10. Gears

This part addresses gear geometry, the kinematic relations, and the forces transmitted by the four principal types of gears: spur, helical, bevel, and worm gears. The forces transmitted between meshing gears supply torsional moments to shafts for motion and power transmission and create forces and moments that affect the shaft and its bearings.

10.1 Types of Gears

Spur gears, illustrated in Fig. (10–1), have teeth parallel to the axis of rotation and are used to transmit motion from one shaft to another, parallel, shaft. Of all types, the spur gear is the simplest and, for this reason, will be used to develop the primary kinematic relationships of the tooth form.

Helical gears, shown in Fig. (10–2), have teeth inclined to the axis of rotation. Helical gears can be used for the same applications as spur gears and, when so used, are not as noisy, because of the more gradual engagement of the teeth during meshing. The inclined tooth also develops thrust loads and bending couples, which are not present with spur gearing. Sometimes helical gears are used to transmit motion between nonparallel shafts.

Bevel gears, shown in Fig. (10–3), have teeth formed on conical surfaces and are used mostly for transmitting motion between intersecting shafts. The figure actually illustrates *straight-tooth bevel gears*. *Spiral bevel gears* are cut so the tooth is no longer straight, but forms a circular arc. *Hypoid gears* are quite similar to spiral bevel gears except that the shafts are offset and nonintersecting.

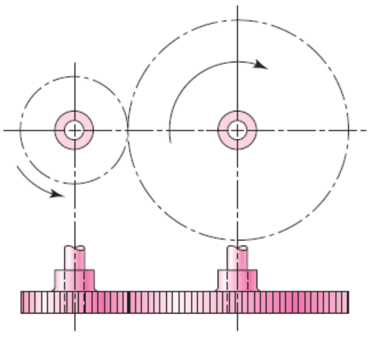

Figure (10–1)

Spur gears are used to transmit rotary motion between parallel shafts

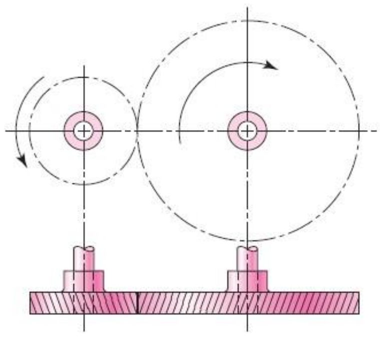

Figure (10–2)

Helical gears are used to transmit motion between parallel or nonparallel shafts

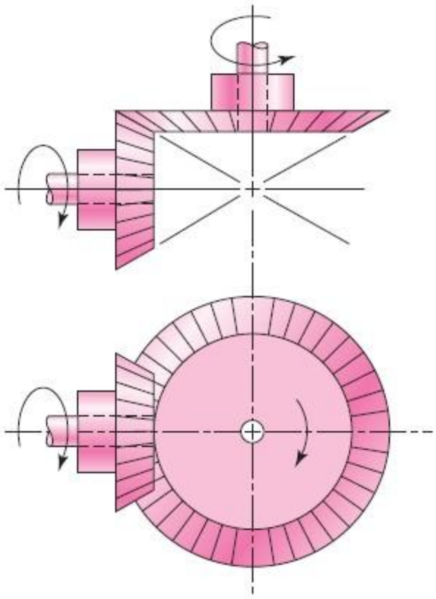

Figure (10–3)

Bevel gears are used to transmit rotary motion between intersecting shafts

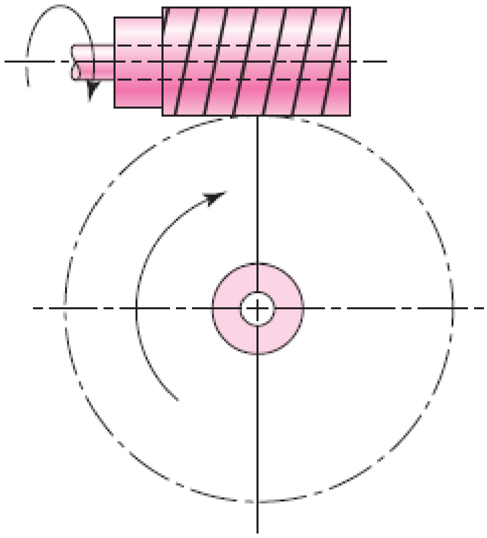

Figure (10–4)

Worm gearsets are used to transmit rotary motion between nonparallel and nonintersecting shafts

Worms and *worm gears,* shown in Fig. (10–4), represent the fourth basic gear type. As shown, the worm resembles a screw. The direction of rotation of the worm gear, also called the worm wheel, depends upon the direction of rotation of the worm and upon whether the worm teeth are cut right-hand or left-hand. Worm-gear sets are also made so that the teeth of one or both wrap partly around the other. Such sets are called *single-enveloping* and *double-enveloping* worm-gear sets. Worm-gear sets are mostly used when the speed ratios of the two shafts are quite high, say, 3 or more.

