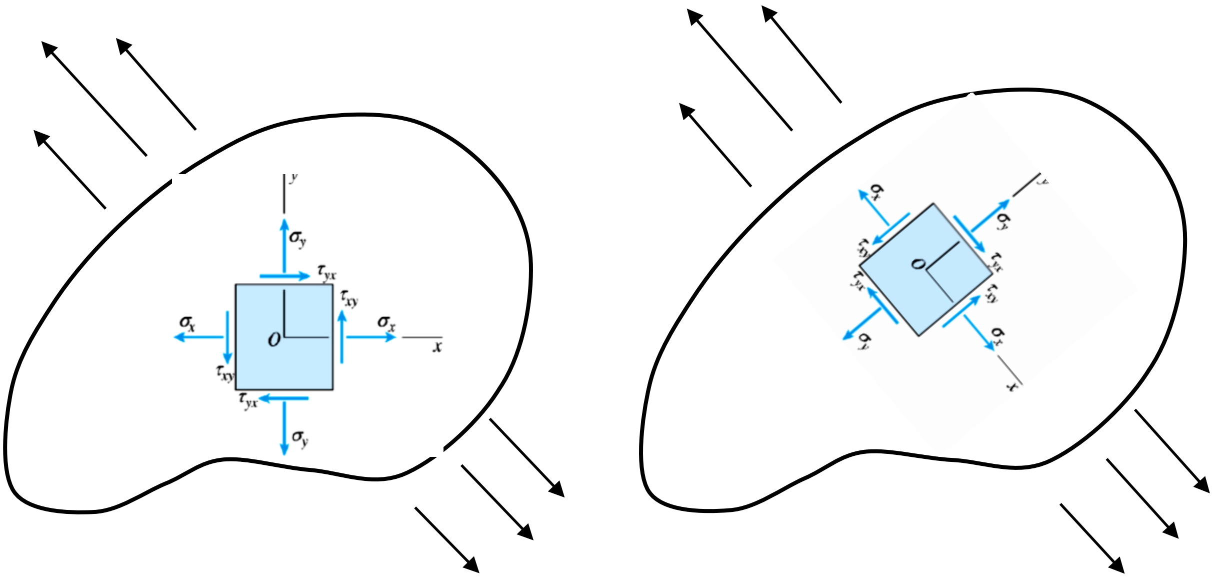
Two Dimensional Problems in Cartesian Coordinate System

INTROUDUCTION

Suppose a body (in *xy-plane*) is under the external force shown in Figure below.



It's required to find the stress (σ_x, σ_y *and* τ_{xy}) at any point inside the body (neglect the body force). First remember the equations of equilibrium:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \text{........(1)}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \text{........(2)}$$

There are two equations, but with three unknown (σ_x, σ_y *and* τ_{xy}). Then the problem is statically indeterminate. One more equation is needed. Use the compatibility of strains:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \text{.......(3)}$$

Or
$$\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) = 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x\, \partial y}$$

Differentiate eq. (1) w.r.t. x and differentiate eq. (2) w.r.t. y and add. Then substitute in eq. (3). Simplify and arrange to get:

$$\nabla^2(\sigma_x + \sigma_y) = 0 \text{........(4)}$$

But this is not a useful equation. Airy introduced the concept of stress function $\phi = \phi(x,y)$, where:

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}\ ,\ \sigma_y = \frac{\partial^2 \phi}{\partial x^2},\ \tau_{xy} = \tau_{yx} = -\frac{\partial^2 \phi}{\partial x \partial y} \text{.......(5)}$$

And $\phi = \phi(x,y)$ is an arbitrary form called *Airy's stress function*. It is easily shown that this form satisfies equilibrium (zero body force case). Substitute eqs. (5) in eq. (3) or in eq. (4) to get:

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = \nabla^4 \phi = 0 \text{.........(6)}$$

Solving this Eq. and applying B.C. give the stress distribution.

An alternatives methods of choosing stress functions is using polynomials from Pascal Triangle.

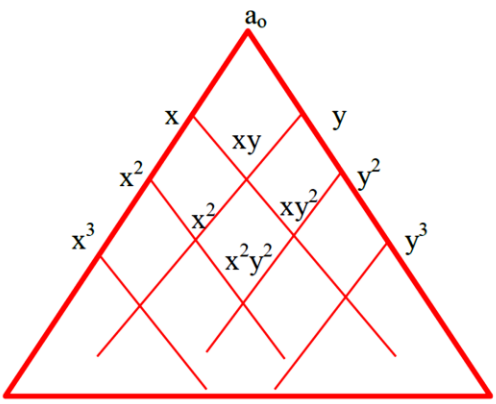
$$(x + y)^0 = 1$$

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$



SOLUTIONS OF TWO-DIMENSIONAL PROBLEMS BY THE USE OF POLYNOMIALS

It has been shown that the solution of two-dimensional problems, when body forces are absent or are constant, is reduced to the integration of the differential equation:

