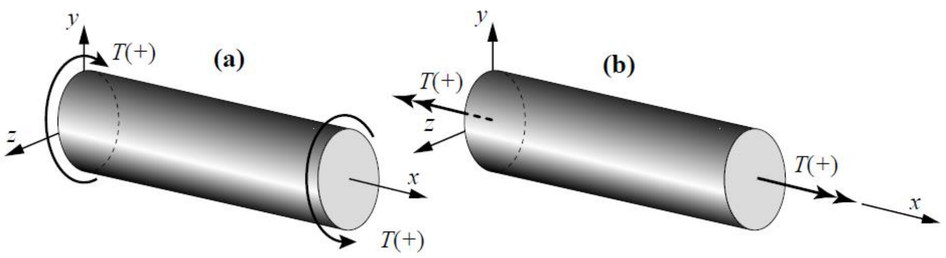
TORSION

TORSION OF PRISMATIC STRIAGHT BARS

Consider a prismatic bar subjected to twisting action,



Let,
$T = M_t$ =torque
u = displacement in the x direction
v = displacement in the y direction
w = displacement in the z direction
$\psi(x,y)$ = the warping function
θ = rotation I unit length
x, y, z = Cartesian co-ordinates

Consider any point in the section, which, owing to the application of T, will rotate and warp, as shown in Figure :

$u = -yz\theta$

$v = xz\theta$

due to rotation, and $w = \theta.\psi(x,y) = \theta\psi$

due to warping. The theory assumes that, $\varepsilon_x=\varepsilon_y=\varepsilon_z=\gamma_{xy}=0$

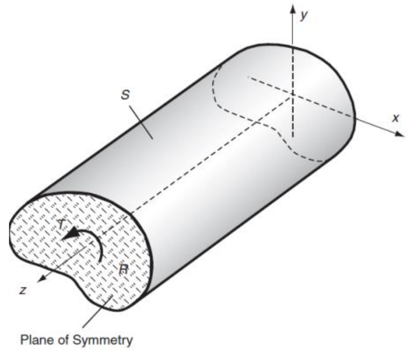
therefore the only shearing strains that exist are γ_{xz}, γ_{yz} which are defined as follows:

γ_{xz} = shear strain in the x-z plane $=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}=\theta\left(\frac{\partial \psi}{\partial x}-y\right)$

γ_{zy} = shear strain in the y-z plane

$$=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=\theta\left(\frac{\partial \psi}{\partial y}+x\right)$$

The corresponding components of stress,



$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

$$\tau_{xz} = G\theta\left(\frac{\partial\psi}{\partial x} - y\right) = -\frac{\partial\phi}{\partial y}$$

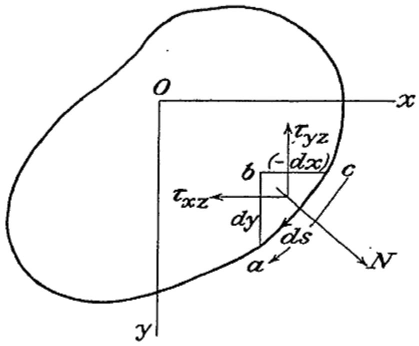
$$\tau_{yz} = G\theta\left(\frac{\partial\psi}{\partial y} + x\right) = \frac{\partial\phi}{\partial x}$$

We find that function ψ must satisfy the equation: $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$

The equilibrium equations in the case of torsion are:

$$\frac{\partial\tau_{yz}}{\partial y} \times dy \times dx \times dz + \frac{\partial\tau_{xz}}{\partial x} \times dx \times dy \times dz = 0$$

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} = 0$$



For the lateral surface of the bar, which is free has normal perpendicular to the *z*-axis, we have

$\bar{X} = \bar{Y} = \bar{Z} = 0$, and $\cos Nz = n = 0$.

$$\tau_{xz} = \frac{\partial\phi}{\partial y} \qquad \tau_{yz} = -\frac{\partial\phi}{\partial x}$$

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} = 0$$

$$\frac{\partial\phi}{\partial y} = G\theta\left(\frac{\partial\psi}{\partial x} - y\right) \qquad -\frac{\partial\phi}{\partial x} = G\theta\left(\frac{\partial\psi}{\partial y} + x\right)$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = F, \qquad F = -2G\theta$$

B.C: $\tau_{xz} l + \tau_{yz} m = 0$

Where: $l = \cos Nx = dy/dx$ *and* $m = \cos Ny = -dx/ds$

At the end of the twisted bar. The normal to the end cross-sections are parallel to the end. Then $l = m = 0$, $n = \pm 1$.

$$\bar{X} = \pm\tau_{xz} \qquad \bar{Y} = \pm\tau_{yz}$$

Let

$$\iint \bar{X}\,dx\,dy = \iint \tau_{xz}\,dx\,dy = \iint \frac{\partial\phi}{\partial y}\,dx\,dy = \int dx \int \frac{\partial\phi}{\partial y}\,dy = 0$$

$$\iint \bar{Y}\,dx\,dy = \iint \tau_{yz}\,dx\,dy = -\iint \frac{\partial\phi}{\partial x}\,dx\,dy = -\int dy \int \frac{\partial\phi}{\partial x}\,dx = 0$$

$$M_t = \iint (\bar{Y}x - \bar{X}y)\,dx\,dy = -\iint \frac{\partial\phi}{\partial x}\,x\,dx\,dy - \iint \frac{\partial\phi}{\partial y}\,y\,dx\,dy$$

