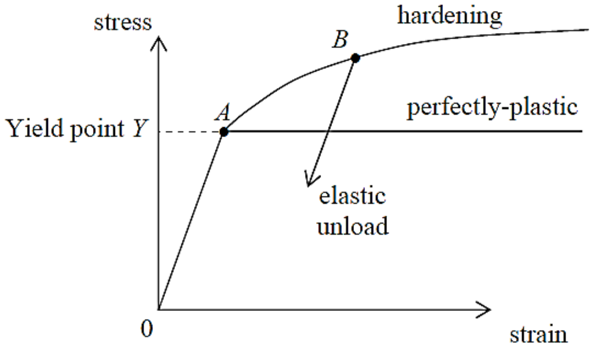
STRAIN HARDENING



In the one-dimensional (uniaxial test) case, a specimen will deform up to yield and then generally harden, Fig. Also shown in the figure is the perfectly-plastic idealization. In the perfectly plastic case, once the stress reaches the yield point (A), plastic deformation ensues, so long as the stress is maintained at Y. If the stress is reduced, elastic unloading occurs. In the hardening case, once yield occurs, the stress needs to be continually increased in order to drive the plastic deformation. If the stress is held constant, for example at B, no further plastic deformation will occur; at the same time, no elastic unloading will occur. Note that this condition cannot occur in the perfectly-plastic case, where there is one of plastic deformation or elastic unloading.

Strain Hardening is when a metal is strained beyond the yield point. An increasing stress is required to produce additional plastic deformation and the metal apparently becomes stronger and more difficult to deform.

These ideas can be extended to the multiaxial case, where the initial yield surface will be of the form

$$f_0(\sigma_{ij}) = 0$$

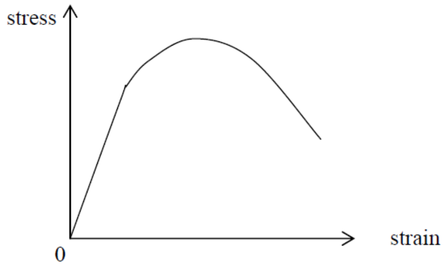
In the perfectly plastic case, the yield surface remains unchanged.. In the more general case, the yield surface may change size, shape and position, and can be described by

$$f(\sigma_{ij}, \mathrm{K}_i) = 0 \qquad \ldots\ldots 1$$

Here, K_i represents one or more hardening parameters, which change during plastic deformation and determine the evolution of the yield surface. They may be scalars or higher-order tensors. At first yield, the hardening parameters are zero, and $f(\sigma_{ij},0) = f_0(\sigma_{ij})$. The description of how the yield surface changes with plastic deformation, Eqn. 1, is called the hardening rule.

STRAIN SOFTENING

The strain-softening of a material is the decline of stress at increasing strain. Strain-softening diagrams are obtained from displacement controlled compression tests on concrete-like materials.



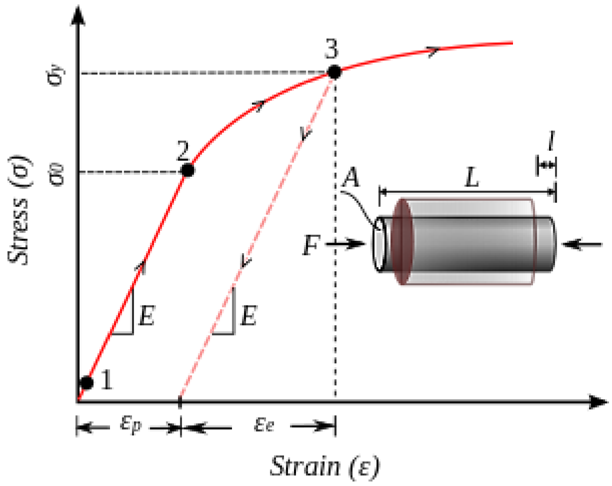
uniaxial stress-strain curve for a strain-softening material

RULES OF PLASTIC FLOW

Flow plasticity is a solid mechanics theory that is used to describe the plastic behavior of materials. Flow plasticity theories are characterized by the assumption that a flow rule exists that can be used to determine the amount of plastic deformation in the material.

