THEORY OF YIELD CRITERION

The failure of a material under stresses is the condition when the material cannot take any more stress.

In the case of multidimensional stress at a point we have a more complicated situation present. Since it is impractical to test every material and every combination of stresses σ_1, σ_2, and σ_3, a failure theory is needed for making predictions on the basis of a material's performance on the tensile test., of how strong it will be under any other conditions of static loading. The "theory" behind the various failure theories is that *whatever is responsible for failure in the standard tensile test will also be responsible for failure under all other conditions of static loading*.

Failure of a material under one normal stress is defined by experimental evidence as:

1. Yield stress σ_y (in tension or compression materials) in ductile materials
2. Rupture or crushing stress in brittle materials

For easier computations, consider a material under principal stresses:

$$\sigma_1 > \sigma_2 > \sigma_3$$

There are many theories of failures:

- Max. principal stress theory – Rankine
- Max. principal strain theory – St. Venants
- Distortional energy – von Mises
- Max. shear stress theory – Tresca
- Mohr-Coulomb envelop theory

Max. Principal Stress Theory – Rankine

For maximum normal stress theory, the failure occurs when one of the principal stresses (σ_1,σ_2 *and* σ_3) equals to the yield strength.

$$\sigma_1 > \sigma_2 > \sigma_3$$

Failure occurs when either $\sigma_1 = \sigma_{yield}$ in ductile material or $\sigma_1 = \sigma_{rupture}$ in brittle material. The shorting's of this theory are the neglect the effects of the other principal stresses (σ_2 and σ_3).

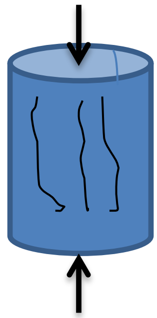
Theory of Maximum Tensile Strain (Saint-Venant)

The tensile strain is:

$\varepsilon_{tensile} = 1/E\ [\sigma_1 - \mu(\sigma_1 - \sigma_3)]$

If only σ_{yield} is acting: $\varepsilon_{tensile} = \sigma_{yield}/E$

Then the failure: $\sigma_1 - \mu(\sigma_1 - \sigma_3) = \sigma_{yield}$ ….. this is acceptable for certain brittle material (concrete).



Theory of Maximum Shearing Stress (Tressca)

Failure occur when: τ_{max} = k (a specified value)

But $\tau_{max} = 1/2\ (\sigma_1 - \sigma_3)$ (from Moher circle)

$\tau_{max} = 1/2\ (\sigma_{yield} - 0)$

The failure is specified by: $\sigma_1 = \sigma_3 = \sigma_{yield}$

This theory is acceptable for ductile materials.

Theory of Maximum Distortion Energy (Von Mises)

It predicts the failure of a specimen subjected to any combination of loads when the strain energy per unit volume due to shear of any portion of the stressed member reaches the failure value of strain energy per unit volume due to shear as determined from an axial or compression test of the same material.

The total strain enery per unit volume is given by the sum of the energy component due to three principal stresses and strains:

- $U_t = ½\, \sigma_1\varepsilon_1 + ½\, \sigma_2\varepsilon_2 + ½\, \sigma_3\varepsilon_3$
- $\varepsilon_1 = \sigma_1 / E - v\sigma_2 / E - v\sigma_3 / E$
- $\varepsilon_2 = \sigma_2 / E - v\sigma_1 / E - v\sigma_3 / E$
- $\varepsilon_3 = \sigma_3 / E - v\sigma_2 / E - v\sigma_1 / E$

=>

- $\mathbf{U_t = 1/2E\, [\, \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\,]}$

---------- (1)

Here the total strain energy can be considered as the sum of two parts, one part representing the energy needed to cause a volume change of the element with no change in shape & the other part representing the energy needed to distort * the element.

$U_t = U_v + U_s$ or $U_s = U_t - U_v$

- $U_v = ½\, \sigma_v\varepsilon_v$

  $= ½\, \sigma_v \{\, 3\, \sigma_v\, (1-2v) / E\, \}$

  $= 3(1-2v)\, \sigma_v^2 / 2E$

  $= 3(1-2v)\, \{(\sigma_1 + \sigma_2 + \sigma_3 /3)^2\} / 2E$

  $= 1-2v\, (\sigma_1 + \sigma_2 + \sigma_3)^2 /6$

$\mathbf{U_s = 1+ v / 6E\, [\, (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$

---------- (2)

