. Elastic Strain

Normal strain ε is given by the equation $\epsilon = \delta/l$, where δ is the total elongation of the bar within the length l. Hooke's law for the tensile specimen is given by the equation

$$\sigma = E\epsilon \qquad 1\text{-}10$$

where the constant E is called *Young's modulus* or the *modulus of elasticity*.

When a material is placed in tension, there exists not only an axial strain, but also negative strain (contraction) perpendicular to the axial strain. Assuming a linear, homogeneous, isotropic material, this lateral strain is proportional to the axial strain. If the axial direction is x, then the lateral strains are $\epsilon_y = \epsilon_z = -\nu\epsilon_x$. The constant of proportionality v is called *Poisson's ratio*, which is about 0.3 for most structural metals.
If the axial stress is in the x direction, then from Eq. (1–10)

$$\epsilon_x = \frac{\sigma_x}{E} \qquad \epsilon_y = \epsilon_z = -\nu\frac{\sigma_x}{E} \qquad 1\text{-}11$$

For a stress element undergoing σ_x , σ_y , and σ_z simultaneously, the normal strains are given by

$$\begin{aligned} \epsilon_x &= \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \\ \epsilon_y &= \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \\ \epsilon_z &= \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \end{aligned} \qquad 1\text{-}12$$

Shear strain γ is the change in a right angle of a stress element when subjected to pure shear stress, and Hooke's law for shear is given by

$$\tau = G\gamma \qquad 1\text{-}13$$

where the constant G is the *shear modulus of elasticity* or *modulus of rigidity*.

It can be shown for a linear, isotropic, homogeneous material; the three elastic constants are related to each other by

$$E = 2G(1 + \nu) \qquad \text{1-14}$$

1-5. Uniformly Distributed Stresses

The assumption of a uniform distribution of stress is frequently made in design. The result is then often called *pure tension*, *pure compression*, or *pure shear*, depending upon how the external load is applied to the body under study. The word *simple* is sometimes used instead of *pure* to indicate that there are no other complicating effects.

For *pure tension* and *pure compression*

$$\sigma = \frac{F}{A} \qquad \text{1-15}$$

and for *pure shear*

$$\tau = \frac{F}{A} \qquad \text{1-16}$$

1-6. Normal Stresses for Beams in Bending

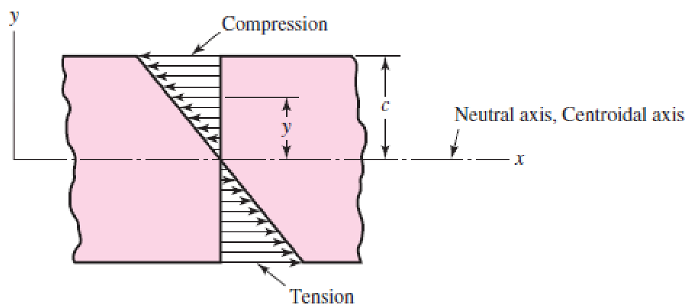

Figure (1-5)

Bending stresses according to Eq. (1–17)

The bending stress varies linearly with the distance from the neutral axis, *y*, and is given by

1-17

$$\sigma_x = -\frac{My}{I}$$

where *I* is the second *moment of area* about the *z* axis. That is

$$I = \int y^2 dA \qquad 1\text{-}18$$

So, the maximum *magnitude* of the bending stress is

$$\sigma_{max} = \frac{Mc}{I} \quad \text{or} \quad \sigma_{max} = \frac{M}{Z} \qquad 1\text{-}19$$

where *Z = I/c* is called the *section modulus.*

1-7. Torsion

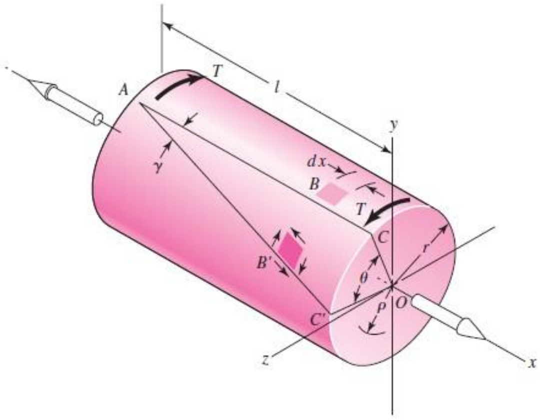

Figure (1-5)

The angle of twist, in radians, for a solid round bar is

$$\theta = \frac{Tl}{GJ} \qquad 1\text{-}20$$

where *T* = torque, *l* = length, *G* = modulus of rigidity, *and J* = polar second moment of area.