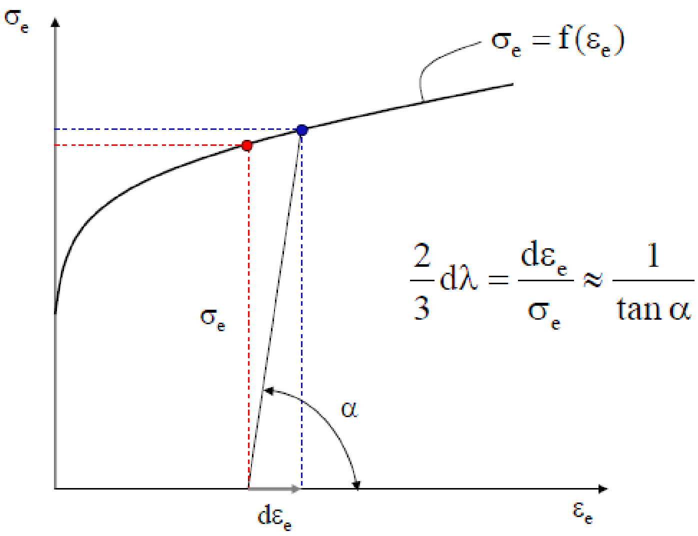
In flow plasticity theories it is assumed that the total strain in a body can be decomposed additively (or multiplicatively) into an elastic part and a plastic part. The elastic part of the strain can be computed from a linear elastic or hyperelastic constitutive model. However, determination of the plastic part of the strain requires a flow rule and a hardening model.

In *metal plasticity*, the assumption that the plastic strain increment and deviatoric stress tensor have the same principal directions is encapsulated in a relation called the flow rule. *Rock plasticity theories* also use a similar concept except that the requirement of pressure-dependence of the yield surface requires a relaxation of the above assumption.



Instead, it is typically assumed that the plastic strain increment and the normal to the pressure-dependent yield surface have the same direction, i.e.,

$d\varepsilon_p = d\lambda \frac{\partial f}{\partial \sigma}$



where $d\lambda$ >0 is a hardening parameter. This form of the flow rule is called an associated flow rule and the assumption of co-directionality is called the normality condition. The function f is also called a plastic potential.

The above flow rule is easily justified for perfectly plastic deformations for which $d\sigma$=0 when $d\varepsilon_p > 0$, i.e., the yield surface remains constant under increasing plastic deformation. This implies that the increment of elastic strain is also zero, $d\varepsilon_e = 0$, because of Hooke's law. Therefore,

$d\sigma : \frac{\partial f}{\partial \sigma} = 0$ and $d\sigma : d\varepsilon_p = 0$

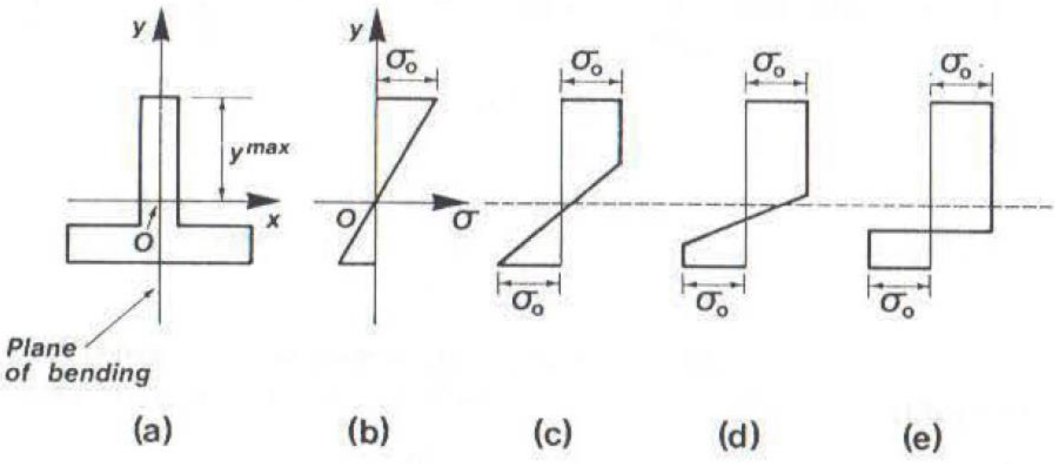
Hence, both the normal to the yield surface and the plastic strain tensor are perpendicular to the stress tensor and must have the same direction.

