PROBLEM 1

Given the following stress function

$$\phi = \frac{P}{\pi} r\theta\cos\theta$$

Determine the stress components σ_r, σ_θ *and* $\tau_{r\theta}$

Solution: The stress components, by definition of ϕ, are given as follows

$$\sigma_r = \left(\frac{1}{r}\right)\frac{\partial\phi}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial^2\phi}{\partial\theta^2} \qquad \text{(i)}$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} \qquad \text{(ii)}$$

$$\tau_{r\theta} = \left(\frac{1}{r^2}\right)\frac{\partial\phi}{\partial\theta} - \left(\frac{1}{r}\right)\frac{\partial^2\phi}{\partial r\partial\theta} \qquad \text{(iii)}$$

The various derivatives are as follows:

$$\frac{\partial\phi}{\partial r} = \frac{P}{\pi}\theta\cos\theta$$

$$\frac{\partial^2\phi}{\partial r^2} = 0$$

$$\frac{\partial\phi}{\partial\theta} = \frac{P}{\pi} r(-\theta\sin\theta + \cos\theta)$$

$$\frac{\partial^2\phi}{\partial\theta^2} = -\frac{P}{\pi} r(\theta\cos\theta + 2\sin\theta)$$

$$\frac{\partial^2\phi}{\partial r\partial\theta} = \frac{P}{\pi}(-\theta\sin\theta + \cos\theta)$$

Substituting the above values in equations (i), (ii) and (iii), we get

$$\sigma_r = \left(\frac{1}{r}\right)\frac{P}{\pi}\theta\cos\theta - \left(\frac{1}{r^2}\right)\frac{P}{\pi} r(\theta\cos\theta + 2\sin\theta)$$

$$= \left(\frac{1}{r}\right)\frac{P}{\pi}\theta\cos\theta - \left(\frac{1}{r}\right)\frac{P}{\pi}\theta\cos\theta - \left(\frac{1}{r}\right)\frac{P}{\pi}2\sin\theta$$

$$\therefore \sigma_r = -\frac{2}{r}\frac{P}{\pi}\sin\theta$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = 0$$

$$\tau_{r\theta} = \left(\frac{1}{r^2}\right)\frac{P}{\pi} r(-\theta\sin\theta + \cos\theta) - \left(\frac{1}{r}\right)\frac{P}{\pi}(-\theta\sin\theta + \cos\theta)$$

$$\therefore \tau_{r\theta} = 0$$

Therefore, the stress components are

$$\sigma_r = -\left(\frac{2}{r}\right)\frac{P}{\pi}\sin\theta$$

$$\sigma_\theta = 0$$

$$\tau_{r\theta} = 0$$

PROBLEM 2

A thick cylinder of inner radius 10*cm* and outer radius 15*cm* is subjected to an internal pressure of 12*MPa*. Determine the radial and hoop stresses in the cylinder at the inner and outer surfaces.

Solution: The radial stress in the cylinder is given by

$$\sigma_r = \left(\frac{p_i a^2 - p_o b^2}{b^2 - a^2}\right) - \left(\frac{p_i - p_o}{b^2 - a^2}\right)\frac{a^2 b^2}{r^2}$$

The hoop stress in the cylinder is given by

$$\sigma_\theta = \left(\frac{p_i a^2 - p_o b^2}{b^2 - a^2}\right) + \left(\frac{p_i - p_o}{b^2 - a^2}\right)\frac{a^2 b^2}{r^2}$$

As the cylinder is subjected to internal pressure only, the above expressions reduce to

$$\sigma_r = \left(\frac{p_i a^2}{b^2 - a^2}\right) - \left(\frac{p_i}{b^2 - a^2}\right)\frac{a^2 b^2}{r^2}$$

and $\sigma_\theta = \left(\frac{p_i a^2}{b^2 - a^2}\right) + \left(\frac{p_i}{b^2 - a^2}\right)\frac{a^2 b^2}{r^2}$

Stresses at inner face of the cylinder (i.e., at r = 10 cm):

Radial stress $= \sigma_r = \left[\frac{12 \times (0.1)^2}{(0.15)^2 - (0.1)^2}\right] - \left[\frac{(0.15)^2 (0.1)^2}{(0.1)^2}\right]\left[\frac{12}{(0.15)^2 - (0.1)^2}\right]$

