METHODS OF PLASTIC ANALYSIS

There are three main approaches for performing a plastic analysis:

The Incremental Method

This is probably the most obvious approach: the loads on the structure are incremented until the first plastic hinge forms. This continues until sufficient hinges have formed to collapse the structure. This is a labour-intensive, 'brute-force', approach, but one that is most readily suited for computer implementation.

The Equilibrium (or Statical) Method

In this method, free and reactant bending moment diagrams are drawn. These diagrams are overlaid to identify the likely locations of plastic hinges. This method therefore satisfies the equilibrium criterion first leaving the two remaining criterion to derived therefrom.

The Kinematic (or Mechanism) Method

In this method, a collapse mechanism is first postulated. Virtual work equations are then written for this collapse state, allowing the calculations of the collapse bending moment diagram. This method satisfies the mechanism condition first, leaving the remaining two criteria to be derived therefrom.

IMPORTANT DEFINITIONS

Load Factor: The load factor for a possible collapse mechanism i, denoted λ_i , is of prime importance in plastic analysis:

$$\lambda_i = \frac{\text{Collapse Load for Mechanism } i}{\text{Working Load}}$$

The working load is the load which the structure is expected to carry in the course of its lifetime. The collapse load factor, λ_c is the load factor at which the structure will actually fail. It is therefore the minimum of the load factors for the n_m different possible collapse mechanisms:

$$\lambda_C = \min_{1 \le i \le n_m} \lambda_i$$

Factor of Safety: This is defined as $$\text{FoS} = \frac{\text{First yield load}}{\text{Working Load}}$$

THEOREMS OF LIMIT ANALYSIS

When the structure is exposed to the load of the proportional nature that gradually increases, at some point it reaches a certain critical value, at which point it comes to plastic failure of the structure (ie, unlimited increase of deformation at constant load), after which a construction is no longer able to receive further increase of the load. This critical state is called the limit state of the construction, and load that causes it is the limit load. Determination of the bearing power of structures (limit load) is an important factor in designing structures. The limit analysis of structures is an alternative analytical method to determine the maximum load parameter or increasing load parameter, which a perfect elastic-plastic construction is able to bear .

Limit analysis is a structural analysis field which is dedicated to the development of efficient methods to directly determine estimates of the collapse load of a given structural model without resorting to iterative or incremental analysis. For this purpose, the field of limit analysis is based on a set of theorems, referred to as limit theorems, which are a set of theorems based on the law of conservation of energy that state properties regarding stresses and strains, lower and upper-bound limits for the collapse load and the exact collapse load.

The theorems of limit analysis can be stated in a form that does not directly refer to any concepts from plasticity theory:

A body will not collapse under a given loading if a possible stress field can be found that is in equilibrium with a loading greater than the given loading.

A body will collapse under a given loading if a velocity field obeying the constraints (or a mechanism) can be found that so that the internal dissipation is less than the rate of work of the given loading.

Compared to the incremental analysis (the step-by-step method), the efficiency of the limit analysis is achieved by observing the final state, state of failure, without paying attention to what was happening with the construction and load from the moment when one section of the structure was completely plasticized (formation of the first plastic joint for solid beam) or one rod lattice was completely plasticized (formation of first plastic truss rod), until the failure. Limit

analysis methods are based on the theorem of plastic failure of an ideal elasto-plastic body. These theorems are known as static (lower) and kinematic (upper) theorems of themarginal analysis of structures. It should be noted that in addition to the limit state of load there are other limit states, which may occur before the state of limit equilibrium and which can be restrictive to the transferring of an external load, such as limit states of usability, or even a marginal state of cracks in structures made of reinforced or pre-stressed concrete.

THE BASIC SETTINGS OF THE LIMIT ANALYSIS

The calculation of structures by applying the theory of plasticity allows plasticization of materials, that is to say, out of the boundaries of elastic behavior. In the area of elastic behavior of the structure, stresses and deformations are proportionally dependent. Increasing the load affecting the structure leads to a gradual increase in stress until a stress level in the most stressed fiber (or fibers, in the case of a symmetrical section) reaches a value of the yield stress. Further increase of load leads to plasticization of the cross section, in other words, it leads to the increase of the plasticity zone, which gradually expands in height and in length of the beam, until it comes to the plasticization of the entire cross section, and therefore the formation of a plastic joint. It is known that, for statically determined beams, plasticization of one section of the structure (by forming a plastic joint in the area of the maximum bending moment) is followed by the loss of load bearing capacity and the transition of a beam into a mechanism. Unlike statically determined beams, with statically indeterminate beams, the formation of a plastic joint does not lead to the formation of a mechanism of failure. The bearing capacity of an n times statically indeterminate structure will be fully depleted when n+1 plastic joints are formed within the structure. For determining the limit loads, the following assumptions are introduced:

- deformations are proportional to the deviation from the neutral axis (Bernoulli hypothesis of straight sections is valid),
- an idealized elasto-plastic dependency for materials applies for tension stress as well as pressure.
- deformations are small,
- section has the necessary ductility,
- conditions of balance of the cross-section are met, of normal forces $\Sigma X=0$, as well as the bending moment $\Sigma M=0$.