$$\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}=\nabla^4\phi=0$$

By taking polynomials of various degrees, and suitably adjusting their coefficients, a number of practically important problems can be solved. Recalling Airy's stress function:

$$\sigma_x=\frac{\partial^2\phi}{\partial y^2}\ ,\ \sigma_y=\frac{\partial^2\phi}{\partial x^2},\ \tau_{xy}=-\frac{\partial^2\phi}{\partial x\partial y}$$

When body forces are weight only. Solve this equation & apply B.C. to get stress distribution.

- *Polynomial of the First Degree* : Let $\varphi_1 = a_1x + b_1y$

Now, the corresponding stresses are :

$$\sigma_x=\frac{\partial^2\phi_1}{\partial y^2}=0$$

$$\sigma_y=\frac{\partial^2\phi_1}{\partial x^2}=0$$

$$\tau_{xy}=-\frac{\partial^2\phi_1}{\partial x\partial y}=0$$

There is no stress

- *Polynomial of the Second Degree* : Let $\phi_2=\frac{a_2}{2}x^2+b_2xy+\frac{c_2}{2}y^2$

The corresponding stresses are :

$$\sigma_x=\frac{\partial^2\phi_2}{\partial y^2}=c_2$$

$$\sigma_y=\frac{\partial^2\phi_2}{\partial x^2}=a_2$$

$$\tau_{xy}=-\frac{\partial^2\phi}{\partial x\partial y}=-b_2$$

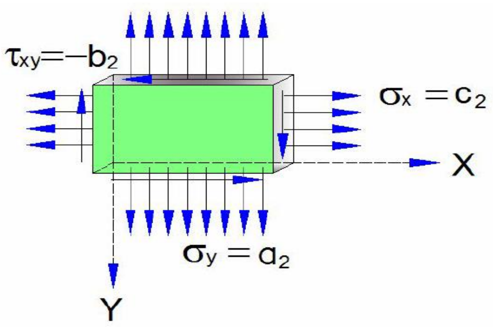


Fig. 1

All three stress components are constant, throughout the body, i.e., the polynomial of 2nd stress function represents a combination of uniform tensions or compressions in two perpendicular directions nd a uniform shear. The forces on the boundaries must equal the stresses at these points; in the case of a rectangular plate with sides parallel to the coordinate axes these forces are shown in Figure 1.

This function represents a combination of uniform tensions or compressions in two perpendicular directions and a uniform shear. For case of pure shear a_2 & c_2 =0

- *Polynomial of the Third Degree* :

The corresponding stresses are :

$$\phi_3 = \frac{a_3}{6}x^3 + \frac{b_3}{2}x^2 y + \frac{c_3}{2}xy^2 + \frac{d_3}{6}y^3$$

$$\sigma_x = \frac{\partial^2 \phi_3}{\partial y^2} = c_3 x + d_3 y$$

$$\sigma_y = \frac{\partial^2 \phi_3}{\partial x^2} = a_3 x + b_3 y$$

$$\tau_{xy} = -\frac{\partial^2 \phi_3}{\partial x \partial y} = -b_3 x - c_3 y$$

This stress function gives a linearly varying stress field. It should be noted that the magnitudes of the coefficients a_3 ,b_3 ,c_3 and d_3 are chosen freely since the expression for φ_3 is satisfied irrespective of values of these coefficients.

Now, if $a_3 = b_3 = c_3 = 0$ except d_3 , we get from the stress components $\sigma_x = d_3$, $\sigma_y = 0$, $\tau_{xy} = 0$. This corresponds to pure bending on the face perpendicular to the x-axis: At $y = -h$, $\sigma_x = -d_3 h$ and $y = h$, $\sigma_x = d_3 h$. The variation of σ_x with y is linear as shown in the Figure 2.

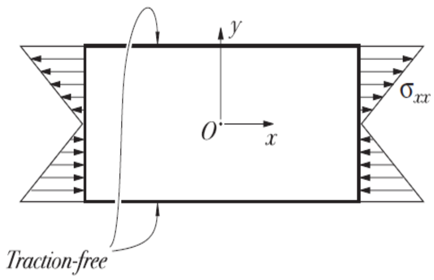


Fig. 2.

