BENDING OF A NARROW CANTILEVER BEAM SUBJECTED TO END LOAD

Consider a cantilever beam of narrow rectangular cross-section carrying a load P at the end as shown in Figure 6. Find corresponding stress, strain and displacement equations.

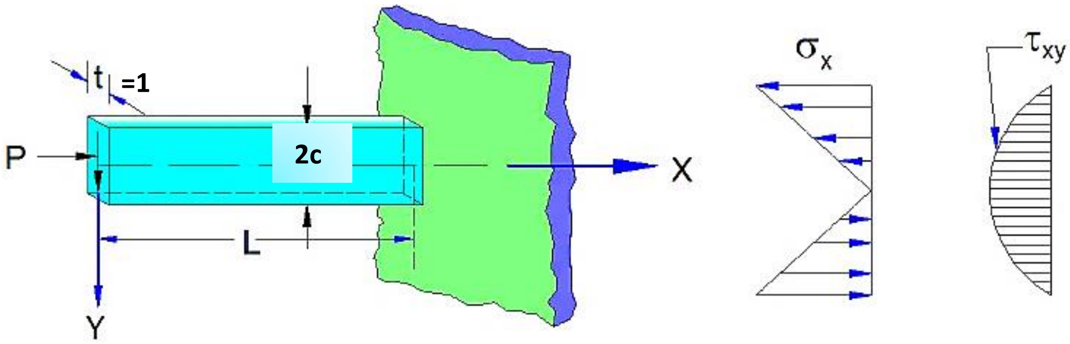


Fig. 6: Cantilever Beam Subjected to End Load

The above problems may be considered as a case of plane stress provided that the thickness of the beam t is small relative to the depth 2c.

The elasticity solution for this problem is again obtained using a Airy stress function. In this case,

$$\Phi = b_2 xy - \frac{d_4}{6} xy^3$$

$$\sigma_x = d_4 xy, \qquad \sigma_y = 0$$

$$\tau_{xy} = -b_2 - \frac{d_4}{2} y^2$$

$$(\tau_{xy})_{y=\pm c} = -b_2 - \frac{d_4}{2} c^2 = 0$$

$$d_4 = -\frac{2b_2}{c^2}$$

These conditions express the fact that the top and bottom edges of the beam are not loaded. Further, the applied load *P* must be equal to the resultant of the shearing forces distributed across the free end.

$$-\int_{-c}^{c} \tau_{xy} \cdot dy = \int_{-c}^{c} \left(b_2 - \frac{b_2}{c^2} y^2\right) dy = P$$

$$b_2 = \frac{3}{4}\frac{P}{c} \qquad \sigma_x = -\frac{3}{2}\frac{P}{c^3} xy, \qquad \sigma_y = 0$$

$$\tau_{xy} = -\frac{3P}{4c}\left(1 - \frac{y^2}{c^2}\right)$$

But $2/3c^3$ is equal to the moment of inertia I of the cross-section, thus $d^4 = P/I$. The stress state in the beam is thus described by

$$\sigma_x = -\frac{Pxy}{I}, \qquad \sigma_y = 0$$
$$\tau_{xy} = -\frac{P}{I}\frac{1}{2}(c^2 - y^2)$$

Note**:** The present solution is an exact one only if the shearing forces are distributed according to the same parabolic law as assumed herein. If the distribution of forces is different from this parabolic law but is equivalent statically, then the above expressions for σ_x and τ_{xy} do not represent a correct solution at the ends of the beam. Away from the ends (say a distance on the order of the depth of the beam), Saint-Venant's principle assures that the solution will be satisfactory.

The strains associated with the problem are related to the stresses through the constitutive relations. Assuming a linear, isotropic elastic material gives

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x}{E} = -\frac{Pxy}{EI}, \qquad \epsilon_y = \frac{\partial v}{\partial y} = -\frac{\nu\sigma_x}{E} = \frac{\nu Pxy}{EI}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G} = -\frac{P}{2IG}(c^2 - y^2)$$

The displacements u and v are obtained by suitably integrating ε_x and ε_x, then substitute in γ_{xy} equation and subjected to appropriate boundary conditions. In particular,

$$u = -\frac{Px^2y}{2EI} + f(y), \qquad v = \frac{\nu Pxy^2}{2EI} + f_1(x)$$

where *f*(*y*) and $f_1(x)$ are as yet unknown functions of *y* and *x*, respectively. Substituting theses equations into γ_{xy} equation gives:

$$-\frac{Px^2}{2EI} + \frac{df(y)}{dy} + \frac{\nu Py^2}{2EI} + \frac{df_1(x)}{dx} = -\frac{P}{2IG}(c^2 - y^2)$$