For a work hardening material, the yield surface can expand with increasing stress. We assume Drucker's second stability postulate which states that for an infinitesimal stress cycle this plastic work is positive, i.e., $d\sigma : d\varepsilon_p \geq 0$

The above quantity is equal to zero for purely elastic cycles. Examination of the work done over a cycle of plastic loading-unloading can be used to justify the validity of the associated flow rule.

MOMENT-ROTATION CHARACTERISTICS OF GENERAL CROSS SECTION

We consider an arbitrary cross-section with a vertical plane of symmetry, which is also the plane of loading. We consider the cross section subject to an increasing bending moment, and assess the stresses at each stage.



Stress distributions in beam with single axis of symmetry

(a) Cross section
(b) At yield moment
(c) At attainment of yield stress on lower face
(d) Plastic zones spreading inwards from both faces
(e) At plastic moment

Stage 1 – Elastic Behavior

The applied moment causes stresses over the cross-section that are all less than the yield stress of the material.

Stage 2 – Yield Moment

The applied moment is just sufficient that the yield stress of the material is reached at the outermost fiber(s) of the cross-section. All other stresses in the cross section are less than the yield stress. This is limit of applicability of an elastic analysis and of elastic design.

Stage 3 – Elasto-Plastic Bending

The moment applied to the cross section has been increased beyond the yield moment. Since by the idealized stress-strain curve the material cannot sustain a stress greater than yield stress, the fibers at the yield stress have progressed inwards towards the center of the beam. Thus over the cross section there is an elastic core and a plastic region.

Stage 4 – Plastic Bending

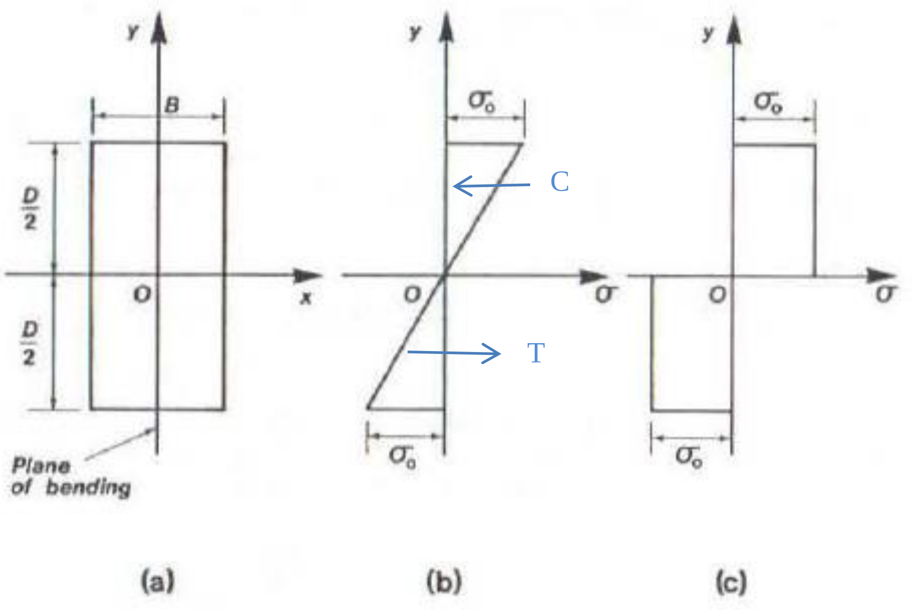
The applied moment to the cross section is such that all fibers in the cross section are at yield stress. This is termed the Plastic Moment Capacity of the section since there are no fibers at an elastic stress. Also note that the full plastic moment requires an infinite strain at the neutral axis and so is physically impossible to achieve. However, it is closely approximated in practice. Any attempt at increasing the moment at this point simply results in more rotation, once the cross-section has sufficient ductility. Therefore in steel members the cross section classification must be plastic and in concrete members the section must be under-reinforced.

Stage 5 – Strain Hardening

Due to strain hardening of the material, a small amount of extra moment can be sustained.