If $\sigma_1 > \sigma_2 > \sigma_3$ are three principal stress, then $\sigma_v = (\sigma_1 + \sigma_2 + \sigma_3)/3$
Where σ_v= mean stress or hydraulic stress. This causes change in volume and no change in angle.
So for the case of Maximum shear/distortion energy theory, the failure occurs when the quantity U_s reaches the value in elastic limit. As for limiting value:

$\sigma_1 = \sigma_{\text{yield}}$ and $\sigma_2 = \sigma_3 = 0$

$$U_Y = \frac{1+\mu}{6E}\left(\sigma_{yield}^2 + \sigma_{yield}^2\right) = \frac{1+\mu}{3E}\left(\sigma_{yield}^2\right) \ldots\ldots\ldots (3)$$

At failure: $U_s = U_Y$

Equating the two energies and simplify:

$$\frac{1+\mu}{6E}(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 = \frac{1+\mu}{3E}\sigma_{yield}^2$$

$$\frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2} = \frac{\sqrt{2}}{3}\sigma_{yield}$$

This theory also states: Failure occurs when $\tau_{oct.}$ reaches a certain value.

$$\tau_{oct.} = \frac{\sqrt{2}}{3}\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3)}$$

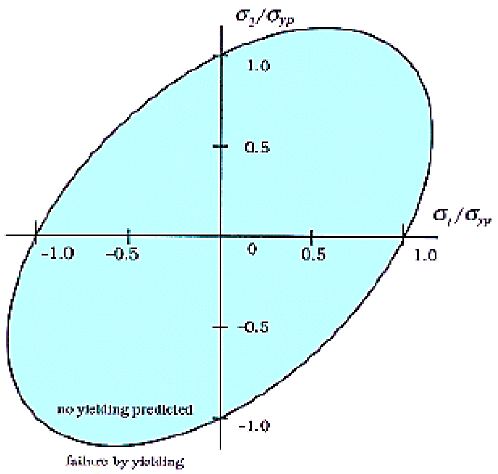
$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$

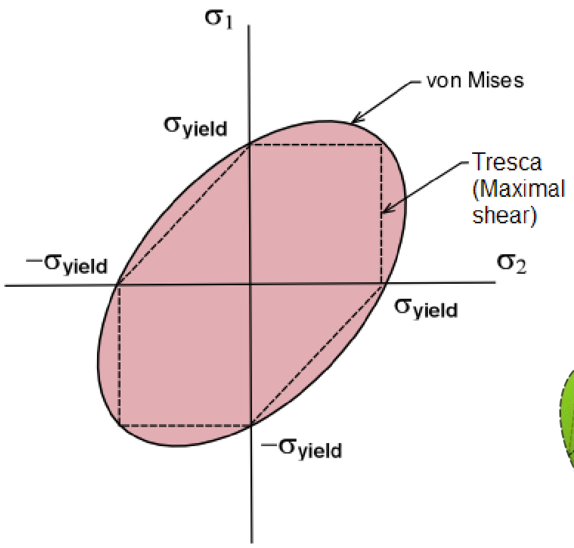
For 2-D Case

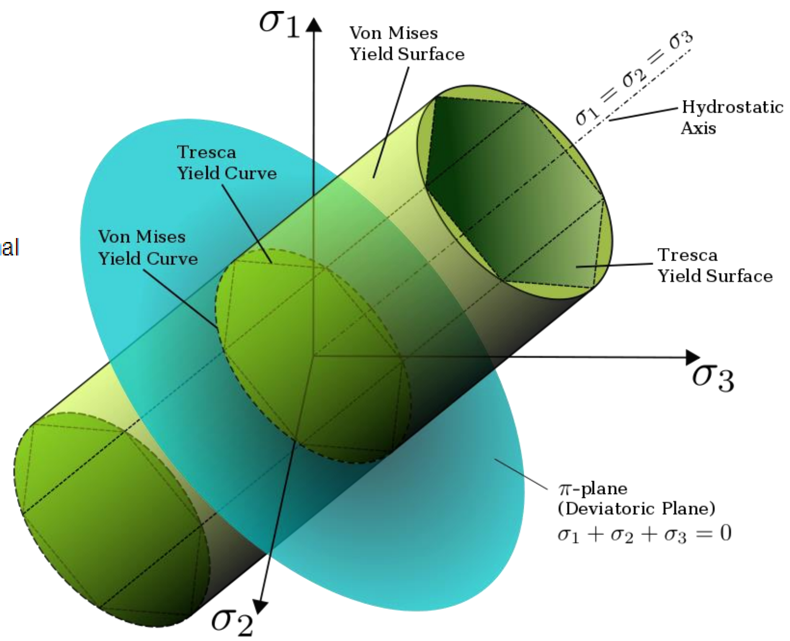
$$\mathbf{1+ v / 3E\,(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2) = 1+ v / 3E}$$