10.2 Nomenclature

The terminology of spur-gear teeth is illustrated in Fig. (10–5). The *pitch circle* is a theoretical circle upon which all calculations are usually based; its diameter is the *pitch diameter*. The pitch circles of a pair of mating gears are tangent to each other. A *pinion* is the smaller of two mating gears. The larger is often called the *gear*.

The *circular pitch p* is the distance, measured on the pitch circle, from a point on one tooth to a corresponding point on an adjacent tooth. Thus the circular pitch is equal to the sum of the *tooth thickness* and the *width of space.*

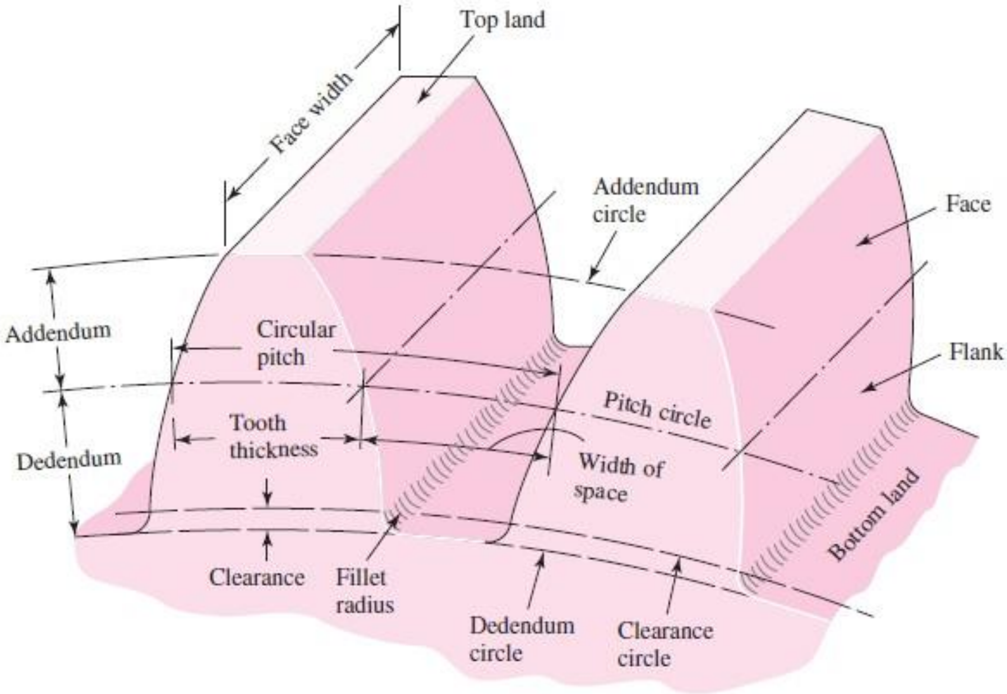

Figure (10–5)

Nomenclature of spur-gear teeth

The *module m* is the ratio of the pitch diameter to the number of teeth. The customary unit of length used is the millimeter. The module is the index of tooth size in SI.

The *diametral pitch P* is the ratio of the number of teeth on the gear to the pitch diameter. Thus, it is the reciprocal of the module. Since diametral pitch is used only with U.S. units, it is expressed as teeth per inch.

The *addendum a* is the radial distance between the *top land* and the pitch circle. The *dedendum b* is the radial distance from the *bottom land* to the pitch circle. The *whole depth ht* is the sum of the addendum and the dedendum.

The *clearance circle* is a circle that is tangent to the addendum circle of the mating gear. The *clearance c* is the amount by which the dedendum in a given gear exceeds the addendum of its mating gear. The *backlash* is the amount by which the width of a tooth space exceeds the thickness of the engaging tooth measured on the pitch circles.

The following relations are useful:

$P = \frac{N}{d}$

10-
1

$$m = \frac{d}{N} \tag{10-2}$$

$$p = \frac{\pi d}{N} = \pi m \tag{10-3}$$

$$pP = \pi \tag{10-4}$$

where
P = diametral pitch, teeth per inch
N = number of teeth
d = pitch diameter, in
m = module, mm
d = pitch diameter, mm
p = circular pitch

10.3 Conjugate Action

When the tooth profiles are designed so as to produce a constant angular velocity ratio during meshing, these are said to have *conjugate action*. In theory, at least, it is possible arbitrarily to select any profile for one tooth and then to find a profile for the meshing tooth that will give conjugate action. One of these solutions is the *involute profile*, which, with few exceptions, is in universal use for gear teeth and is the only one with which we should be concerned.