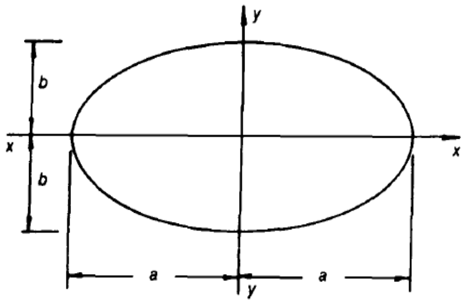
Integrating this by parts, and observing that $\phi=0$ at the boundary, we find

$$M_t = 2\iint \phi\,dx\,dy$$

TORSION OF ELLIPTIC CROSS SECTION

Let the B.C. of the cross-section be given by the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



To satisfy the B.C. and $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = F$ the stress function taking as:

$$\phi = \frac{a^2b^2F}{2(a^2+b^2)}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

$$M_t = \frac{a^2b^2F}{a^2+b^2}\left(\frac{1}{a^2}\iint x^2\,dx\,dy + \frac{1}{b^2}\iint y^2\,dx\,dy - \iint dx\,dy\right)$$

Since

$$\iint x^2\,dx\,dy = I_y = \frac{\pi ba^3}{4}, \qquad \iint y^2\,dx\,dy = I_x = \frac{\pi ab^3}{4},$$

$$\iint dx\,dy = \pi ab$$

$$M_t = -\frac{\pi a^3b^3F}{2(a^2+b^2)}$$

$$F = -\frac{2M_t(a^2+b^2)}{\pi a^3b^3}$$

$$\phi = -\frac{M_t}{\pi ab}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

The stress component are:

$$\tau_{yz} = G\theta\left(\frac{\partial\psi}{\partial y} + x\right) = \frac{\partial\phi}{\partial x}$$

$$\tau_{yz} = \frac{2M_t x}{\pi a^3 b}$$

Similarly, $\tau_{xz} = \dfrac{2M_t y}{\pi a b^3}$

$$\tau = \sqrt{\tau_{yz}^2 + \tau_{xz}^2} = \frac{2M_t}{\pi a^3 b^3}\left[b^4 x^2 + a^4 y^2\right]^{\frac{1}{2}}$$

To determine where the maximum shear stress occurs, substitute for x^2 from:

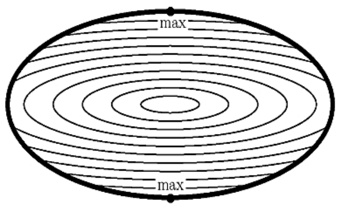
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \longrightarrow \quad \tau = \frac{2M_t}{\pi a^3 b^3}\left[a^2 b^4 + a^2(a^2 - b^2)y^2\right]^{\frac{1}{2}}$$

or $x^2 = a^2\,(1 - y^2/b^2)$

$$\tau = \frac{2M_t}{\pi a^3 b^3}\left[a^2 b^4 + a^2(a^2 - b^2)y^2\right]^{\frac{1}{2}}$$

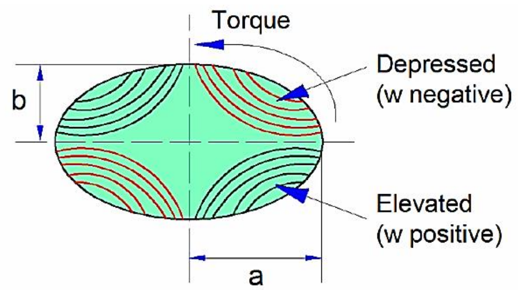
Since all terms under the radical (power 1/2) are positive, the maximum shear stress occurs when y is maximum, i.e., when $y = b$. Thus, maximum shear stress τ_{max} occurs at the ends of the minor axis and its value is

$$\tau_{max} = \frac{2M_t}{\pi a^3 b^3}(a^4 b^2)^{1/2}$$



Therefore, $\tau_{max} = \dfrac{2M_t}{\pi a b^2}$

For $a = b$, this formula coincides with the well-known formula for circular cross-section. Knowing the warping function, the displacement w can be easily determined.



$$w = \theta\psi = \frac{M_t(b^2 - a^2)}{\pi a^3 b^3 G}xy$$

Where: $\theta = \dfrac{4\pi^2 TJ}{A^4 G}$

$J = \dfrac{\pi}{64}[bh^3 + hb^3]$ and A is the area of cross-section $= \pi bh/4$

The contour lines giving w = constant are the hyperbolas shown in the Figure having the principal axes of the ellipse as asymptote.

EQUILATERAL TRIANGULAR SECTION (H.W.)

(a) Show that the stress function for the torsion of a long cylinder of **solid** triangular cross section (shown below) subjected to a given torque T is given by:

$$\Phi(x, y) = C\left(x - \sqrt{3}y - \frac{2}{3}h\right)\left(x + \sqrt{3}y - \frac{2}{3}h\right)\left(x + \frac{1}{3}h\right)$$

(b) Determine the related stress and the maximum shear stress τ_{max} and its position by plot