ELASTIC STRESS-STRAIN RELATIONS

For a linear-elastic isotropic material with all components of stress present:

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] \qquad G = \frac{E}{2(1+\nu)}.$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$
$$\gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{yz} = \frac{2(1+\nu)}{E}\tau_{yz}$$
$$\gamma_{zx} = \frac{\tau_{zx}}{G} \qquad \gamma_{zx} = \frac{2(1+\nu)}{E}\tau_{zx}$$

These equations are the generalized Hooke's law. These Equations may be solved to obtain stress components as a function of strains:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)]$$
$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)]$$
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)]$$
$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = G\gamma_{xy}$$
$$\tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz} = G\gamma_{yz}$$
$$\tau_{zx} = \frac{E}{2(1+\nu)}\gamma_{zx} = G\gamma_{zx}.$$

For the first three relationships one may find:

$$\sigma_x = \frac{E}{(1+\nu)}\left[\varepsilon_x + \frac{\nu}{(1-2\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]$$
$$\sigma_y = \frac{E}{(1+\nu)}\left[\varepsilon_y + \frac{\nu}{(1-2\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]$$
$$\sigma_z = \frac{E}{(1+\nu)}\left[\varepsilon_z + \frac{\nu}{(1-2\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]$$

PRINCIPAL STRAINS AND PLANES

Strain relations can be as function of principal strain written as follows:

$$\varepsilon_1, \varepsilon_2 = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{1}{2}\gamma_{xy}\right)^2 + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2}$$

$$\gamma_{max} = \pm 2\sqrt{\left(\frac{1}{2}\gamma_{xy}\right)^2 + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2}$$

$$2\alpha = \tan^{-1}\frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

by virtue of all shear strains and shear stresses being equal to zero.

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)]$$

$$\varepsilon_2 = \frac{1}{E}[\sigma_2 - \nu(\sigma_3 + \sigma_1)]$$

$$\varepsilon_3 = \frac{1}{E}[\sigma_3 - \nu(\sigma_1 + \sigma_2)]$$

$$\sigma_1 = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_1 + \nu(\varepsilon_2 + \varepsilon_3)]$$

$$\sigma_2 = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_2 + \nu(\varepsilon_3 + \varepsilon_1)]$$

$$\sigma_3 = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_3 + \nu(\varepsilon_1 + \varepsilon_2)]$$

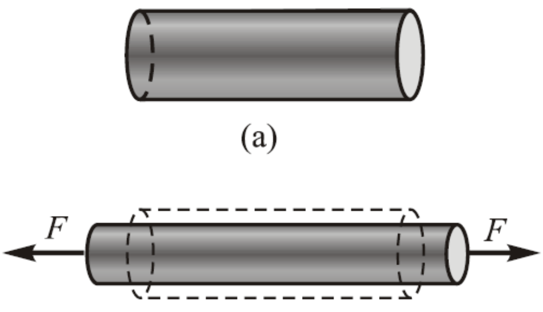
RODS UNDER AXIAL STRESS

As very simple example we may be taken tension of a prismatic bar in the axial direction. Body force are neglected. The eq. of equilibrium satisfied by taking:

$\sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0,$

$\sigma_x = constant = X$

We have a uniform distribution of tensile stress over cross-section.

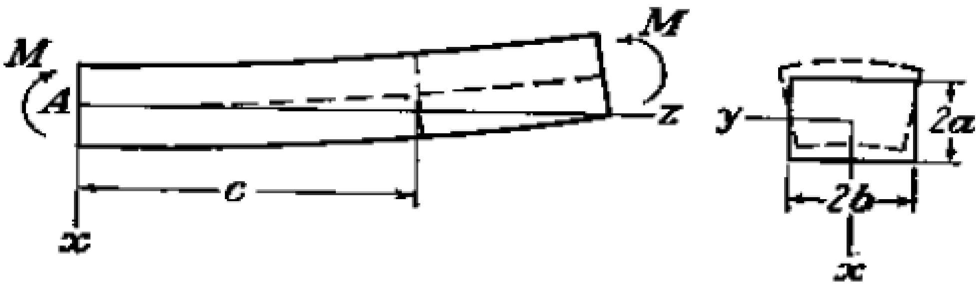


(a)