10.4 Involute Properties

An involute curve may be generated as shown in Fig. (10–6*a*). A partial flange *B* is attached to the cylinder *A*, around which is wrapped a cord *def*, which is held tight. Point *b* on the cord represents the tracing point, and as the cord is wrapped and unwrapped about the cylinder, point *b* will trace out the involute curve *ac*. The radius of the curvature of the involute varies continuously, being zero at point *a* and a maximum at point *c*. At point *b* the radius is equal to the distance *be*, since point *b* is instantaneously rotating about point *e*. Thus the generating line *de* is normal to the involute at all points of intersection and, at the same time, is always tangent to the cylinder *A*. The circle on which the involute is generated is called the *base circle.*

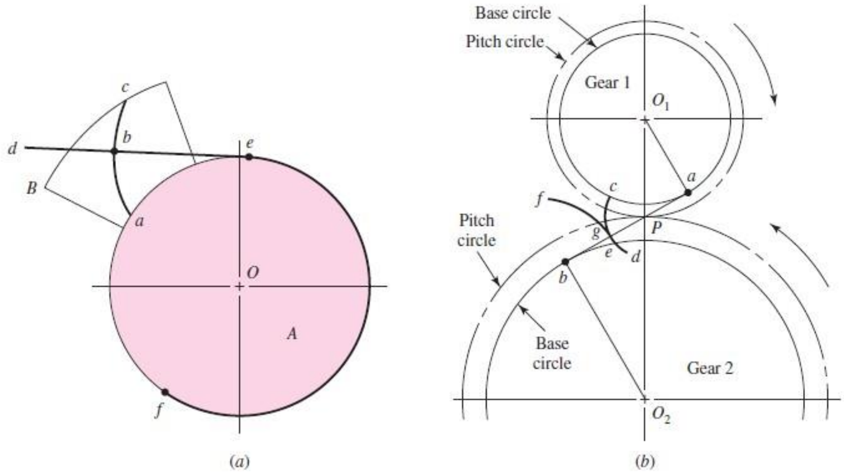

Figure (10–6)
(a) Generation of an involute; (*b*) involute action

Let us now examine the involute profile to see how it satisfies the requirement for the transmission of uniform motion. In Fig. (10–6*b*), two gear blanks with fixed centers at O_1 and O_2 are shown having base circles whose respective radii are O_1a and O_2b. We now imagine that a cord is wound clockwise around the base circle of gear 1, pulled tight between points *a* and *b*, and wound counterclockwise around the base circle of gear 2. If, now, the base circles are rotated in different directions so as to keep the cord tight, a point *g* on the cord will trace out the involutes *cd* on gear 1 and *ef* on gear 2. The involutes are thus generated simultaneously by the tracing point. The tracing point, therefore, represents the point of contact, while the portion of the cord *ab* is the generating line. The point of contact moves along the generating line; the generating line does not change position, because it is always tangent to the base circles; and since the generating line is always normal to the involutes at the point of contact, the requirement for uniform motion is satisfied.

10.5 Fundamentals

When two gears are in mesh, their pitch circles roll on one another without slipping. Designate the pitch radii as r_1 and r_2 and the angular velocities as ω_1 and ω_2, respectively. Then the pitch-line velocity is

$V = |r_1\omega_1| = |r_2\omega_2|$

Thus the relation between the radii on the angular velocities is

$$\left|\frac{\omega_1}{\omega_2}\right| = \frac{r_2}{r_1} \qquad \text{10-5}$$

Suppose now we wish to design a speed reducer such that the input speed is 1800 rev/min and the output speed is 1200 rev/min. This is a ratio of 3:2; the gear pitch diameters would be in the same ratio, for example, a 4-in pinion driving a 6-in gear. The various dimensions found in gearing are always based on the pitch circles.

Suppose we specify that an 18-tooth pinion is to mesh with a 30-tooth gear and that the diametral pitch of the gearset is to be 2 teeth per inch. Then, from Eq. (10–1), the pitch diameters of the pinion and gear are, respectively,

$$d_1 = \frac{N_1}{P} = \frac{18}{2} = 9\text{in} \qquad d_2 = \frac{N_2}{P} = \frac{30}{2} = 15\text{in}$$

The first step in drawing teeth on a pair of mating gears is shown in Fig. (10–7). The center distance is the sum of the pitch radii, in this case 12-in. So locate the pinion and gear centers O_1 and O_2, 12-in apart. Then construct the pitch circles of radii r_1 and r_2. These are tangent at *P*, the *pitch point*. Next draw line *ab*, the common tangent, through the pitch point. We now designate gear 1 as the driver, and since it is rotating counterclockwise, we draw a line *cd* through point *P* at an angle ϕ to the common tangent *ab*. The line *cd* has three names, all of which are in general use. It is called the *pressure line*, the *generating line,* and the *line of action*. It represents the direction in which the resultant force acts between the gears. The angle ϕ is called the *pressure angle,* and it usually has values of 20^o or 25^o, though 14.5^o was once used.