In order of the limit load of a structure to be determined by applying the theory of plasticity, first it is necessary to prove that an applicable limit state will be caused by formation of the mechanism of failure, in other words, it is necessary to eliminate the occurrence of any other limit states. It is necessary to exclude the occurrence of fatigue because of the effects of variable load, then the possibility of local instability prior to reaching full plasticization and exclude the appearance of any effects that would lead to failure of the structure before the formation of a sufficient number of plastic joints for its transition into the mechanism of failure.

In the theory of the limit analysis the following assumptions apply:

- sections where the bending moment is less than the moment of plasticization of the cross-section, are in the elastic range;
- section in which full plastic moment of the cross section (M_p) happened is the perfect plastic joint;
- turning of section, after reaching the plastic moment, grows without limit without further increasing the load,
- body is made of elastic-perfect plastic material with infinite surface flow.

It can be said that one beam is in a state of limit balance when the bearing capability of the construction is fully exhausted, and in a sufficient number of sections the beam behaves completely plastically. Based on this we can conclude that when it comes to forming a sufficient number of plastic joints, deformities are progressive, and the beam transforms into the failure mechanism. The moment that immediately precedes the formation of the mechanism of failure represents the moment of the limit balance of the system.

The Upperbound (Unsafe) Theorem

This can be stated as:

If a bending moment diagram is found which satisfies the conditions of equilibrium and mechanism (but not necessarily yield), then the corresponding load factor is either greater than or equal to the true load factor at collapse.

This is called the unsafe theorem because for an arbitrarily assumed mechanism the load factor is either exactly right (when the yield criterion is met) or is wrong and is too large, *leading a designer to think that the frame can carry more load than is actually possible*.

Think of it like this: ***unless it's exactly right, it's dangerous***.

Since a plastic analysis will generally meet the equilibrium and mechanism criteria by this theorem a plastic analysis is either right or dangerous. This is why plastic analyses are not used as often in practice as one might suppose.

The Lowerbound (Safe) Theorem

This can be stated as:

If a bending moment diagram is found which satisfies the conditions of equilibrium and yield (but not necessarily that of mechanism), then the corresponding load factor is either less than or equal to the true load factor at collapse.

This is a safe theorem because the load factor will be less than (or at best equal to) the collapse load factor once equilibrium and yield criteria are met leading the designer to think that the structure can carry less than or equal to its actual capacity.

Think of it like this: ***it's either wrong and safe or right and safe.***

Since an elastic analysis will always meet equilibrium and yield conditions, an elastic analysis will always be safe. This is the main reason that it is elastic analysis that is used, in spite of the significant extra capacity that plastic analysis offers.

The Uniqueness Theorem

Linking the upper- and lower-bound theorems, we have:

If a bending moment distribution can be found which satisfies the three conditions of equilibrium, mechanism, and yield, then the corresponding load factor is the true load factor at collapse.

So to have identified the correct load factor (and hence collapse mechanism) for a structure we need to meet all three of the criteria:

1. Equilibrium;
2. Mechanism;
3. Yield.

The permutations of the three criteria and the three theorems are summarized in the following table:

Criterion	*Upperbound (Unsafe) Theorem*	*Lowerbound (Safe) Theorem*	*Unique Theorem*
Mechanism	$\lambda \geq \lambda_c$ (Mechanism + Equilibrium)		$\lambda = \lambda_c$ (Mechanism + Equilibrium + Yield)
Equilibrium		$\lambda \leq \lambda_c$ (Equilibrium + Yield)	
Yield			

The Uniqueness Theorem does not claim that any particular collapse mechanism is unique – only that the collapse load factor is unique.

Although rare, it is possible for more than one collapse mechanism to satisfy the Uniqueness Theorem, but they will have the same load factor.

Corollaries of the Theorems

Some other results immediately apparent from the theorems are the following:

1. If the collapse loads are determined for all possible mechanisms, then the actual collapse load will be the lowest of these (Upper-bound Theorem);
2. The collapse load of a structure cannot be decreased by increasing the strength of any part of it (Lower-bound Theorem);
3. The collapse load of a structure cannot be increased by decreasing the strength of any part of it (Upper-bound Theorem);
4. The collapse load is independent of initial stresses and the order in which the plastic hinges form (Uniqueness Theorem);

The first point above is the basis for using virtual work in plastic analysis. However, in doing so, it is essential that the designer considers the actual collapse more. To not do so would lead to an unsafe design by the Upper-bound Theorem.

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