Similarly, if all the coefficients except b_3 are zero, then we get:

$$\sigma_x = 0, \quad \sigma_y = b_3 y, \quad \tau_{xy} = -b_3 x$$

The stresses represented by the above stress field will vary as shown in the Figure 3. The stress y s is constant with x (i.e. constant along the span L of the beam), but varies with y at a particular section. At $y = +h$, $\sigma_y = b_3 h$ (i.e., tensile),

while at $y = -h$, $\sigma_y = -b_3h$ (i.e. compressive).
σ_x is zero throughout. Shear stress τ_{xy} is zero at $x = 0$ and is equal to $-b_3l$ at $x = l$. At any other section, the shear stress is proportional to x.

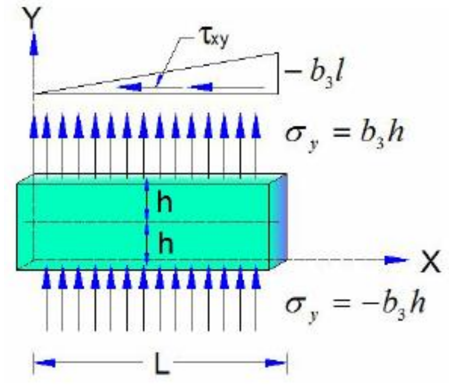


Fig. 3

- *Polynomial of the Fourth Degree*: $\phi_4 = \frac{a_4}{12}x^4 + \frac{b_4}{6}x^3y + \frac{c_4}{2}x^2y^2 + \frac{d_4}{6}xy^3 + \frac{e_4}{12}y^4$

The corresponding stresses are :

$$\sigma_x = c_4x^2 + d_4xy + e_4y^2$$

$$\sigma_y = a_4x^2 + b_4xy + c_4y^2$$

$$\tau_{xy} = -\left(\frac{b_4}{2}\right)x^2 - 2c_4xy - \left(\frac{d_4}{2}\right)y^2$$

Now, taking all coefficients except d_4 equal to zero, we find

$$\sigma_x = d_4xy, \quad \sigma_y = 0, \quad \tau_{xy} = -\frac{d_4}{2}y^2$$

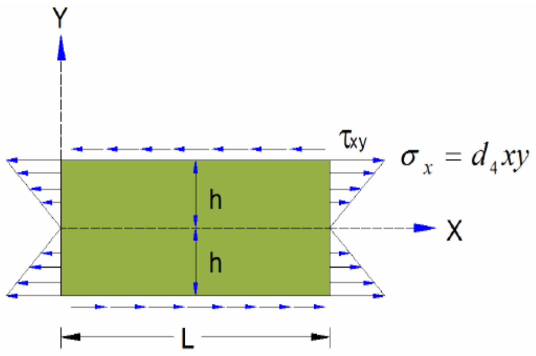


Fig. 4

On the longitudinal sides, y = ±h are uniformly distributed shearing forces. At the ends, the shearing forces are distributed according to a parabolic distribution. The shearing forces acting on the boundary of the beam reduce to the couple.

Therefore, $M = \frac{d_4h^2L}{2}2h - \frac{1}{3}\frac{d_4h^2}{2}2hL$

Or $M = \frac{2}{3}d_4h^3L$

This couple balances the couple produced by the normal forces along the side x = L of the beam.

- *Polynomial of the Fifth Degree* :

$$\varphi_5 = \frac{a_5}{20}x^5 + \frac{b_5}{12}x^4 y + \frac{c_5}{6}x^3 y^2 + \frac{d_5}{6}x^2 y^3 + \frac{e_5}{12}xy^4 + \frac{f_5}{20}y^5$$

The corresponding stresses are :

$$\sigma_x = \frac{\partial^2 \phi_5}{\partial y^2} = \frac{c_5}{3}x^3 + d_5 x^2 y - (2c_5 + 3a_5)xy^2 - \frac{1}{3}(b_5 + 2d_5)y^3$$

$$\sigma_y = \frac{\partial^2 \phi_5}{\partial x^2} = a_5 x^3 + b_5 x^2 y + c_5 xy^2 + \frac{d_5}{3}y^3$$

$$\tau_{xy} = -\frac{\partial^2 \phi_5}{\partial x \partial y} = -\frac{1}{3}b_5 x^3 - c_5 x^2 y - d_5 xy^2 + \frac{1}{3}(2c_5 + 3a_5)y^3$$