Next, on each gear draw a circle tangent to the pressure line. These circles are the *base circles*. Since they are tangent to the pressure line, the pressure angle determines their size. As shown in Fig. (10–8), the radius of the base circle is

$$r_b = r\cos\phi \qquad \text{10-6}$$

where r is the pitch radius.

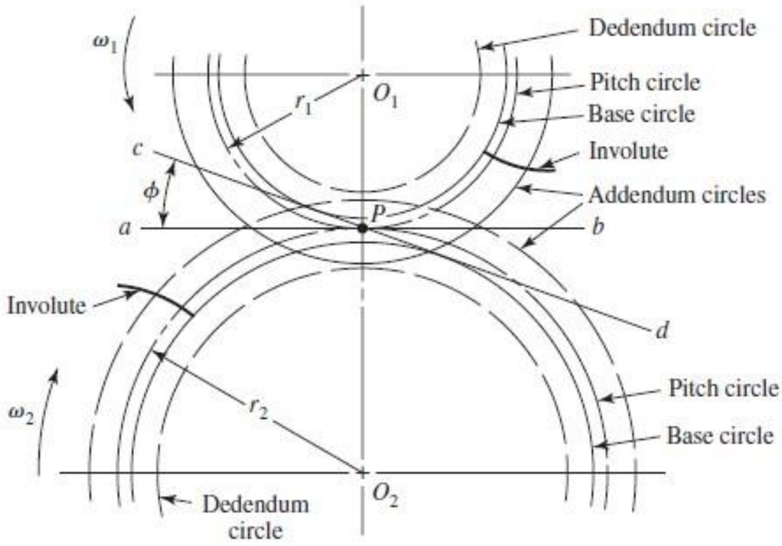

Figure (10–7)
Circles of a gear layout

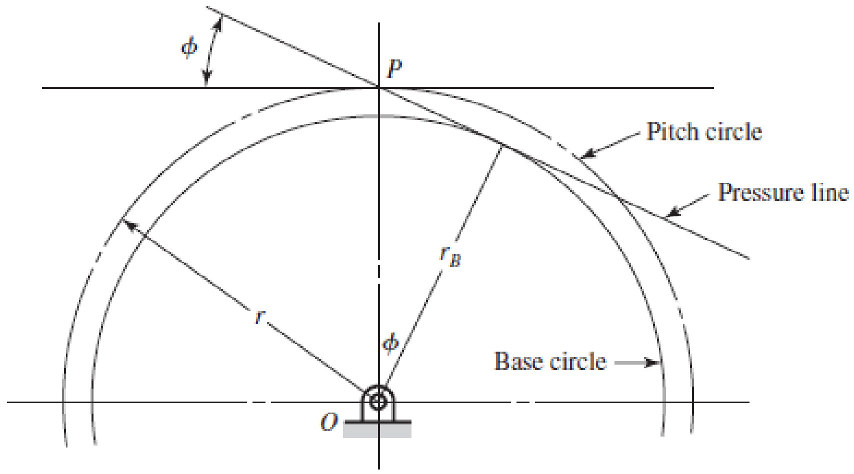

Figure (10–8)

Base circle radius can be related to the pressure angle ϕ

and the pitch circle radius by $r_b = r \cos \phi$

Now generate an involute on each base circle as previously described and as shown in Fig. (10–7). This involute is to be used for one side of a gear tooth. It is not necessary to draw another curve in the reverse direction for the other side of the tooth, because we are going to use a template which can be turned over to obtain the other side.

The addendum and dedendum distances for standard interchangeable teeth are, as we shall learn later, $1/P$ and $1.25/P$, respectively. Therefore, for the pair of gears we are constructing,

$$a = \frac{1}{P} = \frac{1}{2} = 0.5 \text{ in} \qquad b = \frac{1.25}{P} = \frac{1.25}{2} = 0.625 \text{ in}$$

Using these distances, draw the addendum and dedendum circles on the pinion and on the gear as shown in Fig. (10–7).

To draw a tooth, we must know the tooth thickness. From Eq. (10–4), the circular pitch is

$$p = \frac{\pi}{P} = \frac{\pi}{2} = 1.57 \text{in}$$

Therefore, the tooth thickness is

$$t = \frac{p}{2} = \frac{1.57}{2} = 0.785 \text{in}$$

measured on the pitch circle. Using this distance for the tooth thickness as well as the tooth space, draw as many teeth as desired, using the template, after the points have been marked on the pitch circle. In Fig. (10–9) only one tooth has been drawn on each gear. You may run into trouble in drawing these teeth if one of the base circles happens to be larger than the dedendum circle. The reason for this is that the involute begins at the base circle and is undefined below this circle. So, in drawing gear teeth, we usually draw a radial line for the profile below the base circle. The actual shape, however, will depend upon the kind of machine tool used to form the teeth in manufacture, that is, how the profile is generated.

