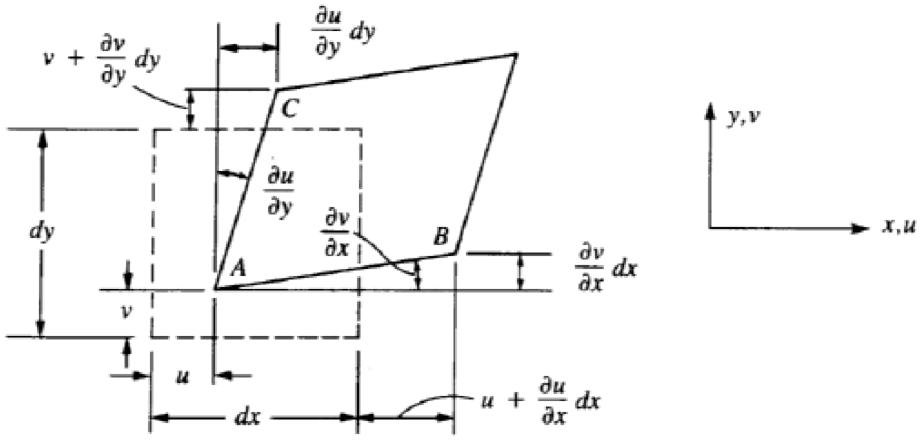
Strain at point: The general two-dimensional state of strain at a point is show below.

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



The two dimensional strain transformation equations are very similar to the 2D stress transformation equations. The analysis is based on a plane strain state in which all strains in the z-direction are zero. The analysis can also be used for a plane stress state with one minor modification. A material cannot have both plane stress and plane strain states at the same time.

The relationship between the strains at a point measured relative to a set of axes x-y and a set x'-y' which have the same origin but are rotated counter-clockwise from the original axes by an angle θ are given by for the normal strains and by for the shearing strain. Note the similarity of form between these equations and the stress transformation equations.

$$\varepsilon_n = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} Cos(2\theta) + \frac{\gamma_{xy}}{2} Sin(2\theta)$$

$$\frac{\gamma_n}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2} Sin(2\theta) + \frac{\gamma_{xy}}{2} Cos(2\theta)$$

for the shearing strain. Note the similarity of form between these equations and the stress transformation equations.

Principal Strains and Maximum Shearing Strain

As with the stresses there are maximum and minimum (principal) values of the normal strains for particular orientations at the point and maximum shearing strains.

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left[\frac{\gamma_{xy}}{2}\right]^2}$$

The principal strains are given by

and the maximum shearing strain is given by

The orientation of the larger principal strain to the positive x-direction is given by

The direction of the smaller principal strain is perpendicular to the first. The

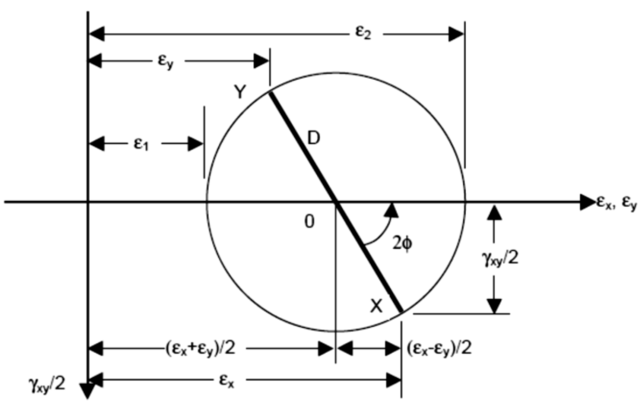
$$\frac{\gamma_{\max}}{2} = \pm\sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left[\frac{\gamma_{xy}}{2}\right]^2}$$

$$\theta_P = \frac{1}{2} Tan^{-1}\left[\frac{\left(\frac{\gamma_{xy}}{2}\right)}{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)}\right]$$

directions involved with the maximum shearing strain are the two directions at 45° to both of the principal directions.