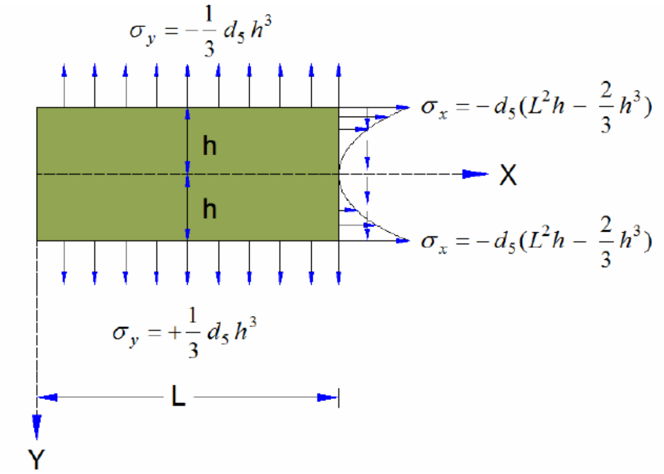
Here the coefficients a_5 ,b_5 ,c_5 ,d_5 are arbitrary, and in adjusting them we obtain solutions for various loading conditions of the beam.

Now, if all coefficients, except d_5 , equal to zero, we find:

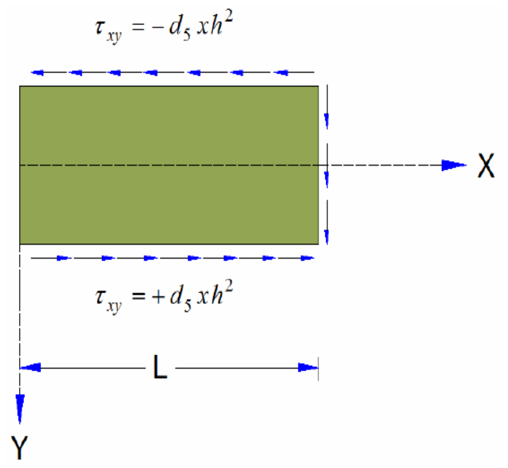
$$\sigma_x = d_5\left(x^2 y - \frac{2}{3}y^3\right)$$

$$\sigma_y = \frac{1}{3}d_5 y^3$$

$$\tau_{xy} = -d_5 xy^2$$



Case (i)



Case (ii)

Fig. 4

SAINT-VENANT'S PRINCIPLE:

In applying the equations for axial loading of members, we have assumed up to this point that we are sufficiently far enough from the point of load application that the distribution of normal stress is uniform. In doing so, we have unknowingly been applying Saint Venant's Principle. This principle states that:

The stresses and strains in a body at points that are sufficiently remote from points of application of load depends only on the static resultant of the loads and not on the distribution of loads.

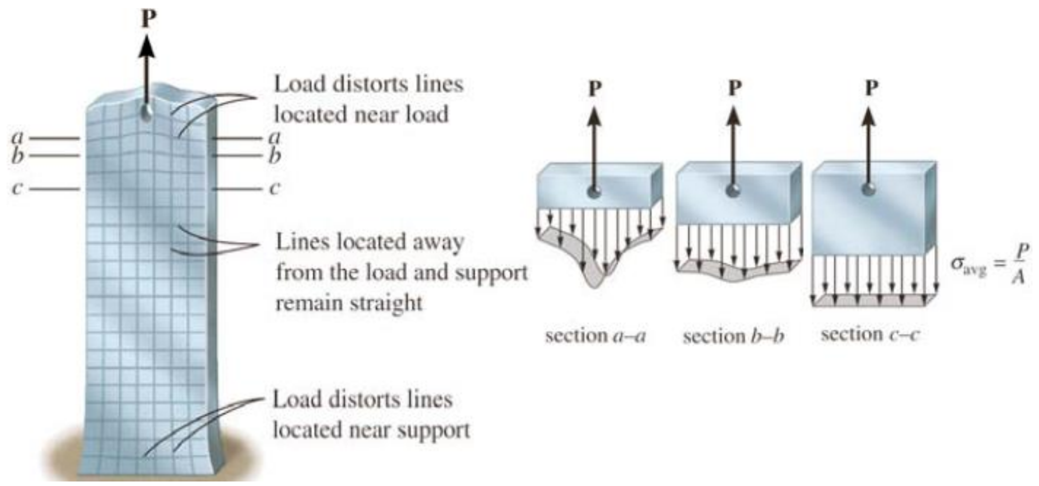


Fig. 5: Saint-Venant's Principle

Point loads on a surface give rise to a stress concentration near the point of application. A stress concentration is an increase in stress along the cross-section that may be caused either by such a point load or by another discontinuity, such as a hole in the material or an abrupt change in the cross-sectional shape. Since we have already shown strain to be proportional to stress, we can get a good idea about the magnitude of normal stress by examining the normal strain in a material as it is being subjected to some loads. To allow this, we can draw lines parallel to the normal plane and see if they remain plane during load application. In each of the following cases, witness how near the discontinuity there is a non-uniform distribution in the strain (and therefore stress) field, while farther away the distribution is linear (ie. the lines remain straight).