This equation contains terms that are functions of x only and of y only, and one term that is independent of both x and y. It is thus re-written as

$$F(x) + G(y) = K$$

Where:

$$F(x) = -\frac{Px^2}{2EI} + \frac{df_1(x)}{dx}, \qquad G(y) = \frac{df(y)}{dy} + \frac{\nu Py^2}{2EI} - \frac{Py^2}{2IG}$$
$$K = -\frac{Pc^2}{2IG}$$

The form of equation means that *F(x)* must be some constant *d* and *G(y)* must be some constant *e*. The equations are thus re-written as:

$$e + d = -\frac{Pc^2}{2IG}$$

$$\frac{df_1(x)}{dx} = \frac{Px^2}{2EI} + d, \qquad \frac{df(y)}{dy} = -\frac{Py^2}{2EI} + \frac{Py^2}{2IG} + e$$

Integrating these equations gives:

$$f(y) = -\frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + ey + g$$

$$f_1(x) = \frac{Px^3}{6EI} + dx + h$$

where *g* and *h* are constants of integration. Substituting in the expressions for *u* and *v* gives:

$$u = -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + ey + g$$

$$v = \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} + dx + h$$

Assuming that the centroid of the cross-section at the end is fixed, it follows that for all boundary conditions $u = v = 0$ at $x = L$ and $y = 0$

$$g = 0, \qquad h = -\frac{Pl^3}{6EI} - dl \qquad (v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^3}{6EI} - d(l - x)$$

Case 1: The first possible constraint condition assumes that an element of the longitudinal beam axis is fixed; i.e., horizontal element at the end remain horizontal: $\partial v/\partial y = 0$ at $x = l$ and $y = 0$. Differentiating *v* equation and evaluating it at $x = L$ and $y = 0$ gives:

$$\left(\frac{\partial v}{\partial x}\right)_{\substack{x=l\\y=0}} = 0 \qquad d = -\frac{Pl^2}{2EI}$$

$$e = \frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}$$

$$u = -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + \left(\frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}\right)y$$

$$v = \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI}$$

The general equation for the elastic curve (deflection curve) becomes:

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI}$$

$$(u)_{x=l} = -\frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} - \frac{Pc^2y}{2IG}$$

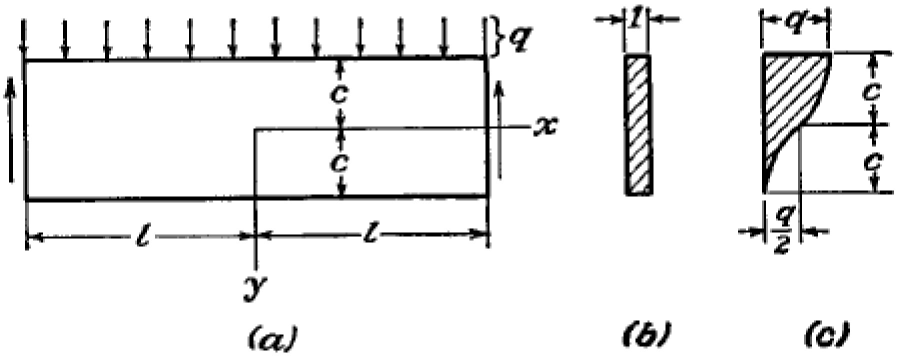
$$\left(\frac{\partial u}{\partial y}\right)_{x=l} = -\frac{\nu Py^2}{2EI} + \frac{Py^2}{2IG} - \frac{Pc^2}{2IG}$$

$$\left(\frac{\partial u}{\partial y}\right)_{\substack{x=l\\y=0}} = -\frac{Pc^2}{2IG} = -\frac{3}{4}\frac{P}{cG}$$

Problem 1 ***:*** find the elastic curve (deflection curve) for *case 2*: The constraint condition at the fixed end and assumes that a vertical element of the cross-section is fixed; i.e., $\partial u/\partial y = 0$ at $x = l$ and compare between results of *case 1*.

Problem 2: Let a beam of narrow rectangular cross section of unit width, supported at the end, be bent by a uniformly distributed load of intensity q, and subjected to shear load along its sides Calculate the relating stress, strain, displacements and elastic curve (deflection curve) for following B.C.

$$(\tau_{xy})_{y=\pm c} = 0, \qquad (\sigma_y)_{y=+c} = 0, \qquad (\sigma_y)_{y=-c} = -q$$



<u>Problem 3</u>: Show that : $\emptyset = \left(\frac{q}{8c^3}\right)\left[x^2(y^3 - 3c^2y + 2c^3) - \left(\frac{1}{5}\right)y^3(y^2 - 2c^2)\right]$ is a stress function and find problem it solves when applied to the region in y = ± ½ c and x = 0 on the side x positive.