$= 9.6 - 21.6$

or $\sigma_r = -12\ MPa$

Hoop stress $= \sigma_\theta = \left[\frac{12 \times (0.1)^2}{(0.15)^2 - (0.1)^2}\right] + \left[\frac{12}{(0.15)^2 - (0.1)^2}\right]\left[\frac{(0.15)^2 (0.1)^2}{(0.1)^2}\right]$

$= 9.6 + 21.6$

or $\sigma_\theta = 31.2\ MPa$

Stresses at outerface of the cylinder (i.e., at r = 15 cm):

Radial stress $= \sigma_r = \left[\frac{12 \times (0.1)^2}{(0.15)^2 - (0.1)^2}\right] - \left[\frac{12}{(0.15)^2 - (0.1)^2}\right]\left[\frac{(0.1)^2 (0.15)^2}{(0.15)^2}\right]$

$\sigma_r = 0$

Hoop stress $= \sigma_\theta = \left[\frac{12 \times (0.1)^2}{(0.15)^2 - (0.1)^2}\right] + \left[\frac{(0.1)^2 (0.15)^2}{(0.15)^2}\right]\left[\frac{12}{(0.15)^2 - (0.1)^2}\right]$

$= 9.6 + 9.6$

or $\sigma_\theta = 19.2\ MPa$

PROBLEM 3

A steel tube, which has an outside diameter of 10cm and inside diameter of 5cm, is subjected to an internal pressure of 14 *MPa* and an external pressure of 5.5 *MPa*. Calculate the maximum hoop stress in the tube.

***Solution*:** The maximum hoop stress occurs at $r = a$.

Therefore, Maximum hoop stress = $(\sigma_\theta)_{max} = \left[\frac{p_i a^2 - p_0 b^2}{b^2 - a^2}\right] + \left[\frac{p_i - p_0}{b^2 - a^2}\right]\left[\frac{a^2 b^2}{a^2}\right]$

$$= \left[\frac{p_i a^2 - p_0 b^2}{b^2 - a^2}\right] + \left[\frac{p_i - p_0}{b^2 - a^2}\right] b^2$$

$$= \frac{p_i a^2 - p_0 b^2 + p_i b^2 - p_0 b^2}{b^2 - a^2}$$

$$(\sigma_\theta)_{max} = \frac{p_i (a^2 + b^2) - 2 p_0 b^2}{b^2 - a^2}$$

Therefore, $(\sigma_\theta)_{max} = \frac{14[(0.05)^2 + (0.1)^2] - 2 \times 5.5 \times (0.1)^2}{(0.1)^2 - (0.05)^2}$

Or $(\sigma_\theta)_{max} = 8.67\ MPa$

DISPLACEMENT FOR SYMMETRIC LOADED CASES

By using physical equations, we obtain strain of axisymmetrical problems as follow:

$$\sigma_r = \frac{A}{r^2} + B(1 + 2\log r) + 2C$$

$$\sigma_\theta = -\frac{A}{r^2} + B(3 + 2\log r) + 2C$$

$$\tau_{r\theta} = \tau_{\theta r} = 0$$

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}$$

$$\frac{\partial u_r}{\partial r} = \varepsilon_r = \frac{1}{E}\left[(1+\mu)\frac{A}{r^2} + (1-3\mu)B + 2(1-\mu)B\log r + 2(1-\mu)C\right]$$

$$\frac{u_r}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \varepsilon_\theta = \frac{1}{E}\left[-(1+\mu)\frac{A}{r^2} + (3-\mu)B + 2(1-\mu)B\log r + 2(1-\mu)C\right]$$

$$\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \gamma_{r\theta} = 0$$

By integration, we obtain displacements: (using above 3 equations)

$$u_r = \frac{1}{E}\left[-(1+\mu)\frac{A}{r} + 2(1-\mu)Br(\log r - 1) + (1-3\mu)Br + 2(1-\mu)Cr\right] + f(\theta),$$

$$v = \frac{4Br\theta}{E} - \int f(\theta)d\theta + f_1(r)$$

$$\gamma_{r\theta} = 0 \Rightarrow f_1(r) - r\frac{df_1(r)}{dr} - \frac{df(\theta)}{d\theta} - \int f(\theta)d\theta = 0$$

where $f(\theta)$ and $f_1(r)$ are respectively arbitrary functions of θ and r

$$\begin{cases} f_1(r) = Fr \\ f(\theta) = H\cos\theta + K\sin\theta \end{cases}$$ F, H, K are a constants