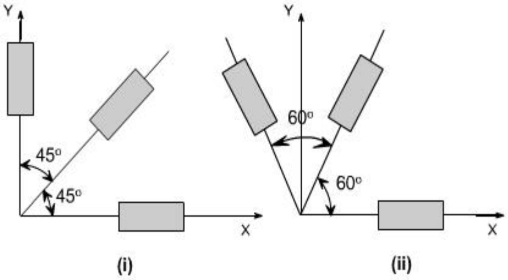
Mohr's Circle for Strain

A Mohr's Circle mapping between the strains acting with respect to a set of *x-y* axes at a point and a point in the strain plane can be made. The same rules apply as for the stress circle with ε replacing σ and $\gamma/2$ replacing τ. This makes the radius of the circle equal to half the in-plane maximum shearing strain)



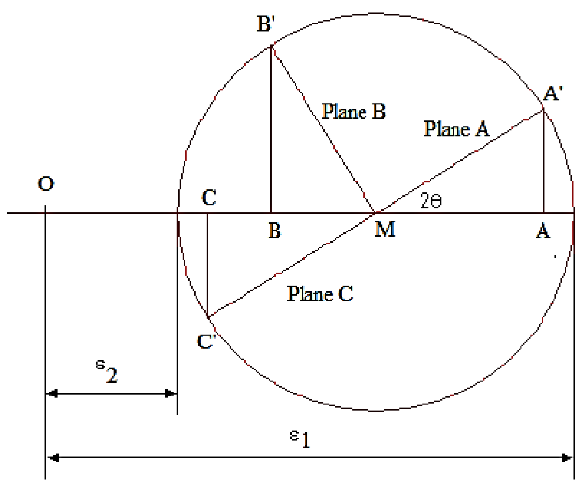
Strain Gauge Rosette

A strain gauge rosette is a term for an arrangement of two or more strain gauges that are positioned closely to measure strains along different directions of the component under evaluation. Single strain gauges can only measure strain effectively in one direction, so the use of multiple strain gauges enables more measurements to be taken, providing a more precise evaluation of strain on the surface being measured.



Case 1: Construction of The 45°Circle

We shall use a numerical example to explain the construction of the circle. Suppose the three strains are $\varepsilon_A = 700\ \mu\varepsilon$ $\varepsilon_B = 300\ \mu\varepsilon$ $\varepsilon_C = 200\mu\varepsilon$

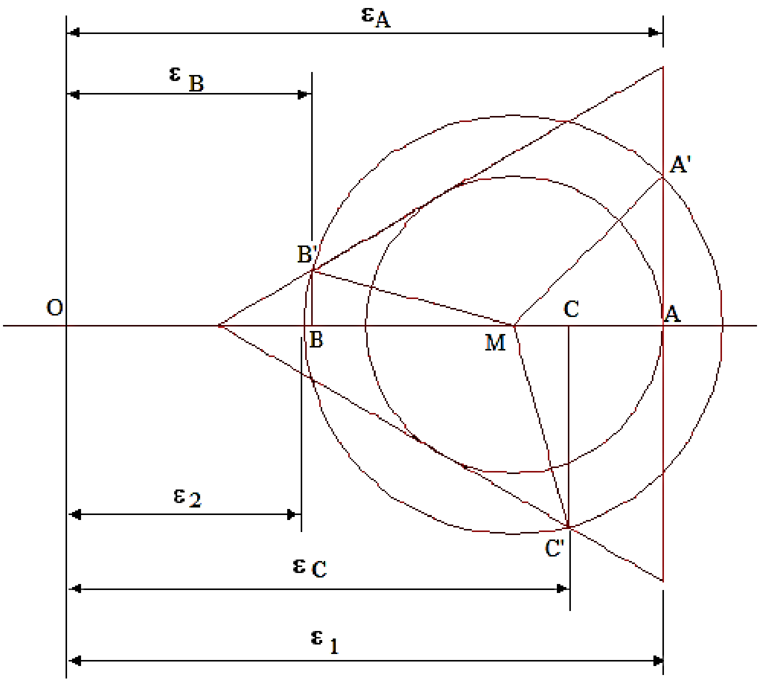


1. Choose a suitable origin O
2. Scale off horizontal distances from O for ε_A, ε_B and ε_C and mark them as A, B and C.
3. Mark the centre of the circle M half way between A and C.
4. Construct vertical lines through A, B and C
5. Measure distance BM
6. Draw lines A A' and C C' equal in length to BM
7. Draw circle centre M and radius M A' = M C'
8. Draw B B'

Scaling off the values we find $\varepsilon_1 = 742\ \mu\varepsilon$, $\varepsilon_2 = 158\ \mu\varepsilon$ and the angle $2\theta = 30^{o}$
The first principal plane is hence 15^{o} clockwise of plane A.

