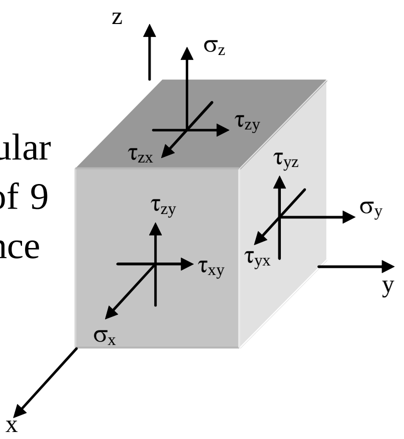
3D ELASTICITY

SPECIFICATION OF STRESS AT POINT AND EQUATIONS OF EQUILIBIRIUM



σ_x, σ_y and σ_z are normal stresses, the rest 6 are the

shear stresses. τ_{xy} is the stress on the face perpendicular to the x-axis and points in the +ve y direction Total of 9 stress components of which only 6 are independent since

$$\tau_{xy} = \tau_{yx}$$
$$\tau_{yz} = \tau_{zy}$$
$$\tau_{zx} = \tau_{xz}$$

The stress vector is therefore:

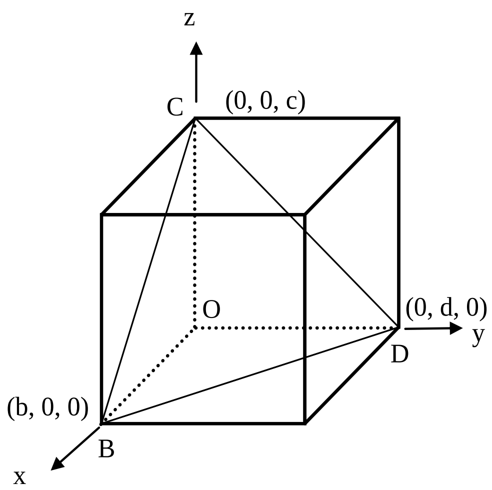
$$\underline{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

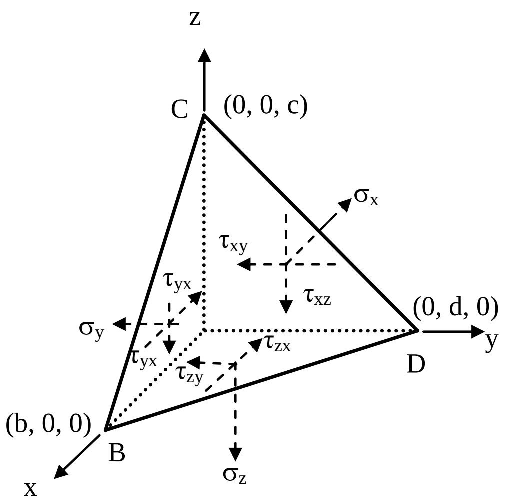
Consider the equilibrium of a differential volume element to obtain the 3 equilibrium equations of elasticity:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0$$

These equations must satisfied all point throughout the volume of the body.

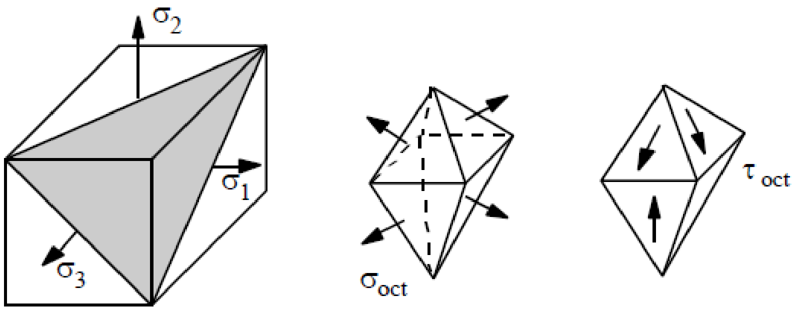
The stress vary over the volume of the body, and when we arrive at the surface they must be in such equilibrium with external forces in the surface of the body. This conditions of equilibrium with external forces on the surface can be obtained taking a tetrahedron *OBCD*





OCTAHEDRAL PLANES AND STRESSES

The Figure below illustrates the orientation of one of the eight octahedral planes which are associated with a given stress state.



