Introduction

***Theory of Elasticity*:** determination of stress and displacement in a body as a result of applied (mechanical or thermal) loading. All structural materials possess to a certain extent the property of elasticity , i.e, if external forces, producing deformation of a structure, do not exceed a certain limit, the deformation disappears with removal of the forces.

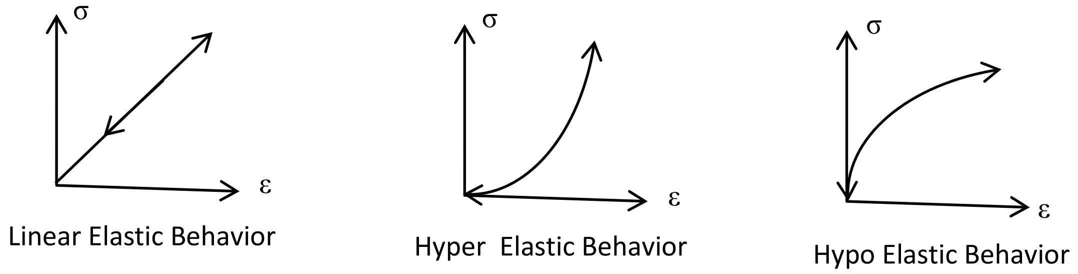


Fig. 1: Stress-strain relationship

Perfectly Elastic: the body resume its initial form (i.e. its shape) completely after removal of deforming force.

Elastic Property: has the same path in loading and unloading

Assumption :

1. The body is homogenous and continuously distribution over its volume
2. The body is isotropic, i.e., the elastic properties are the same in all directions.
3. The action of external forces are perfectly elastic.

Sign Convention: Right hand rule

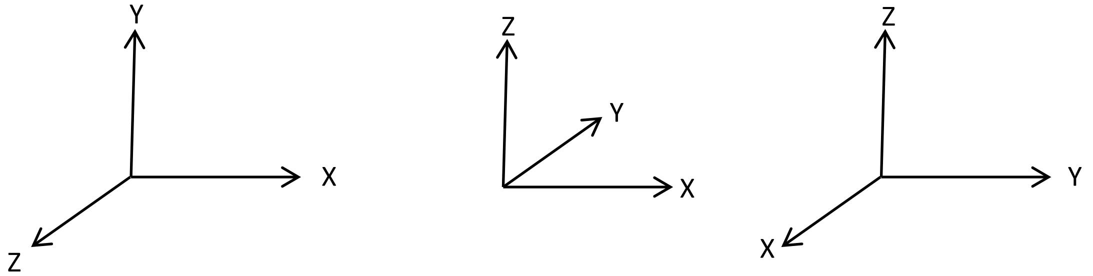
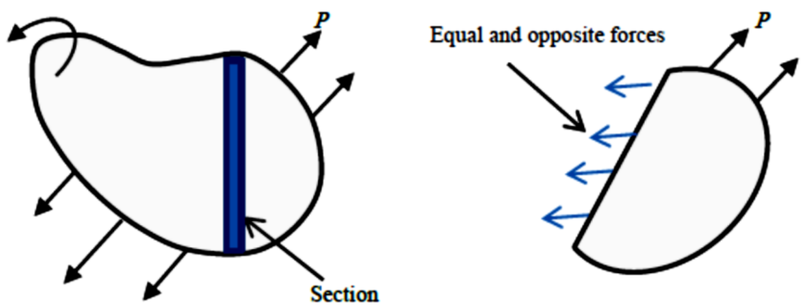


Fig. 2: Cartesian coordinates

Stresses:

Stress is a measure of the diffusion of external forces into the body.



Components of Stress:

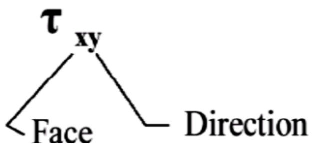
In general: $Stress = \frac{\delta p}{\delta A}$ (MPa, kN/mm^2)

Where: ***p*** is the applied load, and ***A*** is the cress section area

The most general state of stress at a point may be represented by 6 components:

Normal Stresses: $\sigma_x\ \sigma_y\ \sigma_z$ and Shear Stresses: $\tau_{xy}\ \tau_{yz}\ \tau_{xz}$