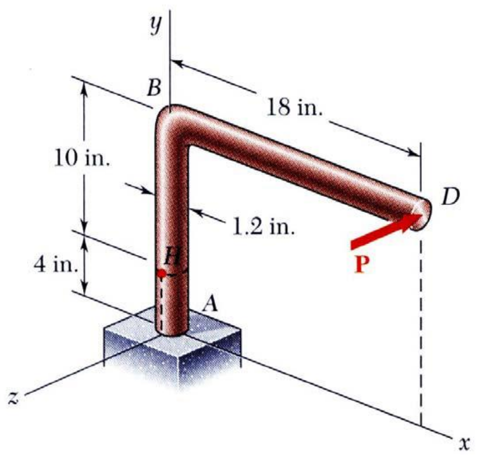
Case 2: Construction of The 60 ° Circle

1. Select a suitable origin O
2. Scale of A, B and C to represent the three strains ε_A, ε_B and ε_C
3. Calculate OM = (A + B + C)/3
4. Draw the inner circle radius MA
5. Draw the triangle (60° each corner)
6. Draw outer circle passing through B' and C'.
7. Make sure that the planes A, B and C are in an anti clockwise direction because you can obtain an upside down version for clockwise directions.
8. Scale off principal strains.

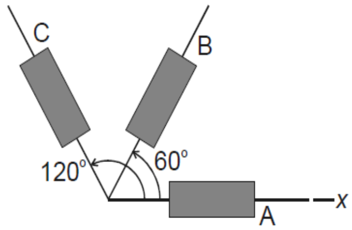


Problem 1:

A single horizontal force *P* of 150 lb magnitude is applied to end D of lever *ABD*. Determine (a) the normal and shearing stresses on an element at point *H* having sides parallel to the *x*and *y*axes, (b) the principal planes and principal stresses at the point *H*.

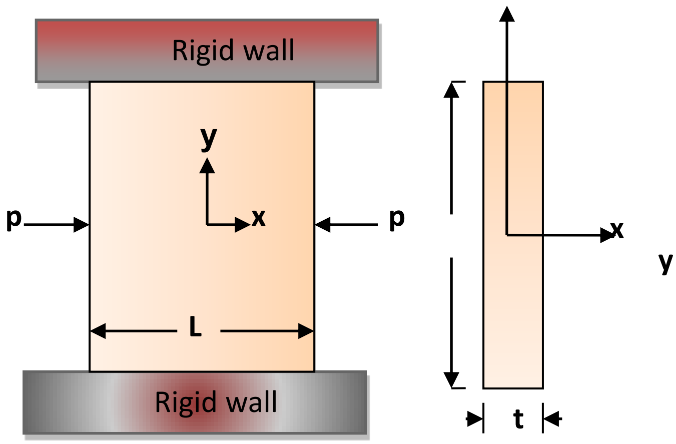


Problem 2: A "0°-60°-120°" strain gauge rosette is bonded to the surface of a
thin steel plate. Under one loading condition, the strain measurements
are $\varepsilon_A = 60\ \mu\varepsilon$, $\varepsilon_B = 135\ \mu\varepsilon$, $\varepsilon_C = 264\ \mu\varepsilon$. Find the principal strains, their orientations, and the principal stresses.



Problem 3: A thin plate of width (b), thickness (t) and length (L) is subjected to an axial compressive force (p) as shown in figure below. Find:

a- the shortening of the plate parallel to the force p.
b- the component of normal strain in the thin direction.



Problem 4: show that the line elements at the point x, y that have the maximum and minimum rotation are those in the two perpendicular direction θ determined by:

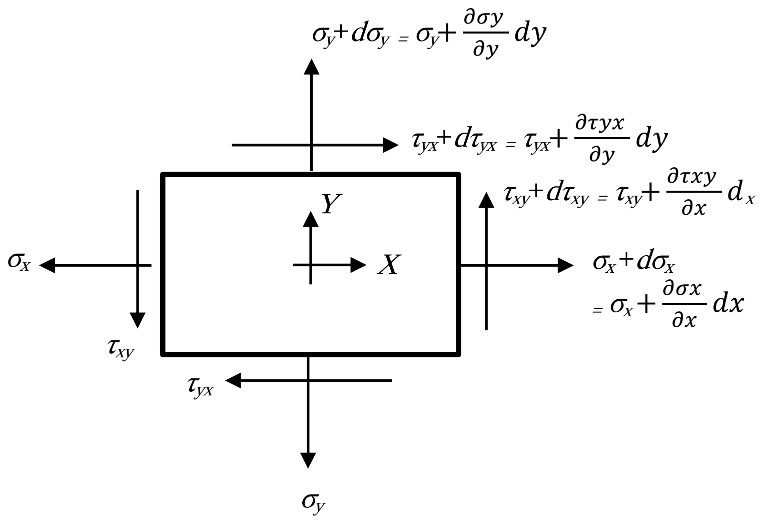
$$\tan 2\theta = \frac{\partial v/\partial y - \partial u/\partial x}{\partial v/\partial x + \partial u/\partial y}$$