Each of the octahedral planes cut across one of the corners of a principal element, so that the eight planes together form an octahedron. The stresses acting on these planes have interesting and significant characteristics. First of all, identical normal stresses act on all eight planes. By themselves the normal stresses are therefore said to be hydrostatic and tend to compress or enlarge the octahedron but not distort it.

$$\sigma_{oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

Shear stresses are also identical. These distort the octahedron without changing its volume. Although the octahedral shear stress is smaller than the highest principal shear stress, it constitutes a single value that is influenced by all three principal shear stresses. Thus, it is important as a criterion for predicting yielding of a stressed material.

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

In cases in which σ_x, σ_y, σ_z, τ_{xy}, τ_{xz} and τ_{yz} are known:

$$\sigma_{oct} = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right)\right]^{1/2}$$

CONDITION OF COMPATIBILITY

Six independent strain components can be notice:
We have 3 displacement functions and 6 strain components. But strains are derivable from displacement

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_y = \frac{\partial v}{\partial y},\ \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},\ \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

We have 6 compatibility equations :

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y}, \qquad 2\frac{\partial^2 \epsilon_x}{\partial y\, \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z}, \qquad 2\frac{\partial^2 \epsilon_y}{\partial x\, \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial x\, \partial z}, \qquad 2\frac{\partial^2 \epsilon_z}{\partial x\, \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

These differential relations are called the conditions of compatibility.

Using Hooks law these conditions can transformed into relations between the components of stress.

Take, for instance, the condition: $$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z}$$

$$\epsilon_y = \frac{1}{E}[(1+\nu)\sigma_y - \nu\Theta]$$

$$\epsilon_z = \frac{1}{E}[(1+\nu)\sigma_z - \nu\Theta]$$

$$\gamma_{yz} = \frac{2(1+\nu)\tau_{yz}}{E}$$

Substitute in eq. above

Where: $\Theta = \sigma_x + \sigma_y + \sigma_z$

$$(1+\nu)\left(\frac{\partial^2 \sigma_y}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial y^2}\right) - \nu\left(\frac{\partial^2 \Theta}{\partial z^2} + \frac{\partial^2 \Theta}{\partial y^2}\right) = 2(1+\nu)\frac{\partial^2 \tau_{yz}}{\partial y\, \partial z}$$

Using equations of equilibrium: $$\frac{\partial \tau_{yz}}{\partial y} = -\frac{\partial \sigma_z}{\partial z} - \frac{\partial \tau_{xz}}{\partial x} - Z$$

$$\frac{\partial \tau_{yz}}{\partial z} = -\frac{\partial \sigma_y}{\partial y} - \frac{\partial \tau_{xy}}{\partial x} - Y$$

$$2\frac{\partial^2 \tau_{yz}}{\partial y\, \partial z} = -\frac{\partial^2 \sigma_z}{\partial z^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial}{\partial x}\left(\frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial y}\right) - \frac{\partial Z}{\partial z} - \frac{\partial Y}{\partial y}$$

$$2\frac{\partial^2 \tau_{yz}}{\partial y\, \partial z} = \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \sigma_z}{\partial z^2} + \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$(1+\nu)\left(\nabla^2\Theta - \nabla^2\sigma_x - \frac{\partial^2 \Theta}{\partial x^2}\right) - \nu\left(\nabla^2\Theta - \frac{\partial^2 \Theta}{\partial x^2}\right)$$

$$= (1+\nu)\left(\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right)$$

$$(1-\nu)\nabla^2\Theta = -(1+\nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)$$

$$\nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2 \Theta}{\partial x^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - 2\frac{\partial X}{\partial x}$$