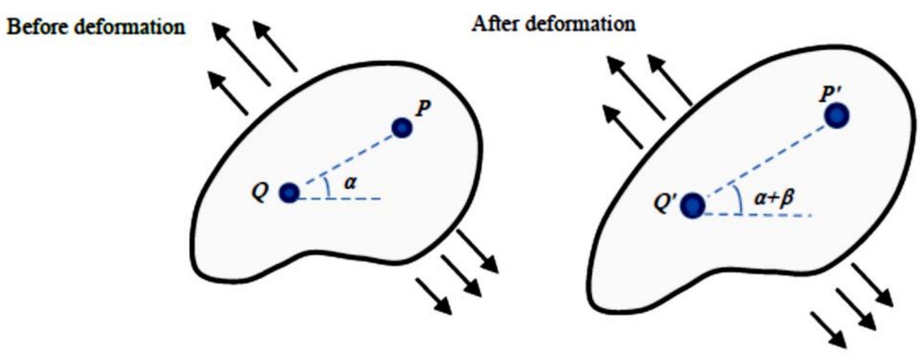
From equilibrium principles: $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{zy} = \tau_{yz}$



Strains:

Strain is a measure of deformation in body. Take two point P and Q in a body before deformation. After deformation, these two points move to P' and Q'. There are two types of deformation:

1. Change of length which cause normal or direct strain i.e. $PQ \neq P'Q'$
2. Change of angle which cause shear strain i.e. $\alpha \rightarrow \alpha+\beta$



Components of Strain:

Strain is the intensity of deformation. In the example (Figure 3), we consider an elastic bar of length L. If the bar is subjected by an axial force F, it will stretch an amount δ as shown in Figure 3b). The quantity δ/L is a measure of the change in length relative to the original length and is defined to be the axial strain for the bar. In Figure 3d), a shear load is applied that is parallel to the top surface as shown. The angle θ measures the amount the original right angle in Figure 3c) has changed from a right angle, and the angle θ is related to the shear strain.

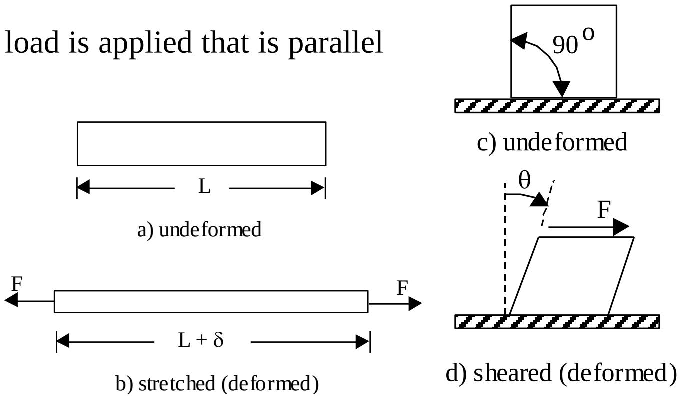
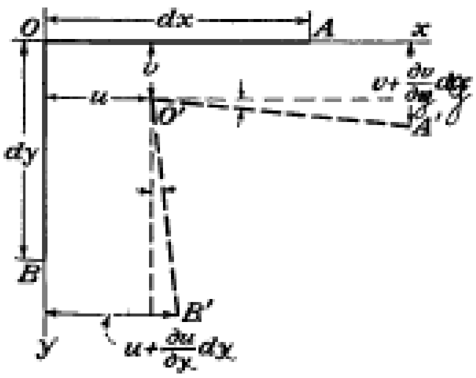


Fig. 3: axial and shear deformations

What causes strain?

- Mechanical loads (forces, pressures, etc.)
- Temperature change (thermal expansion)
- Moisture absorption
- Stress (relationship between stress and strain is the
- constitutive relationship)

General case: u,v displacement of point o



$$\varepsilon_x = \frac{u+\left(\partial u/\partial x\right)dx-u}{dx} \qquad \varepsilon_{x}=\partial u/\partial x$$

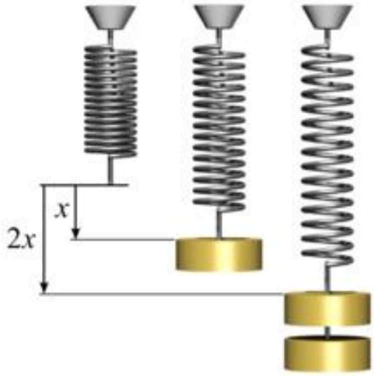
$$\varepsilon_y = \frac{v+\left(\partial v/\partial y\right)dy-v}{dy} \qquad \varepsilon_{y}=\partial v/\partial y$$

$$\gamma_{xy}=\left(v+\frac{\partial v}{\partial x}dx-v\right)*\frac{1}{dx}+\left(u+\frac{\partial u}{\partial y}dy-u\right)*\frac{1}{dy} \qquad \gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}$$