The portion of the tooth between the clearance circle and the dedendum circle includes the fillet. In this instance the clearance is

$$c = b - a = 0.625 - 0.5 = 0.125 \text{ in}$$

The construction is finished when these fillets have been drawn.

Referring again to Fig. (10–9), the pinion with center at O_1 is the driver and turns counterclockwise. The pressure, or generating, line is the same as the cord used in Fig. (10–6*a*) to generate the involute, and contact occurs along this line. The initial contact will take place when the flank of the driver comes into contact with the tip of the driven tooth. This occurs at point *a* in Fig. (10–9), where the addendum circle of the driven gear crosses the pressure line. If we now construct tooth profiles through point *a* and draw radial lines from the intersections of these profiles with the pitch circles to the gear centers, we obtain the *angle of approach* for each gear.

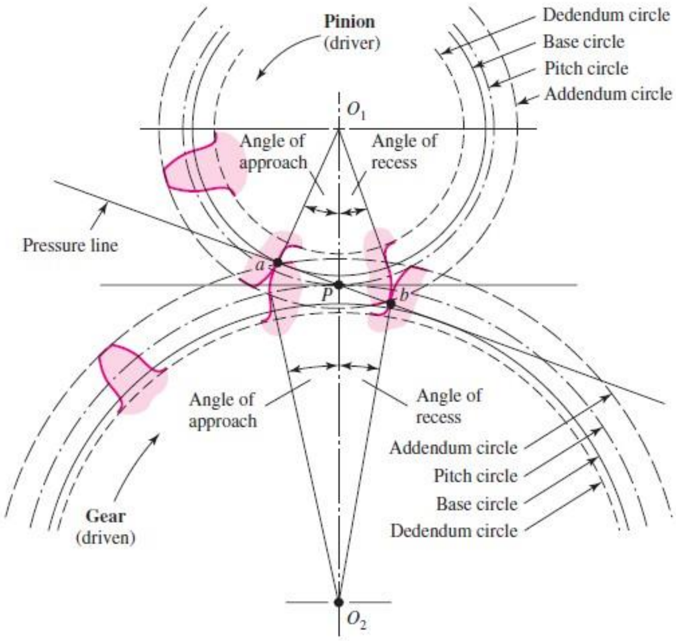

Figure (10–9)
Tooth action

As the teeth go into mesh, the point of contact will slide up the side of the driving tooth so that the tip of the driver will be in contact just before contact ends. The final point of contact will therefore be where the addendum circle of the driver crosses the pressure line. This is point *b* in Fig. (10–9). By drawing another set of tooth profiles through *b*, we obtain the *angle of recess* for each gear in a manner similar to that of finding the angles of approach. The sum of the angle of approach and the angle of recess for either gear is called the *angle of action.* The line *ab* is called *the line of action.*

We may imagine a *rack* as a spur gear having an infinitely large pitch diameter. Therefore, the rack has an infinite number of teeth and a base circle which is an infinite distance from the pitch point. The sides of involute teeth on a rack are straight lines making an angle to the line of centers equal to the pressure angle. Figure (10–10) shows an involute rack in mesh with a pinion. Corresponding sides on involute teeth are parallel curves; the *base pitch* is the constant and fundamental distance between them along a common normal as shown in Fig. (10–10). The base pitch is related to the circular pitch by the equation

$$p_b = p_c \cos \phi \qquad 10\text{-}7$$

where p_b is the base pitch.

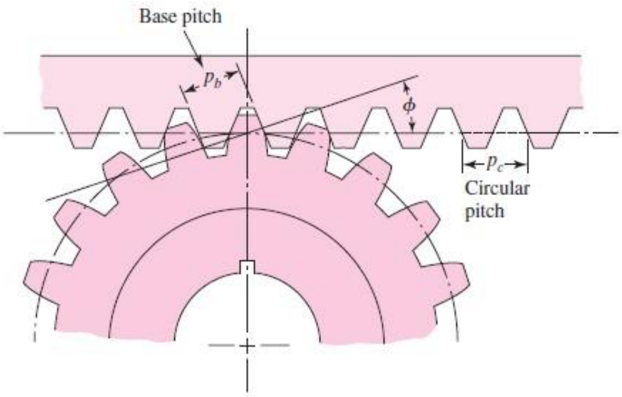

Figure (10–10)

Involute-toothed pinion and rack

Figure (10–11) shows a pinion in mesh with an *internal,* or *ring, gear*. Note that both of the gears now have their centers of rotation on the same side of the pitch point. Thus the positions of the addendum and dedendum circles with respect to the pitch circle are reversed; the addendum circle of the internal gear lies *inside* the pitch circle. Note, too, from Fig. (10–11), that the base circle of the internal gear lies inside the pitch circle near the addendum circle.

