig. 25.3.

TABLE 25.1 Comparison of true and approximate values of the integral of $y' = -2x^3 + 12x^2 - 20x + 8.5$, with the initial condition that $y = 1$ at $x = 0$. The approximate values were computed using Euler's method with a step size of 0.5. The local error refers to the error incurred over a single step. It is calculated with a Taylor series expansion as in Example 25.2. The global error is the total discrepancy due to past as well as present steps.

x	y_{true}	y_{Euler}	Percent Relative Error	
			Global	Local
0.0	1.00000	1.00000		
0.5	3.21875	5.25000	-63.1	-63.1
1.0	3.00000	5.87500	-95.8	-28.1
1.5	2.21875	5.12500	-131.0	-1.4
2.0	2.00000	4.50000	-125.0	20.3
2.5	2.71875	4.75000	-74.7	17.2
3.0	4.00000	5.87500	-46.9	3.9
3.5	4.71875	7.12500	-51.0	-11.3
4.0	3.00000	7.00000	-133.3	-53.1

**FIGURE 25.3**

Comparison of the true solution with a numerical solution using Euler's method for the integral of $y' = -2x^3 + 12x^2 - 20x + 8.5$ from $x = 0$ to $x = 4$ with a step size of 0.5. The initial condition at $x = 0$ is $y = 1$.

Note that, although the computation captures the general trend of the true solution, the error is considerable. As discussed in the next section, this error can be reduced by using a smaller step size.

25.3 RUNGE-KUTTA METHODS

Runge-Kutta (RK) methods achieve the accuracy of a Taylor series approach without requiring the calculation of higher derivatives. Many variations exist but all can be cast in the generalized form of Eq. (25.1):

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h \quad (25.28)$$

where $\phi(x_i, y_i, h)$ is called an *increment function*, which can be interpreted as a representative slope over the interval. The increment function can be written in general form as

$$\phi = a_1k_1 + a_2k_2 + \cdots + a_nk_n \quad (25.29)$$

25.3.3 Fourth-Order Runge-Kutta Methods

The most popular RK methods are fourth order. As with the second-order approaches, there are an infinite number of versions. The following is the most commonly used form, and we therefore call it the *classical fourth-order RK method*:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \quad (25.40)$$

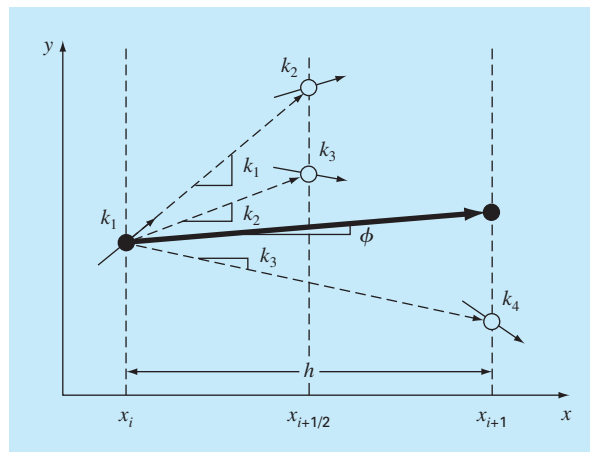
where

$$k_1 = f(x_i, y_i) \quad (25.40a)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h\right) \quad (25.40b)$$

FIGURE 25.15

Graphical depiction of the slope estimates comprising the fourth-order RK method.



$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h\right) \quad (25.40c)$$

$$k_4 = f(x_i + h, y_i + k_3h) \quad (25.40d)$$

Notice that for ODEs that are a function of x alone, the classical fourth-order RK method is similar to Simpson's 1/3 rule. In addition, the fourth-order RK method is similar to the Heun approach in that multiple estimates of the slope are developed in order to come up with an improved average slope for the interval. As depicted in Fig. 25.15, each of the k 's represents a slope. Equation (25.40) then represents a weighted average of these to arrive at the improved slope.

EXAMPLE 25.7

Classical Fourth-Order RK Method

Problem Statement.

- (a) Use the classical fourth-order RK method [Eq. (25.40)] to integrate

$$f(x, y) = -2x^3 + 12x^2 - 20x + 8.5$$

using a step size of $h = 0.5$ and an initial condition of $y = 1$ at $x = 0$.

- (b) Similarly, integrate

$$f(x, y) = 4e^{0.8x} - 0.5y$$

using $h = 0.5$ with $y(0) = 2$ from $x = 0$ to 0.5 .

Solution.

- (a) Equations (25.40a) through (25.40d) are used to compute $k_1 = 8.5$, $k_2 = 4.21875$, $k_3 = 4.21875$ and $k_4 = 1.25$, which are substituted into Eq. (25.40) to yield

$$\begin{aligned} y(0.5) &= 1 + \left\{ \frac{1}{6} [8.5 + 2(4.21875) + 2(4.21875) + 1.25] \right\} 0.5 \\ &= 3.21875 \end{aligned}$$

which is exact. Thus, because the true solution is a quartic [Eq. (PT7.16)], the fourth-order method gives an exact result.

- (b) For this case, the slope at the beginning of the interval is computed as

$$k_1 = f(0, 2) = 4e^{0.8(0)} - 0.5(2) = 3$$

This value is used to compute a value of y and a slope at the midpoint,

$$y(0.25) = 2 + 3(0.25) = 2.75$$

$$k_2 = f(0.25, 2.75) = 4e^{0.8(0.25)} - 0.5(2.75) = 3.510611$$

This slope in turn is used to compute another value of y and another slope at the midpoint,

$$y(0.25) = 2 + 3.510611(0.25) = 2.877653$$

$$k_3 = f(0.25, 2.877653) = 4e^{0.8(0.25)} - 0.5(2.877653) = 3.446785$$

Next, this slope is used to compute a value of y and a slope at the end of the interval,

$$y(0.5) = 2 + 3.071785(0.5) = 3.723392$$

$$k_4 = f(0.5, 3.723392) = 4e^{0.8(0.5)} - 0.5(3.723392) = 4.105603$$

Finally, the four slope estimates are combined to yield an average slope. This average slope is then used to make the final prediction at the end of the interval.

$$\phi = \frac{1}{6}[3 + 2(3.510611) + 2(3.446785) + 4.105603] = 3.503399$$

$$y(0.5) = 2 + 3.503399(0.5) = 3.751699$$

which compares favorably with the true solution of 3.751521.

PROBLEMS

25.1 Solve the following initial value problem over the interval from $t = 0$ to 2 where $y(0) = 1$. Display all your results on the same graph.

$$\frac{dy}{dt} = yt^2 - 1.1y$$

- (a) Analytically.
- (b) Euler's method with $h = 0.5$ and 0.25 .
- (c)
- (d) Fourth-order RK method with $h = 0.5$.

25.2 Solve the following problem over the interval from $x = 0$ to 1 using a step size of 0.25 where $y(0) = 1$. Display all your results on the same graph.

$$\frac{dy}{dt} = (1 + 4t)\sqrt{y}$$

- (a) Analytically.
- (b) Euler's method.
- (c)
- (d)
- (e) Fourth-order RK method.