Shear stresses develop throughout the cross section. For a round bar in torsion, these stresses are proportional to the radius ρ and are given by

$$\tau = \frac{T\rho}{J} \qquad 1\text{-}21$$

Designating *r* as the radius to the outer surface, we have

$$\tau_{max} = \frac{Tr}{J} \qquad (1\text{-}22)$$

Equation (1–22) applies *only* to circular sections. For a solid round section,

$$J = \frac{\pi d^4}{32} \qquad \text{1-23}$$

where *d* is the diameter of the bar. For a hollow round section,

$$J = \frac{\pi}{32}\left(d_o^4 - d_i^4\right) \qquad \text{1-24}$$

where the subscripts *o* and *i* refer to the outside and inside diameters, respectively.

In using Eq. (1–22) it is often necessary to obtain the torque *T* from a consideration of the power and speed of a rotating shaft. For convenience when U. S. Customary units are used, three forms of this relation are

$$H = \frac{FV}{33\,000} = \frac{2\pi Tn}{33\,000(12)} = \frac{Tn}{63\,025} \qquad \text{1-25}$$

where H = power, hp, T = torque, lbf · in, n = shaft speed, rev/min, F = force, lbf, and V = velocity, ft/min.

When SI units are used, the equation is

$$H = T\omega \qquad \text{1-26}$$

where H =power, W, T =torque, N.m, and ω=angular velocity, rad/s

The torque *T* corresponding to the power in watts is given approximately by

$$T = 9.55\frac{H}{n} \qquad \text{1-27}$$

where *n* is in revolutions per minute.

There are some applications in machinery for noncircular-cross-section members and shafts where a regular polygonal cross section is useful in transmitting torque to a gear or pulley that can have an axial change in position. Because no key or keyway is needed, the possibility of a lost key is avoided. Saint Venant (1855) showed that the maximum shearing stress in a rectangular $b \times c$ section bar occurs in the middle of the *longest* side b and is of the magnitude

$$\tau_{max} = \frac{T}{\alpha b c^2} \doteq \frac{T}{bc^2}\left(3 + \frac{1.8}{b/c}\right) \qquad 1\text{-}28$$

where b is the longer side, c the shorter side, and α a factor that is a function of the ratio b/c as shown in the following table. The angle of twist is given by

$$\theta = \frac{Tl}{\beta b c^3 G} \qquad 1\text{-}29$$

where β is a function of b/c, as shown in the table.

b/c	1.00	1.50	1.75	2.00	2.50	3.00	4.00	6.00	8.00	10	∞
α	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333
β	0.141	0.196	0.214	0.228	0.249	0.263	0.281	0.299	0.307	0.313	0.333

In Eqs. (1–28) and (1–29) b and c are the width (long side) and thickness (short side) of the bar, respectively. They cannot be interchanged. Equation (1–28) is also approximately valid for equal-sided angles; these can be considered as two rectangles, each of which is capable of carrying half the torque.

EXAMPLE 1-2

Figure (1–6) shows a crank loaded by a force F = 300 lbf that causes twisting and bending of a 3/4-in-diameter shaft fixed to a support at the origin of the reference system. In actuality, the support may be an inertia that we wish to rotate, but for the purposes of a stress analysis we can consider this a statics problem.

(a) Draw separate free-body diagrams of the shaft *AB* and the arm *BC*, and compute the values of all forces, moments, and torques that act. Label the directions of the coordinate axes on these diagrams.

(b) Compute the maxima of the torsional stress and the bending stress in the arm *BC* and indicate where these act.

(c) Locate a stress element on the top surface of the shaft at *A*, and calculate all the stress components that act upon this element.

(d) Determine the maximum normal and shear stresses at *A*.

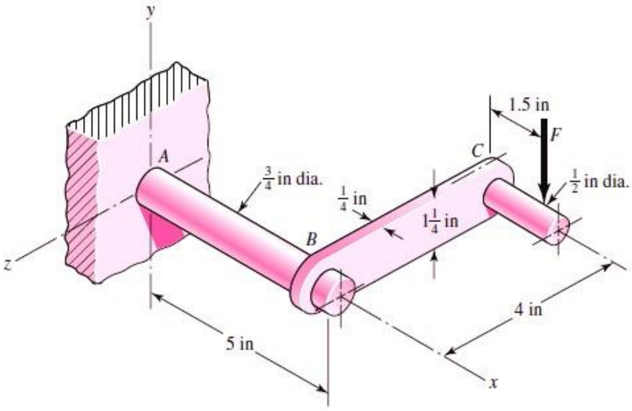

Figure (1-6)