Finally, we have the displacement solutions for the axisymmetric problems:

$$u_r = \frac{1}{E}\left[-(1+\mu)\frac{A}{r} + 2(1-\mu)Br(\log r - 1) + (1-3\mu)Br + 2(1-\mu)Cr\right] + K\cos\theta + H\sin\theta$$

$$v = \frac{4Br\theta}{E} + Fr - K\sin\theta + H\cos\theta$$

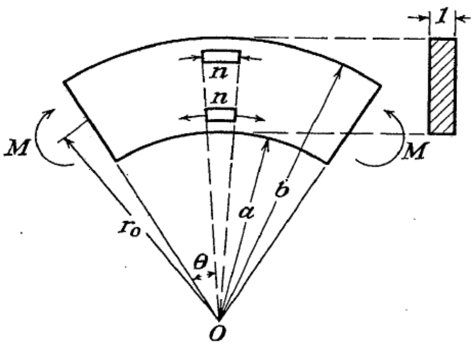
The arbitrary constants *A, B, C, H, F, K* can be determined by known (boundary) conditions. It is noted that the displacements are usually not symmetrical about the z axis.

For plane strain problem, the solution is obtained by simply replace

$E \rightarrow \dfrac{E}{(1-\mu^2)}$, $\mu \rightarrow \dfrac{\mu}{(1-\mu)}$.

PURE BENDING OF CURVED BAR

Consider a curved bar with a constant narrow rectangular cross section and a circular axis in the plane of curvature by couples *M* applied at the ends, as shown in Figure.



$\phi = A \log r + Br^2 \log r + Cr^2 + D$

$$\sigma_r = \frac{A}{r^2} + B(1 + 2\log r) + 2C$$

$$\sigma_\theta = -\frac{A}{r^2} + B(3 + 2\log r) + 2C$$

The B.C. are:

$$\sigma_r = 0 \qquad \text{for } r = a \text{ and } r = b$$

$$\int_a^b \sigma_\theta \, dr = 0 \qquad \int_a^b \sigma_\theta r \, dr = -M$$

$$\tau_{r\theta} = 0 \qquad \text{at the boundary}$$

$$\frac{A}{a^2} + B(1 + 2 \log a) + 2C = 0$$

$$\frac{A}{b^2} + B(1 + 2 \log b) + 2C = 0$$

To have the bending couple equal to *M*, the condition

$$\int_a^b \sigma_\theta r \, dr = \int_a^b \frac{\partial^2 \phi}{\partial r^2} r \, dr = -M$$

must be fulfilled. We have

$$\int_a^b \frac{\partial^2 \phi}{\partial r^2} r \, dr = \left| \frac{\partial \phi}{\partial r} r \right|_a^b - \int_a^b \frac{\partial \phi}{\partial r} dr = \left| \frac{\partial \phi}{\partial r} r \right|_a^b - |\phi|_a^b$$

$$\left| \frac{\partial \phi}{\partial r} r \right|_a^b = 0$$

Substituting in ϕ expression

$$|\phi|_a^b = M$$

$$A \log \frac{b}{a} + B(b^2 \log b - a^2 \log a) + C(b^2 - a^2) = M$$

$$A = -\frac{4M}{N} a^2 b^2 \log \frac{b}{a} \qquad B = -\frac{2M}{N}(b^2 - a^2)$$

$$C = \frac{M}{N}[b^2 - a^2 + 2(b^2 \log b - a^2 \log a)]$$

where for simplicity we have put

$$N = (b^2 - a^2)^2 - 4a^2b^2\left(\log \frac{b}{a}\right)^2$$

$$\sigma_r = -\frac{4M}{N}\left(\frac{a^2b^2}{r^2}\log\frac{b}{a} + b^2 \log\frac{r}{b} + a^2 \log\frac{a}{r}\right)$$

$$\sigma_\theta = -\frac{4M}{N}\left(-\frac{a^2b^2}{r^2}\log\frac{b}{a} + b^2 \log\frac{r}{b} + a^2 \log\frac{a}{r} + b^2 - a^2\right)$$

$$\tau_{r\theta} = 0$$

The displacement solutions for the axisymmetric problems:

$$u_r = \frac{1}{E}\left[-(1+\mu)\frac{A}{r} + 2(1-\mu)Br(\log r - 1) + (1-3\mu)Br + 2(1-\mu)Cr\right] + K\cos\theta + H\sin\theta$$

$$v = \frac{4Br\theta}{E} + Fr - K\sin\theta + H\cos\theta$$

In the case of pure bending, the conditions of constraint are:

$$u = 0 \qquad v = 0 \qquad \frac{\partial v}{\partial r} = 0 \qquad \text{for } \theta = 0 \text{ and } r = r_0 = \frac{a+b}{2}$$

$$\frac{1}{E}\left[-\frac{(1+\nu)A}{r_0} + 2(1-\nu)Br_0\log r_0 - B(1+\nu)r_0 + 2C(1-\nu)r_0\right] + K = 0$$

$$Fr_0 + H = 0$$

$$F = 0$$

From this it follows that $F = H = 0$, and for the displacement v we obtain

$$v = \frac{4Br\theta}{E} - K\sin\theta$$

$$K = \frac{(1+\mu)A}{r_o} - 2(1-\mu)Br_o log r_o + B(1+\mu)r_o - 2C(1-\mu)r_o$$

The arbitrary constants *A, B, C* can be determined by known (boundary) conditions from stress equations.

Note: 1. v displacement equation consists of two parts:

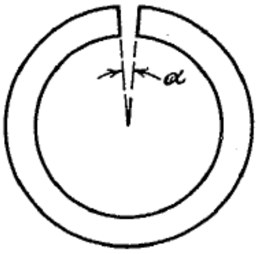
- Translator displacement $-K\sin\theta$ "same for all points in the same sections"
- Rotation of cross-section by the angle $4B\theta/E$

2. Plane cross sections remain plane in pure bending, i.e.

$$\partial v/_{\partial r} = \ constant\ as\ assumed.$$

3. The final displacements are not axisymmetric, while we started assuming everything is axisymmetric.

The constant B in the case of symmetrical stress distribution in full ring was taken as zero. If apportion of the ring between two adjacent cross-sections is cut out and joined again by welding or other means, a ring with initial stresses is obtained. If α is the small angle measuring the portion of the ring that was cut out, the tangential displacement necessary to bring the end of ring together is: $v = \alpha r$



$$v = 2\pi \frac{4Br}{E}$$

$$B = \frac{\alpha E}{8\pi}$$

The bending moment necessary to bring the ends of ring together is:

$$M = -\frac{\alpha E}{8\pi}\frac{(b^2 - a^2)^2 - 4a^2b^2[\log(b/a)]^2}{2(b^2 - a^2)}$$

The initial stresses in the ring can easily be calculated from this by using The solution for pure bending moment.

$$\sigma_r = -\frac{4M}{N}\left(\frac{a^2b^2}{r^2}\log\frac{b}{a} + b^2\log\frac{r}{b} + a^2\log\frac{a}{r}\right)$$

$$\sigma_\theta = -\frac{4M}{N}\left(-\frac{a^2b^2}{r^2}\log\frac{b}{a} + b^2\log\frac{r}{b} + a^2\log\frac{a}{r} + b^2 - a^2\right)$$

$$\tau_{r\theta} = 0$$

STRESS DISTRIBUTION IN ROTATING CIRCULAR DISK

The stress distribution in rotating circular disks is of great practical importance. If the thickness of the disk is small in comparison with its radius, the variation of radial and tangential stresses over the thickness can be neglected and the problem can be easily solved. If the thickness of the disk is constant Eq.

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0$$

$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0$$

$F_r=\rho\omega^2 r \qquad F_\theta = 0$

Where ρ is the mass per unit volume of the material of the disk and ω the angular velocity of the disk. The equations above can then be written in the form

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho\omega^2 r^2 = 0$$

This equation is satisfied if we derive the stress components from a stress function F in the following manner: $r\sigma_r = F, \qquad \sigma_\theta = \frac{dF}{dr} + \rho\omega^2 r^2$

The strain components in the case of symmetry are, $\epsilon_r = \frac{du}{dr} \qquad \epsilon_\theta = \frac{u}{r}$

$$\epsilon_\theta - \epsilon_r + r\frac{d\epsilon_\theta}{dr} = 0$$

Substituting for the strain components their expressions in terms of the stress components,

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

we find that the stress function F should satisfy the following equation:

$$r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr} - F + (3+\nu)\rho\omega^2 r^3 = 0$$

It can be verified by substitution that the general solution of this equation is $F = Cr + C_1\frac{1}{r} - \frac{3+\nu}{8}\rho\omega^2 r^3$

$$\sigma_r = C + C_1\frac{1}{r^2} - \frac{3+\nu}{8}\rho\omega^2 r^2$$

$$\sigma_\theta = C - C_1\frac{1}{r^2} - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

The integration constants C and C_1 are determined from the boundary conditions.