In matrix form:

$$\varepsilon = \begin{bmatrix} \partial u/_{\partial x} & \left(\partial v/_{\partial x} + \partial u/_{\partial y}\right)/_2 & \left(\partial w/_{\partial x} + \partial u/_{\partial z}\right)/_2 \\ \left(\partial u/_{\partial y} + \partial v/_{\partial x}\right)/_2 & \partial v/_{\partial y} & \left(\partial w/_{\partial y} + \partial v/_{\partial z}\right)/_2 \\ \left(\partial u/_{\partial z} + \partial w/_{\partial x}\right)/_2 & \left(\partial v/_{\partial z} + \partial w/_{\partial y}\right)/_2 & \partial w/_{\partial z} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy}/_2 & \gamma_{xz}/_2 \\ \gamma_{yx}/_2 & \varepsilon_{yy} & \gamma_{yz}/_2 \\ \gamma_{zx}/_2 & \gamma_{zy}/_2 & \varepsilon_{zz} \end{bmatrix}$$

Hook's Law (Stress – strain relationship)



Hooke's law is a principle of physics that states that the force (F) needed to extend or compress a spring by some distance X is proportional to that distance. That is: $F = kX$, where k is a constant factor characteristic of the spring: its stiffness, and X is small compared to the total possible deformation of the spring.

The shape of a body will distort when a force is applied to it. Bodies which are "elastic" distort by compression or tension, and return to their original, or equilibrium, position when the distorting force is removed (unless the distorting force exceeds the elastic limit of the material). Hooke's Law states that if the distortion of an elastic body is not too large, the force tending to restore the body to equilibrium is proportional to the displacement of the body from equilibrium.

Under state of uniaxial stress $\sigma_x = E\varepsilon_x$ or $\varepsilon_x = \sigma_x/E$ and the lateral strain component (contraction) is $\varepsilon_y = \varepsilon_z = -\mu\sigma_x/E$

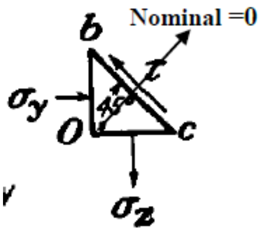
Where E: Modulus of elasticity and μ: Poisson's ratio

- For the general 3D stress state, we can get the relations between direct stresses and strain by method of superposition

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$
$$\epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

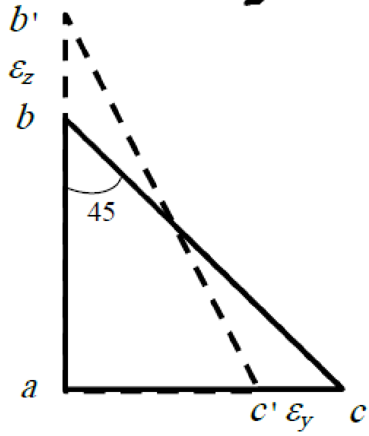
- If $\sigma_x=0$ and $\sigma_y= - \sigma_z$

Summing up forces along and perpendicular to bc:



$\tau = \sigma_z \cos^2 45 - \sigma_y \cos^2 45 = 1/2\ (\sigma_z - \sigma_y) = \sigma_z =$ Max. pure shear

$$\tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \frac{\tan\frac{\pi}{4} - \tan\frac{\gamma}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{\gamma}{2}} = \frac{1 - \frac{\gamma}{2}}{1 + \frac{\gamma}{2}}$$



$$\gamma = \frac{\tau}{G}$$

$prove\ that: \gamma = \frac{E\tau}{2(1+\mu)}$ ***and*** $G = \frac{2(1+\mu)}{E}$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}, \qquad \gamma_{zx} = \frac{1}{G}\tau_{zx}$$

All relations between stress and strains derived previously can be gathered to form the stress-strain relationship in 3 dimensions:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\mu & -\mu & 0 & 0 & 0 \\ -\mu & 1 & -\mu & 0 & 0 & 0 \\ -\mu & -\mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\mu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$