Another interesting observation concerns the fact that the operating diameters of the pitch circles of a pair of meshing gears need not be the same as the respective design pitch diameters of the gears, though this is the way they have been constructed in Fig. (10–9). If we increase the center distance, we create two new operating pitch circles having larger diameters because they must be tangent to each other at the pitch point. Thus the pitch circles of gears really do not come into existence until a pair of gears are brought into mesh.

Changing the center distance has no effect on the base circles, because these were used to generate the tooth profiles. Thus the base circle is basic to a gear. Increasing the center distance increases the pressure angle and decreases the length of the line of action, but the teeth are still conjugate, the requirement for uniform motion transmission is still satisfied, and the angular-velocity ratio has not changed.

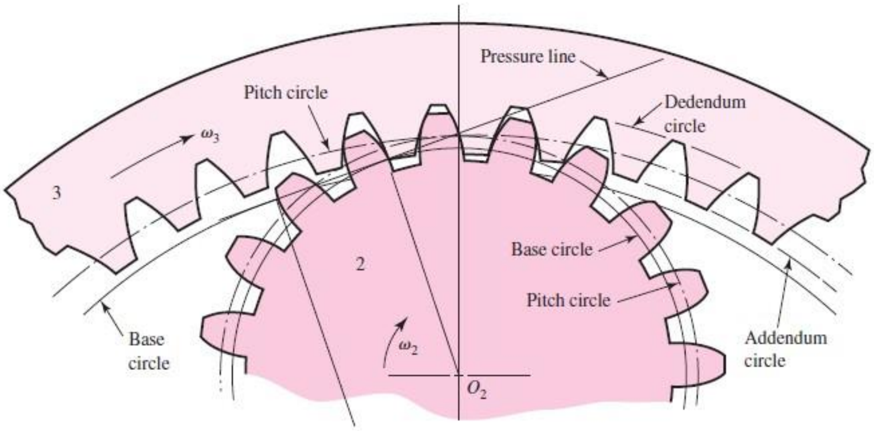

Figure (10–11)
Internal gear and pinion

EXAMPLE 10–1

A gearset consists of a 16-tooth pinion driving a 40-tooth gear. The diametral pitch is 2, and the addendum and dedendum are $1/P$ and $1.25/P$, respectively. The gears are cut using a pressure angle of 20^o.
(a) Compute the circular pitch, the center distance, and the radii of the base circles.
(b) In mounting these gears, the center distance was incorrectly made 1/4 in larger. Compute the new values of the pressure angle and the pitch-circle diameters.

Solution

(*a*) $$p = \pi/P = \pi/2 = 1.57 \text{ in}$$

The pitch diameters of the pinion and gear are, respectively,

$$d_P = 16/2 = 8 \text{ in} \qquad d_G = 40/2 = 20 \text{ in}$$

Therefore the center distance is

$$(d_P + d_G)/2 = (8 + 20)/2 = 14 \text{ in}$$

Since the teeth were cut on the 20° pressure angle, the base-circle radii are found to be, using $r_b = r \cos \phi$,

$$r_b \text{ (pinion)} = (8/2) \cos 20^\circ = 3.76 \text{ in}$$

$$r_b \text{ (gear)} = (20/2) \cos 20^\circ = 9.4 \text{ in}$$

(b) Designating d'_P and d'_G as the new pitch-circle diameters, the 1/4-in increase in the center distance requires that

$$(d'_P + d'_G)/2 = 14.25 \qquad (1)$$

Also, the velocity ratio does not change, and hence

$$d'_P/d'_G = 16/40 \qquad (2)$$

Solving Eqs. (1) and (2) simultaneously yields

$$d'_P = 8.143 \text{ in} \qquad d'_G = 20.357 \text{ in}$$

Since $r_b = r \cos \phi$, the new pressure angle is

$$\phi' = \cos^{-1} [r_b \text{ (pinion)}]/(d'_P/2) = \cos^{-1} [3.76/(8.143/2)] = 22.56^\circ$$

10.6 Contact Ratio

The zone of action of meshing gear teeth is shown in Fig. (10–12). We recall that tooth contact begins and ends at the intersections of the two addendum circles with the pressure line. In Fig. (10–12) initial contact occurs at a and final contact at b. Tooth profiles drawn through these points intersect the pitch circle at A and B, respectively. As shown, the distance AP is called the *arc of approach* q_a , and the distance PB, the *arc of recess* q_r . The sum of these is the *arc of action* q_t .

Now, consider a situation in which the arc of action is exactly equal to the circular pitch, that is, $q_t = p$. This means that one tooth and its space will occupy the entire arc AB. In other words, when a tooth is just beginning contact at a, the previous tooth is simultaneously ending its contact at b. Therefore, during the tooth action from a to b, there will be exactly one pair of teeth in contact.