$$(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2) = \sigma_{yield}^2$$

$$\left(\sigma_1/\sigma_{yield}\right)^2 + \left(\sigma_2/\sigma_{yield}\right)^2 - \left(\sigma_1/\sigma_{yield}\right)\left(\sigma_2/\sigma_{yield}\right) = 1$$

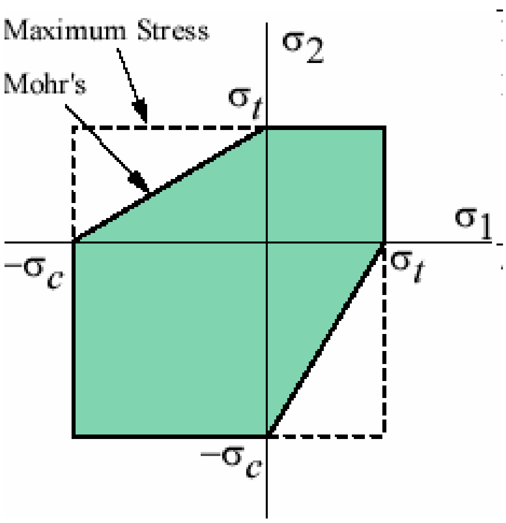






* Distortion is the deformation from shearing stresses, deviator stresses causes change in angles and no change in volume.

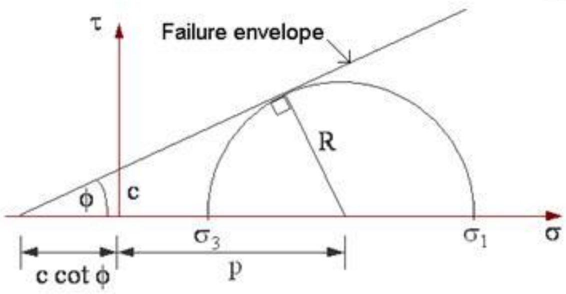
Mohr – Coulomb Failure Envelop Theory

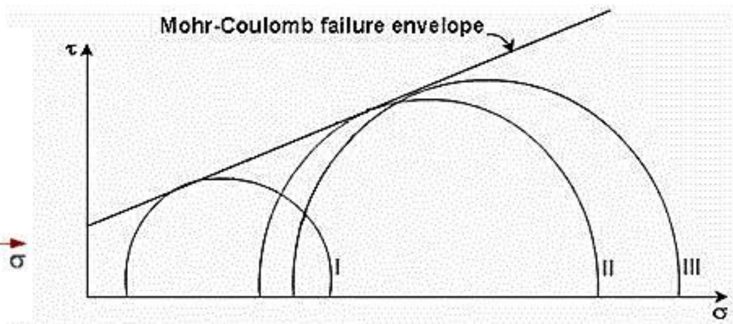


Some materials such as rocks, concrete, cast iron has much greater strength in compression then in tension. Mohr's proposed that, in 1st and 3rd quadrant of the failure Maximum Principal Stress Theory was appropriate based on the ultimate strength of the material in tension or compression respectively. In 2nd & 4th quadrant the Maximum Shear Stress Theory should be applied.

Take different values of σ_1 and σ_3 (triaxial test) plot Mohr circles.

$$\tau_f = c + \sigma_f . \tan\phi$$





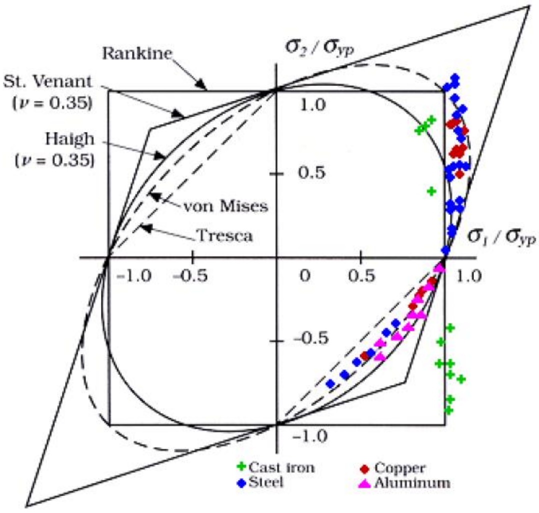
Where τ_f = shear stress on the failure plane
c = apparent cohesion
σ_f =normal stress on the failure plane
f = angle of internal friction

If the stress condition for any other soil sample is represented by a Mohr circle that lies below the failure envelope, every plane within the sample experiences a shear stress which is smaller than the shear strength of the sample. Thus, the point of tangency of the envelope to the Mohr circle at failure gives a clue to the determination of the inclination of the failure plane.