(b)

PURE BENDING OF PRISMATIC RODS

Consider a prismatic bar bent in one of its principal planes by two equal and opposite couples *M*. Taking the origin of the coordinates at the centroid of the cross-section and the *xz*-plane in the principal plane of bending, the stress components given by the usual elementary theory of bending are:



$$\sigma_y = \sigma_x = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0 \qquad \sigma_z = \frac{Ex}{R}$$

The bending moment *M* is given by the equation:

$$M = \int \sigma_z x \, dA = \int \frac{Ex^2 \, dA}{R} = \frac{EI_y}{R} \qquad \frac{1}{R} = \frac{M}{EI_y}$$

$$\epsilon_z = \frac{\partial w}{\partial z} = \frac{x}{R} \qquad w = \frac{xz}{R} + w_0$$

$$\epsilon_x = \frac{\partial u}{\partial x} = -\nu \frac{x}{R} \qquad \epsilon_y = \frac{\partial v}{\partial y} = -\nu \frac{x}{R}$$

$$u_0 = -\frac{\nu x^2}{2R} + f_1(y), \qquad v_0 = -\frac{\nu xy}{R} + f_2(x)$$

Integrate the strain-displacement relations and apply boundary conditions to obtain the displacement field.

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$$

$$\frac{\partial u}{\partial z} = -\frac{z}{R} - \frac{\partial w_0}{\partial x}, \qquad \frac{\partial v}{\partial z} = -\frac{\partial w_0}{\partial y}$$

$$u = -\frac{z^2}{2R} - z\frac{\partial w_0}{\partial x} + u_0, \qquad v = -z\frac{\partial w_0}{\partial y} + v_0$$

$$-z\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial u_0}{\partial x} = -\frac{\nu x}{R}, \qquad -z\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial v_0}{\partial y} = -\nu\frac{x}{R} \qquad \frac{\partial^2 w_0}{\partial x^2} = 0, \qquad \frac{\partial^2 w_0}{\partial y^2} = 0$$

$$2z\frac{\partial^2 w_0}{\partial x\,\partial y} - \frac{\partial f_1(y)}{\partial y} - \frac{\partial f_2(x)}{\partial x} + \frac{\nu y}{R} = 0 \qquad \frac{\partial^2 w_0}{\partial x\,\partial y} = 0, \qquad \frac{\partial f_1(y)}{\partial y} + \frac{\partial f_2(x)}{\partial x} - \frac{\nu y}{R} = 0$$

It's easy to show that all these equations are satisfied by assume $f_1(y)$, $f_2(x)$& w_o as following

$$w_0 = mx + ny + p$$

$$f_1(y) = \frac{\nu y^2}{2R} + \alpha y + \gamma$$

$$f_2(x) = -\alpha x + \beta$$

$$u = -\frac{z^2}{2R} - mz - \frac{\nu x^2}{2R} + \frac{\nu y^2}{2R} + \alpha y + \gamma$$

$$v = -nz - \frac{\nu xy}{R} - \alpha x + \beta$$

$$w = \frac{xz}{R} + mx + ny + p$$

At $x=y=z=0$ $\quad u = v = w = 0, \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial v}{\partial x} = 0$

These conditions are satisfied by taking all the arbitrary constants equal to zero. Then

$$u = -\frac{1}{2R}[z^2 + \nu(x^2 - y^2)], \qquad v = -\frac{\nu xy}{R}, \qquad w = \frac{xz}{R}$$

To obtain the deflection curve of the axis of the bar , substitute *x=y=0* in equations above. Then

$$u = -\frac{z^2}{2R} = -\frac{Mz^2}{2EI_y}, \qquad v = w = 0$$

Problem 1: Prove that

$$\begin{aligned} u' &= a + by - cz \\ v' &= d - bx + ez \\ w' &= f + cx - ey \end{aligned}$$

Problem 2: Drive an expression for displacement for a prismatic bar of length "*l* " and cross-section "*A*" hangs under its own weight *"ρg"*