ANALYSIS OF RECTANGULAR CROSS SECTION

Since we now know that a cross section can sustain more load than just the yield moment, we are interested in how much more. In other words we want to find the yield moment and plastic moment, and we do so for a rectangular section. Taking the stress diagrams from those of the moment-rotation curve examined previously, we have:



Stress distributions in beam of rectangular cross section

(a) Cross section
(b) At yield moment
(c) At plastic moment

Elastic Moment

From the diagram: $M_Y = C \times \frac{2}{3} d$

But, the force (or the volume of the stress block) is: $C = T = \frac{1}{2} \sigma_Y \frac{d}{2} b$

Hence:
$$M_Y = \left(\frac{1}{2} \sigma_Y \frac{d}{2} b \right) \left(\frac{2}{3} d \right)$$
$$= \sigma_Y \cdot \frac{bd^2}{6} \quad = \sigma_Y \cdot S$$

The term $bd^2/6$ is thus a property of the cross section called the *elastic section modulus* and it is termed *S*.

Plastic Moment

From the stress diagram: $M_P = C \times \frac{d}{2}$

And the force is: $C = T = \sigma_Y \frac{d}{2} b$

Hence:
$$M_P = \left(\sigma_Y \frac{bd}{2} \right) \left(\frac{d}{2} \right)$$
$$= \sigma_Y \cdot \frac{bd^2}{4} = \sigma_Y \cdot Z$$

The term $bd^2/4$ is a property of the cross section called the *plastic section modulus*, termed *Z*.

Shape Factor

Thus the ratio of elastic to plastic moment capacity is:

$$\frac{M_P}{M_Y} = \frac{\sigma_Y \times Z}{\sigma_Y \times S} = \frac{d^2 b/4}{d^2 b/6} = 1.5$$

This ration is termed the *shape factor*, *f*, and is a property of a cross section alone. For a rectangular cross-section, we have:

$$f = \frac{Z}{S} = 1.5$$

And so a rectangular section can sustain 50% more moment than the yield moment, before a plastic hinge is formed. Therefore the shape factor is a good measure of the efficiency of a cross section in bending. Shape factors for some other cross sections are:

Circle: $f = 1.698$;

Diamond: $f = 2.0$;

Steel I-beam: f is between 1.12 and 1.15.

MOMENT ROTATION CURVE OF A RECTANGULAR SECTION

It is of interest to examine the moment-rotation curve as the moment approaches the plastic moment capacity of the section. We begin by recalling the relationship between strain, ε , and distance from the neutral axis, y:

$$\varepsilon = \kappa y$$

This is a direct consequence of the assumption that plane sections remain plane and is independent of any constitutive law (e.g. linear elasticity). We next identify the yield strain (that corresponds to the yield stress, σ_Y) as ε_Y . The curvature that occurs at the yield moment is therefore:

$$\kappa_Y = \frac{\varepsilon_Y}{(d/2)} = 2\frac{\varepsilon_Y}{d}$$

For moments applied beyond the yield moment, the curvature can be found by noting that the yield strain, ε_Y , occurs at a distance from the neutral axis of $\alpha\, d/2$, giving:

$$\kappa = \frac{\varepsilon_Y}{(\alpha d/2)} = 2\frac{\varepsilon_Y}{\alpha d}$$

Thus, the ratio curvature to yield curvature is:

$$\frac{\kappa}{\kappa_Y} = \frac{2\varepsilon_Y/\alpha d}{2\varepsilon_Y/d} = \frac{1}{\alpha}$$

From which $\alpha = \kappa_Y/\kappa$.

Also, the ratio of elasto-plastic moment to yield moment is:

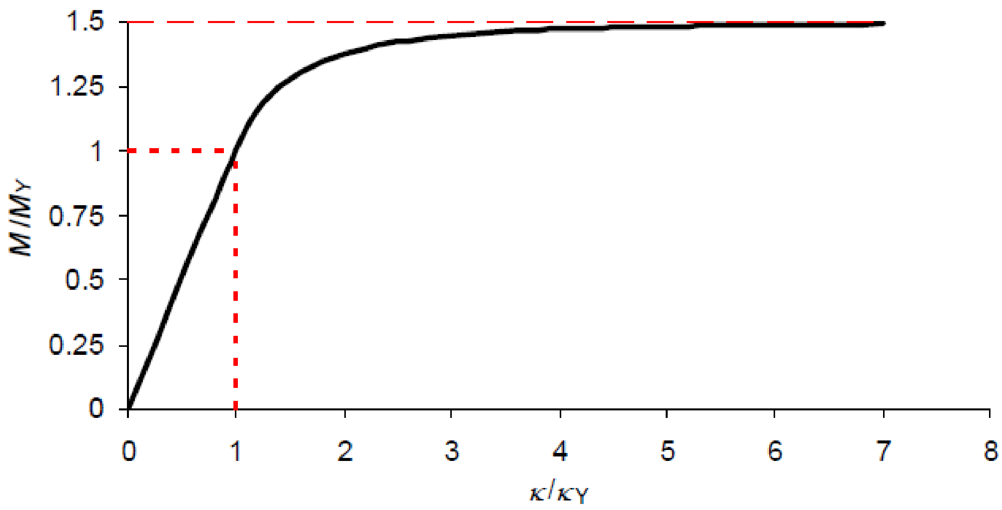
$$\frac{M}{M_Y} = \frac{\sigma_Y \frac{bd^2}{6} \cdot \frac{(3-\alpha^2)}{2}}{\sigma_Y \frac{bd^2}{6}} = \frac{(3-\alpha^2)}{2}$$