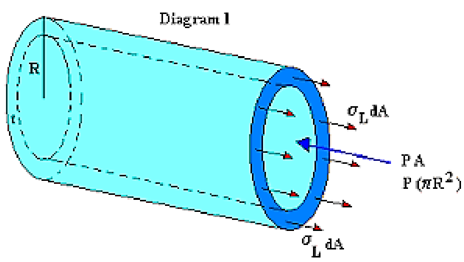
Conclusion

1. Materials does not fail under hydrostatic stress system i.e $\sigma_1 = \sigma_2 = \sigma_3$
2. None of the theories agrees with the test perform for all types of materials and combinations of loads.
3. There is a good agreement between the maximum distortion energy theory and experimental result for ductile materials.
4. The max. principal stress theory appears to be the best for brittle materials
5. Max. shear stress or max. strain energy theories give the good approximation for ductile materials but the max. shear stress criterion is somewhat more conservative.
6. The max. strain theory should not be used in general as it only gives the reliable results in particular cases.
7. If the brittle material has a stress strain diagram, that is different in tension and compression, then the MOHR'S Failure
8. Criterion may be used to predict the failure.

The orientation of the failure plane can be finally determined by the pole method as shown in Figure.

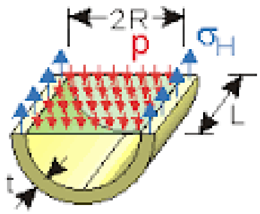


Examples: Consider a thin closed cylinder under pressure p. find p for failure.



Solution: There are two principal stresses in the wall of the cylinder

- Hoop stress*(σ_h)

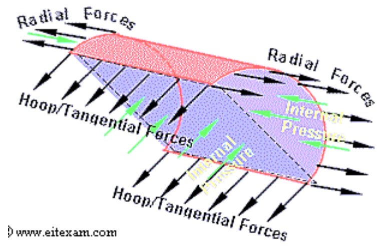


2 σ_h.t.l = p(D.l)

$$\sigma_h = \frac{pD}{2t}$$

*The hoop stress is the force exerted circumferentially (perpendicular both to the axis and to the radius of the object) in both directions on every particle in the cylinder wall.

- Longitude stress*(σ_l)



$$\sigma_l.(\pi D t) = \left(\frac{\pi}{4} D^2\right) \ldots\ldots\ldots \sigma_l = \frac{pD}{4t}$$

Then $\sigma_1 = \frac{pD}{2t}$ $\qquad \sigma_2 = \frac{pD}{4t}$ $\qquad \sigma_3 = 0$

a) Rankine method

$\sigma_1 = \sigma_{yield}$ …… $\sigma_{yield} = \frac{pD}{2t}$ ……… $p = \frac{2t\sigma_{yield}}{D}$

b) Tersca method

$(\sigma_1 - \sigma_3) = \sigma_{yield}$ ……… $\sigma_{yield} = \frac{pD}{2t} - 0$

$$p = \frac{2t\sigma_{yield}}{D}$$

c) Von-Mises

$$\frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \frac{\sqrt{2}}{3}\sigma_{yield}$$

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_{yield}^2$$

$$\left(\frac{pD}{2t} - \frac{pD}{4t}\right)^2 + \left(\frac{pD}{4t} - 0\right)^2 + \left(0 - \frac{pD}{2t}\right)^2 = 2\sigma_{yield}^2$$

$$\frac{p^2D^2}{16t^2}+\frac{p^2D^2}{16t^2}+\frac{p^2D^2}{4t^2}=2\sigma_{yield}^2$$

$$\frac{6p^2D^2}{16t^2}=2\sigma_{yield}^2 \quad \dots\dots\dots \quad \frac{\sqrt{3}pD}{4t}=\sigma_{yield}$$

$$p=\frac{4t\sigma_{yield}}{\sqrt{3}D}$$

* Longitude stress is defined as the total circumferential force exerted along the entire radial thickness