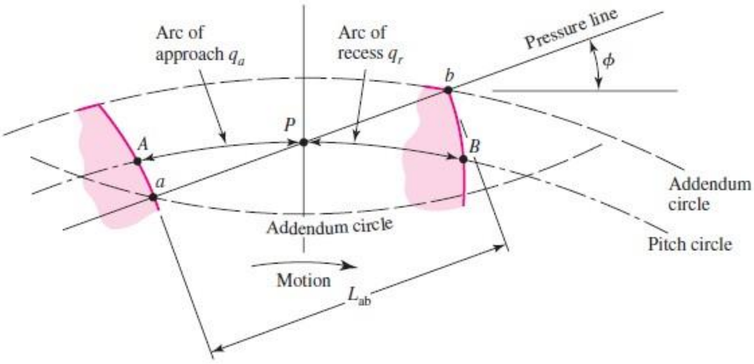

Figure (10–12)
Definition of contact ratio

Next, consider a situation in which the arc of action is greater than the circular pitch, but not very much greater, say, $q_t = 1.2p$. This means that when one pair of teeth is just entering contact at *a*, another pair, already in contact, will not yet have reached *b*.

Thus, for a short period of time, there will be two teeth in contact, one in the vicinity of *A* and another near *B*. As the meshing proceeds, the pair near *B* must cease contact, leaving only a single pair of contacting teeth, until the procedure repeats itself.

Because of the nature of this tooth action, either one or two pairs of teeth in contact, it is convenient to define the term *contact ratio* m_c as

$$m_c = q_t/p \qquad \text{10-8}$$

a number that indicates the average number of pairs of teeth in contact. Note that this ratio is also equal to the length of the path of contact divided by the base pitch. Gears should not generally be designed having contact ratios less than about 1.2, because inaccuracies in mounting might reduce the contact ratio even more, increasing the possibility of impact between the teeth as well as an increase in the noise level.

An easier way to obtain the contact ratio is to measure the line of action *ab* instead of the arc distance *AB*. Since *ab* in Fig. (10–12) is tangent to the base circle when extended, the base pitch p_b must be used to calculate m_c instead of the circular pitch as in Eq. (10–8). If the length of the line of action is L_{ab}, the contact ratio is

$$m_c = L_{ab}/p\cos\phi \qquad \text{10-9}$$

in which Eq. (10–7) was used for the base pitch.

10.7 Interference

The contact of portions of tooth profiles that are not conjugate is called *interference*. Consider Fig. (10–13). Illustrated are two 16-tooth gears that have been cut to the now obsolete 14.5^o pressure angle. The driver, gear 2, turns clockwise. The initial and final points of contact are designated *A* and *B*, respectively, and are located on the pressure line. Now notice that the points of tangency of the pressure line with the base circles *C* and *D* are located *inside* of points *A* and *B*. Interference is present.

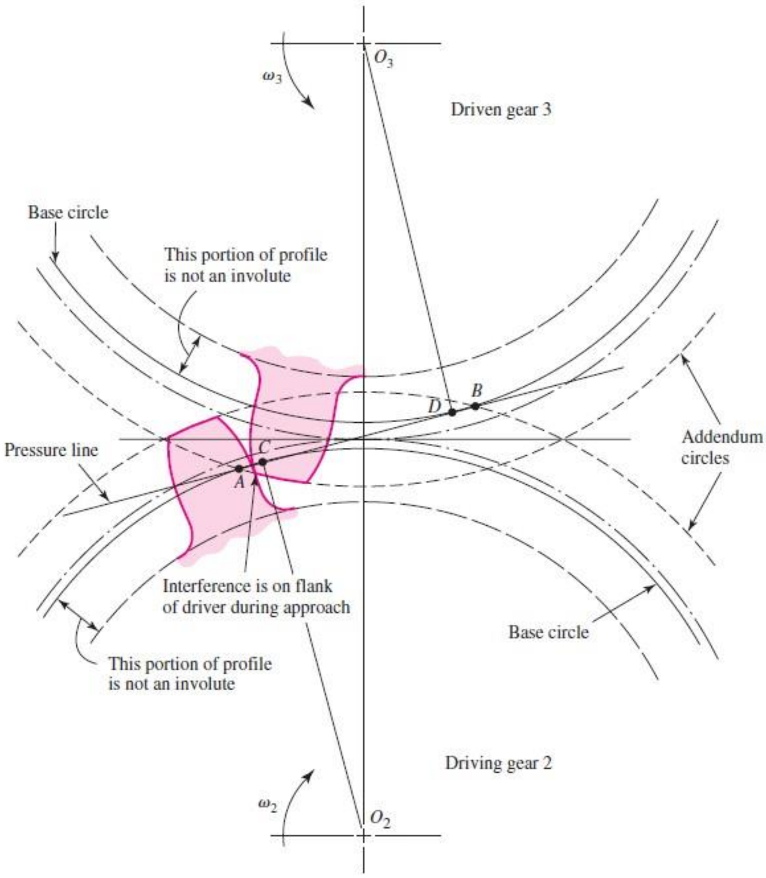

Figure (10–13)