If we now substitute the value $\alpha = \kappa_Y / \kappa$ we find: $\frac{M}{M_Y} = \frac{1}{2}\left[3 - \left(\frac{\kappa_Y}{\kappa}\right)^2\right]$

And so finally we have:

$$\frac{M}{M_Y} = 1.5 - 0.5\left(\frac{\kappa}{\kappa_Y}\right)^{-2}$$

Plotting this gives:



There are some important observations to be made from this graph:

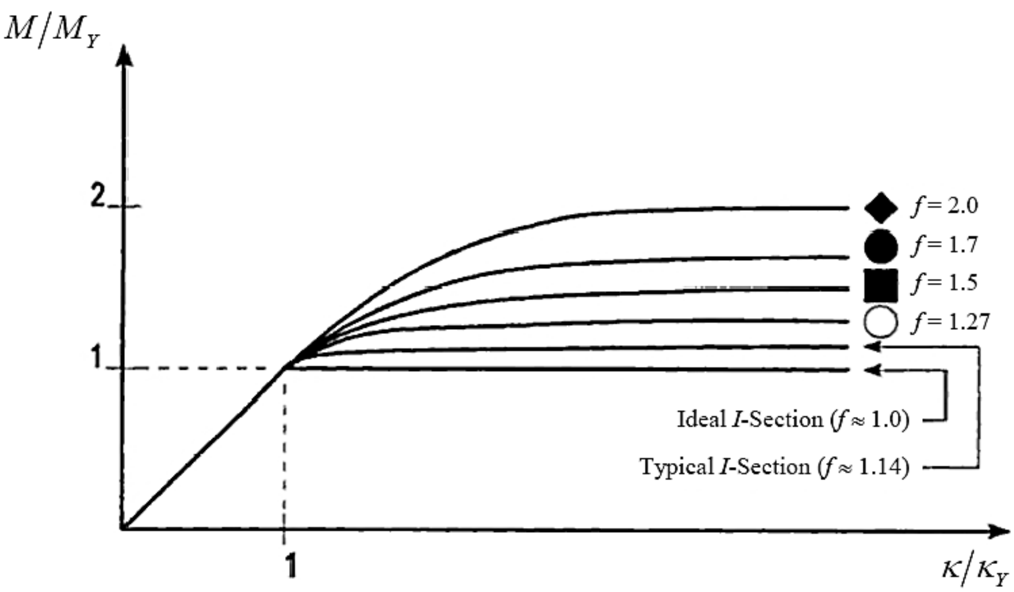
• To reach the plastic moment capacity of the section requires large curvatures. Thus the section must be ductile.

• The full cross-section plasticity associated with the plastic moment capacity of a section can only be reached at infinite curvature (or infinite strain). Since this is impossible, we realize that the full plastic moment capacity is unobtainable.

To show that the idea of the plastic moment capacity of section is still useful, we examine this further. Firstly we note that strain hardening in mild steel begins to occur at a strain of about 10 ε_Y . At this strain, the corresponding moment ratio is:

$$\frac{M}{M_Y} = 1.5 - 0.5(10)^{-2} = 1.495$$

Since this is about 99.7% of the plastic moment capacity, we see that the plastic moment capacity of a section is a good approximation of the section's capacity. These calculations are based on a ductility ration of 10. This is about the level of ductility a section requires to be of use in any plastic collapse analysis. Lastly, for other cross-section shapes we have the moment-curvature relations shown in the following figure.



(Adapted from Bruneau et al (1998))