In the same manner the remaining three conditions can transformed into equations of the following manner:

$$\nabla^2\tau_{yz} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\,\partial z} = -\left(\frac{\partial Z}{\partial y} + \frac{\partial Y}{\partial z}\right)$$

If there are no body forces or if the body forces are constant then these equations become:

$$\left.\begin{aligned}
&(1+\nu)\nabla^2\sigma_x + \frac{\partial^2\Theta}{\partial x^2} = 0, && (1+\nu)\nabla^2\tau_{yz} + \frac{\partial^2\Theta}{\partial y\,\partial z} = 0\\
&(1+\nu)\nabla^2\sigma_y + \frac{\partial^2\Theta}{\partial y^2} = 0, && (1+\nu)\nabla^2\tau_{xz} + \frac{\partial^2\Theta}{\partial x\,\partial z} = 0\\
&(1+\nu)\nabla^2\sigma_z + \frac{\partial^2\Theta}{\partial z^2} = 0, && (1+\nu)\nabla^2\tau_{xy} + \frac{\partial^2\Theta}{\partial x\,\partial y} = 0
\end{aligned}\right\}\text{ Prove them}$$

DETERMINATION OF DISPLACEMENT

To find *u*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial\epsilon_x}{\partial x}, \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial\gamma_{xy}}{\partial y} - \frac{\partial\epsilon_y}{\partial x}, \qquad \frac{\partial^2 u}{\partial z^2} = \frac{\partial\gamma_{xz}}{\partial z} - \frac{\partial\epsilon_z}{\partial x}$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial\epsilon_x}{\partial y}, \qquad \frac{\partial^2 u}{\partial x\,\partial z} = \frac{\partial\epsilon_x}{\partial z}, \qquad \frac{\partial^2 u}{\partial y\,\partial z} = \frac{1}{2}\left(\frac{\partial\gamma_{xz}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z} - \frac{\partial\gamma_{yz}}{\partial x}\right).$$

$$u' = a + by - cz \quad \text{(Prove it)}$$

The second derivatives for the two other components of displacement *v* and *w* can be obtained cyclical interchange in the equation above of letter *x*, *y*, *z*. the additional linear functions have the form:

$$v' = d - bx + ez$$
$$w' = f + cx - ey$$

The constants *a*, *d*, *f* represent a translatory motion of the body, and *b*, *c*, *e* are the three rotations of the rigid body around the coordinate axes. These constants can easily be calculated so as to satisfy the condition of constraint.

PRINCIPAL OF SUPERPOSITION

This principle states that For a given body made up of a material that obeys isotropic Hooke's law, in static equilibrium and whose magnitude of displacement is small.
if $\{u^{(1)}, \sigma^{(1)}\}$ is a solution to the prescribed body forces, $b^{(1)}$ and boundary conditions, $\{u_b^{(1)}, t_{(n)}^{(1)}\}$ and $\{u^{(2)}, \sigma^{(2)}\}$ is a solution to the prescribed body forces, $b_{(2)}$ and boundary conditions, $\{u_b^{(2)}, t_{(n)}^{(2)}\}$ then:
$\{u^{(1)} + u^{(2)}, \sigma^{(1)} + \sigma^{(2)}\}$ will be a solution to the problem with body forces, $b^{(1)} + b^{(2)}$ and boundary conditions, $\{u_b^{(1)} + u_b^{(2)}, t_{(n)}^{(1)} + t_{(n)}^{(2)}\}$. This is one of the most often used principles to solve problems in engineering.

UNIQUENESS OF SOLUTION

There could at most be one solution that could satisfy the prescribed displacement or mixed boundary conditions, in a given body made of a material that obeys Hooke's law and in static equilibrium with no body forces acting on it. If traction boundary condition is specified, we shall see that the stress is uniquely determined but the displacement is not for the bodies made of a material that obeys Hooke's law and in static equilibrium with no body forces acting on it.