Solution

(a) The two free-body diagrams are shown in Fig. (1–7). The results are

At end *C* of arm *BC*: $\mathbf{F} = -300\mathbf{j}$ lbf, $\mathbf{T}_C = -450\mathbf{k}$ lbf · in

At end *B* of arm *BC*: $\mathbf{F} = 300\mathbf{j}$ lbf, $\mathbf{M}_1 = 1200\mathbf{i}$ lbf · in, $\mathbf{T}_1 = 450\mathbf{k}$ lbf · in

At end *B* of shaft *AB*: $\mathbf{F} = -300\mathbf{j}$ lbf, $\mathbf{T}_2 = -1200\mathbf{i}$ lbf · in, $\mathbf{M}_2 = -450\mathbf{k}$ lbf · in At end *A* of shaft *AB*: $\mathbf{F} = 300\mathbf{j}$ lbf, $\mathbf{M}_A = 1950\mathbf{k}$ lbf · in, $\mathbf{T}_A = 1200\mathbf{i}$ lbf · in

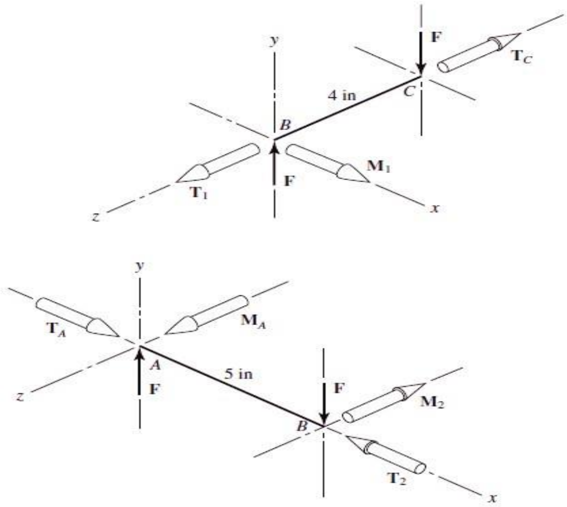

Figure (1-7)

(b) For arm *BC*, the bending moment will reach a maximum near the shaft at *B*. If we assume this is 1200 lbf · in, then the bending stress for a rectangular section will be

Ans.
$$\sigma = \frac{M}{I/c} = \frac{6M}{bh^2} = \frac{6(1200)}{0.25(1.25)^2} = 18\,400 \text{ psi}$$

Of course, this is not exactly correct, because at *B* the moment is actually being transferred into the shaft, probably through a weldment.

For the torsional stress, use Eq. (1–28). Thus

$$\tau_{\max} = \frac{T}{bc^2}\left(3 + \frac{1.8}{b/c}\right) = \frac{450}{1.25(0.25^2)}\left(3 + \frac{1.8}{1.25/0.25}\right) = 19\,400 \text{ psi}$$

Ans.

This stress occurs at the middle of the $1\frac{1}{4}$-in side.

(c) For a stress element at *A*, the bending stress is tensile and is

$$\sigma_x = \frac{M}{I/c} = \frac{32M}{\pi d^3} = \frac{32(1950)}{\pi(0.75)^3} = 47\,100 \text{ psi}$$ *Ans.*

The torsional stress is

$$\tau_{xz} = \frac{-T}{J/c} = \frac{-16T}{\pi d^3} = \frac{-16(1200)}{\pi(0.75)^3} = -14\,500 \text{ psi}$$ *Ans.*

where the reader should verify that the negative sign accounts for the direction of τ_{xz} .

(d) Point *A* is in a state of plane stress where the stresses are in the *xz* plane. Thus the principal stresses are given by Eq. (1–6) with subscripts corresponding to the *x*, *z* axes.

The maximum normal stress is then given by

$$\sigma_1 = \frac{\sigma_x + \sigma_z}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2}$$

$$= \frac{47.1 + 0}{2} + \sqrt{\left(\frac{47.1 - 0}{2}\right)^2 + (-14.5)^2} = 51.2 \text{ kpsi}$$

Ans.

The maximum shear stress at A occurs on surfaces different than the surfaces containing the principal stresses or the surfaces containing the bending and torsional shear stresses. The maximum shear stress is given by Eq. (1–7), again with modified subscripts, and is given by

$$\tau_1 = \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2} = \sqrt{\left(\frac{47.1 - 0}{2}\right)^2 + (-14.5)^2} = 27.7 \text{ kpsi}$$

Ans.