Equilibrium Equations

There are two types of forces acting on a body:

1. Surface forces (or traction forces): they act on the surface of a body.
2. Body forces: they act within a body, like gravity force (or self-weight)

Consider a small rectangular block element which is subjected to body forces *X*,*Y*. The stresses changes but the equilibrium must be satisfied in *x* and *y* directions.



The equilibrium Eqs. for forces in *x* and *y* directions will be:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0.......(1)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0........(2)$$

Where: *X*,*Y* denote the components of body force per unit volume. In practical applications the weight of body is usually the only body force (*ρ=mass/unit volume*). Then the Eq. become:

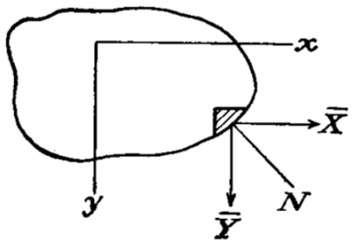
$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho g = 0$$

These are the differential equations of equilibrium for *2D* problems.

Boundary Conditions:

Eqs. of equilibrium must satisfied at all point throughout the volume of the body. Taking small triangular prism as shown in Figure. $\bar{X}$ and $\bar{Y}$ are surface force components per unite area, at this point of the boundary, we have:



$$\bar{X} = l\sigma_x + m\tau_{xy}$$
$$\bar{Y} = m\sigma_y + l\tau_{xy}$$

Where: *l* and *m* are the direction cosines of the normal *N* to boundary. If taking the side of plate parallel to the sides of plate the Eqs. above can be simplified, since for this part of boundary the normal N parallel to y-axis; hence *l*= 0 and *m*= ±1. Then

$$\bar{X} = \pm\tau_{xy}, \qquad \bar{Y} = \pm\sigma_y$$

It is seen that at the boundary the force components equal to the components of the surface forces per unit area of the boundary.

Strain Compatibility:

In 2D-dimentional problems only three components need to be considered, namely,

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Differentiating the first of Eqs. twice with respect to *y*, the second twice with respect to *x* and the third once each with respect to *x* and *y* yields

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \text{.......(3)}$$ Condition of compatibility

- *In case of plane stress*: $\sigma_z = 0$

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \\ \varepsilon_y &= \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \\ \gamma &= \frac{\tau}{G} = \frac{1}{E}\left[2(1+\nu)\tau\right] \end{aligned}\right]$$ Substituting in Eq. (3), we find:

$$\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) = 2(1+\nu)\frac{\partial^2\tau_{xy}}{\partial x\, \partial y}$$

By differentiating 1st eq. of equilibrium w.r.t *x* and differentiating 2nd eq. of equilibrium w.r.t *y*,

$$\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial y^2} + \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = -2\frac{\partial^2\tau_{xy}}{\partial x\partial y}$$ ……. Hence the compatibility Eq. for plane stress becomes:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)$$

…..(4)

- *In case of plane strain*: $\varepsilon_z = 0$, $\sigma_z = \nu(\sigma_x + \sigma_y)$

$$\epsilon_x = \frac{1}{E}[(1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y]$$

$$\epsilon_y = \frac{1}{E}[(1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x]$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

Similar we get: $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)$ …..(5)

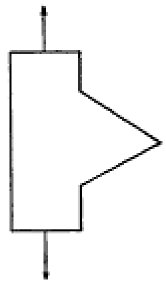
- For body forces involving gravity only (self-weight)

  X=0 and Y= ρg

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0$$ …… (6) For both plane stress and strain

<u>Problem 5:</u> The Figure represent a tooth on a plate in state of plane stress

In the plane of the paper. The face of tooth (the two straight lines) are free from force. Prove that there is no stress at all at the apex of the tooth.



<u>Problem 6:</u> Using stress-strain relations and equations of equilibrium, show that in the absence of body force the displacements in problems of plane stress must satisfy

$$\partial^2 u/\partial^2 x + \partial^2 u/\partial^2 y + \frac{1+\mu}{1-\mu}.\frac{\partial}{\partial x}\left(\partial u/\partial x + \partial v/\partial y\right) = 0$$