- For the case of a *solid disk* we must take $C_1 = 0$ since otherwise the stresses become infinite at the center. The constant C is determined from the condition at the periphery ($r = b$) of the disk. If there are no forces applied there, we have

$$(\sigma_r)_{r=b} = C - \frac{3+\nu}{8}\rho\omega^2 b^2 = 0$$

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(b^2 - r^2)$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2 b^2 - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

These stresses are greatest at the center of the disk, where

$$\sigma_r = \sigma_\theta = \frac{3+\nu}{8}\rho\omega^2 b^2$$

- In the case of existing of small hole of diameter *a* in the center of disk. If there are no forces acting on these boundaries, we have

$$(\sigma_r)_{r=a} = 0, \qquad (\sigma_r)_{r=b} = 0$$

from which we find that: $C = \frac{3+\nu}{8}\rho\omega^2(b^2 + a^2); \qquad C_1 = -\frac{3+\nu}{8}\rho\omega^2 a^2 b^2$

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(b^2 + a^2 - \frac{a^2b^2}{r^2} - r^2\right)$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2\left(b^2 + a^2 + \frac{a^2b^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2\right)$$

We find the maximum radial stress at r = √ab, where

$$(\sigma_r)_{\max.} = \frac{3+\nu}{8} \cdot \rho\omega^2(b-a)^2$$

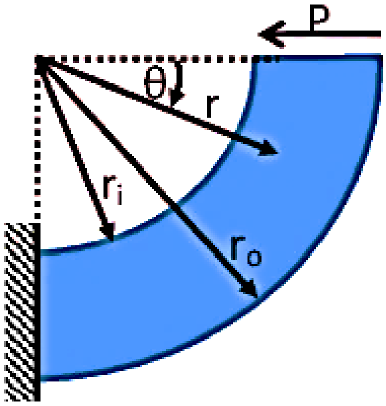
The maximum tangential stress is at the inner boundary, where

$$(\sigma_\theta)_{\max.} = \frac{3+\nu}{4}\rho\omega^2\left(b^2 + \frac{1-\nu}{3+\nu}a^2\right)$$

It will be seen that this stress is larger than (σ_r) max.

When the radius a of the hole approaches zero, the maximum tangential stress approaches a value twice as great as that for a solid disk; i.e., by making a small circular hole at the center of a solid rotating disk we double the maximum stress.

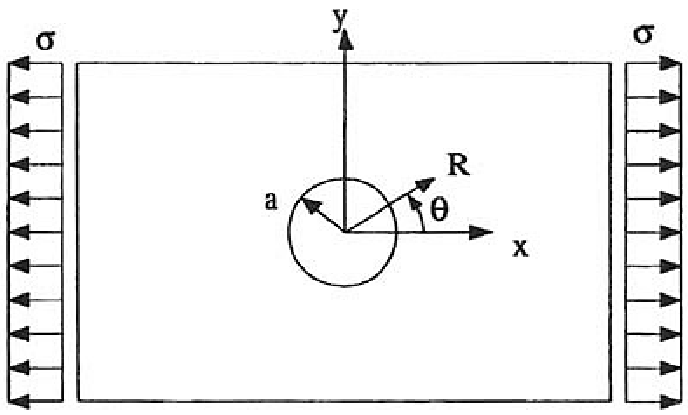
Problem 1: Drive an expression for bending of curved bar by forces at end.



Problem 2: A thin plate with small hole relative to the plate dimension is subjected to a normal stress $\sigma_x = \sigma$ as shown in Figure below. The stresses in the neighborhood of the hole are to be determined. Assume area outside circle of radius R to be unaffected by hole existence i.e. sufficiently far from the hole we may assume that the stresses are as for a plate without the hole.

1. Determine an expression for σ_θ, σ_r and $\tau_{r\theta}$
2. Determine stress at hole face

If the stress function for this case is: $\emptyset = (Ar^2 + Br^4 + \frac{C}{r^2} + D)\cos 2\theta$



Problem 3: Consider a semi – finite medium under a normal line load p (per unit width). Determine an expression for σ_θ, σ_r and $\tau_{r\theta}$ inside the media using the following stress function:

$\emptyset = Cr\theta \sin\theta$

Note: check if the stress function satisfy $\nabla^4\emptyset = 0$