Interference in the action of gear teeth

The interference is explained as follows. Contact begins when the tip of the driven tooth contacts the flank of the driving tooth. In this case the flank of the driving tooth first makes contact with the driven tooth at point *A*, and this occurs *before* the involute portion of the driving tooth comes within range. In other words, contact is occurring below the base circle of gear 2 on the *noninvolute* portion of the flank. The actual effect is that the involute tip or face of the driven gear tends to dig out the noninvolute flank of the driver.

In this example the same effect occurs again as the teeth leave contact. Contact should end at point *D* or before. Since it does not end until point *B*, the effect is for the tip of the driving tooth to dig out, or interfere with, the flank of the driven tooth.

When gear teeth are produced by a generation process, interference is automatically eliminated because the cutting tool removes the interfering portion of the flank. This effect is called *undercutting;* if undercutting is at all pronounced, the undercut tooth is considerably weakened. Thus the effect of eliminating interference by a generation process is merely to substitute another problem for the original one.

The smallest number of teeth on a spur pinion and gear, one-to-one gear ratio, which can exist without interference is N_P . This number of teeth for spur gears is given by

$$N_P = \frac{2k}{3\sin^2\phi}\left(1+\sqrt{1+3\sin^2\phi}\right) \qquad 10\text{-}10$$

where k = 1 for full-depth teeth, 0.8 for stub teeth and ϕ = pressure angle.

For a 20° pressure angle, with k = 1,

$$N_P = \frac{2(1)}{3\sin^2 20^\circ}\left(1+\sqrt{1+3\sin^2 20^\circ}\right) = 12.3 = 13 \text{ teeth}$$

Thus 13 teeth on pinion and gear are interference-free. Realize that 12.3 teeth is possible in meshing arcs, but for fully rotating gears, 13 teeth represents the least number. For a 14.5° pressure angle, N_P = 23 teeth, so one can appreciate why few 14.5°-tooth systems are used, as the higher pressure angles can produce a smaller pinion with accompanying smaller center-to-center distances.

If the mating gear has more teeth than the pinion, that is, $m_G = N_G / N_P = m$ is more than one, then the smallest number of teeth on the pinion without interference is given by

$$N_P = \frac{2k}{(1+2m)\sin^2\phi}\left(m+\sqrt{m^2+(1+2m)\sin^2\phi}\right) \qquad 10\text{-}11$$

For example, if $m = 4$, $\phi = 20^\circ$,

$$N_P = \frac{2(1)}{[1+2(4)]\sin^2 20^\circ}\left(4+\sqrt{}\right.$$

$4^2 + [1 + 2(4)]\sin^2 20^\circ \Big) = 15.4 = 16$ teeth

Thus a 16-tooth pinion will mesh with a 64-tooth gear without interference.

The largest gear with a specified pinion that is interference-free is

$$N_G = \frac{N_P^2 \sin^2 \phi - 4k^2}{4k - 2N_P \sin^2 \phi} \qquad \text{10-12}$$

For example, for a 13-tooth pinion with a pressure angle ϕ of 20°,

$$N_G = \frac{13^2 \sin^2 20^\circ - 4(1)^2}{4(1) - 2(13)\sin^2 20^\circ} = 16.45 = 16 \text{ teeth}$$

For a 13-tooth spur pinion, the maximum number of gear teeth possible without interference is 16.

The smallest spur pinion that will operate with a rack without interference is

$$N_P = \frac{2k}{\sin^2 \phi} \qquad \text{10-13}$$

For a 20° pressure angle full-depth tooth the smallest number of pinion teeth to mesh with a rack is

$$N_P = \frac{2(1)}{\sin^2 20^\circ} = 17.1 = 18 \text{ teeth}$$

Since gear-shaping tools amount to contact with a rack, and the gear-hobbing process is similar, the minimum number of teeth to prevent interference to prevent undercutting by the hobbing process is equal to the value of N_P when N_G is infinite.

The importance of the problem of teeth that have been weakened by undercutting cannot be overemphasized. Of course, interference can be eliminated by using more teeth on the pinion. However, if the pinion is to transmit a given amount of power, more teeth can be used only by increasing the pitch diameter.

Interference can also be reduced by using a larger pressure angle. This results in a smaller base circle, so that more of the tooth profile becomes involute. The demand for smaller pinions with fewer teeth thus favors the use of a 25^{o} pressure angle even though the frictional forces and bearing loads are increased and the contact ratio decreased.