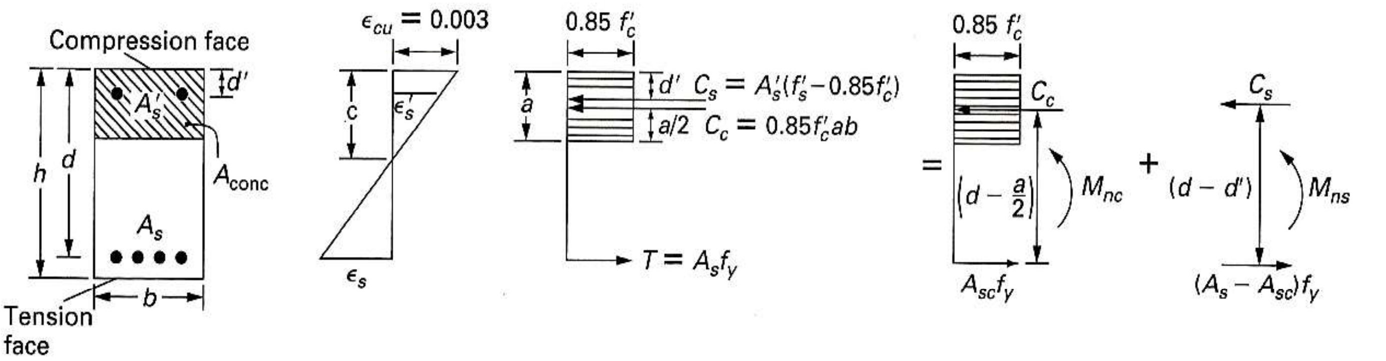
4.10 DOUBLY REINFORCED CONCRETE SECTIONS (SECTIONS WITH COMPRESSION REINFORCEMENT).

Flexural members are designed for tension reinforcement. Any additional moment capacity required in the section is usually provided by increasing the section size or the amount of tension reinforcement.

However, the cross-sectional dimensions in some applications may be limited by architectural or functional requirements (architectural limitations restrict the beam web depth at midspan, or the midspan section dimensions are not adequate to carry the support negative moment even when tensile steel at the support is sufficiently increased), and the extra moment capacity may have to be provided by additional tension and compression reinforcement. The extra steel generates an internal force couple, adding to the sectional moment capacity without changing the ductility of the section. In such cases, the total moment capacity consists of two components:

1. moment due to the tension reinforcement that balances the compression concrete, M_{nc}, and
2. moment generated by the internal steel force couple consisting of compression reinforcement and equal amount of additional tension reinforcement, M_{ns} as illustrated in figure below.



(a) Cross section (b) Strain diagram (c) Stress diagram (d) Moment carried as a singly reinforced beam, M_{nc} (e) Moment carried by the compression steel, M_{ns}

Notation:

ε_s' − strain in compression steel.

$f_s' = E_s\varepsilon_s' \le f_y$ − compression steel stress

A_s' − area fo compression steel

d' − distance from extreme compression fiber to centroid of compression steel

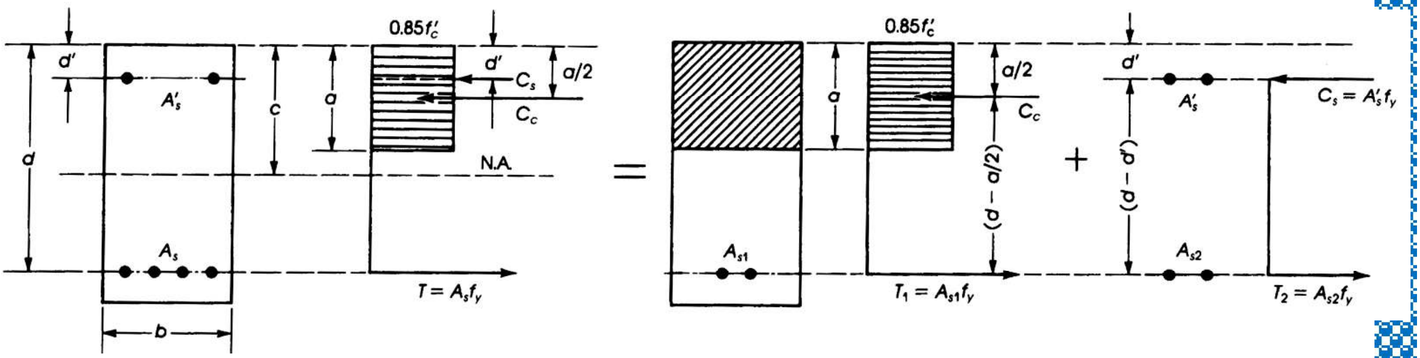
$\rho' = \dfrac{A_s'}{bd}$ − compression steel reinforcement ratio.

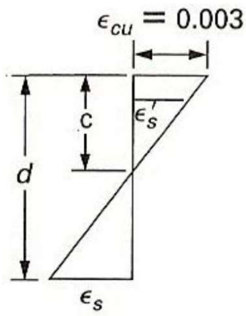
A_{sc} − part of the tension steel that match C_c.

C_c − concrete compression resultant for a beam without compression reinforcement.

C_s − compression steel resultant as if A_s' were stressed at $(f_s' - 0.85f_c')$.

4.10.1 Analysis of doubly reinforced concrete sections





- **Compression steel is yielded**

Compression steel is yielded when $\varepsilon_s' \geq \varepsilon_y = \dfrac{f_y}{E_s}$

$\varepsilon_s' = 0.003\left(\dfrac{c-d'}{c}\right)$, $\quad c = \dfrac{a}{\beta_1}$

$A_{s1} = A_s - A_{s2}$, $\quad A_{s2} = A_s'$

$T_1 = A_{s1}f_y = C_c \quad and \quad M_{nc} = A_{s1}f_y\left(d-\dfrac{a}{2}\right) \quad or \quad M_{nc} = (A_s - A_s')f_y\left(d-\dfrac{a}{2}\right)$

where $\quad a = \dfrac{A_{s1}f_y}{0.85\,f_c'b} = \dfrac{(A_s - A_s')f_y}{0.85\,f_c'b} \qquad \rho = \dfrac{A_s}{bd} \qquad \rho' = \dfrac{A_s'}{bd}$

substituting "a" into $\quad$ c $= \dfrac{a}{\beta_1} = \dfrac{(A_s - A_s')f_y}{\beta_1 \cdot 0.85\,f_c'b} = \dfrac{(\rho-\rho')f_y d}{\beta_1 \cdot 0.85\,f_c'}$

substituting "c" into $\quad \varepsilon_s' = 0.003\left(1-\dfrac{d'}{c}\right) = 0.003\left[1-\dfrac{0.85\beta_1 f_c' d'}{(\rho-\rho')d f_y}\right]$

Compression steel is yielded when

$$\varepsilon_s' \geq \varepsilon_y = \frac{f_y}{E_s}$$

$$0.003\left[1-\frac{0.85\beta_1 f_c' d'}{(\rho-\rho')d f_y}\right] \geq \frac{f_y}{E_s = 200000\ MPa}$$

or in the form

$$\rho - \rho' \geq \frac{0.85 f_c' d'}{d f_y}\beta_1\left(\frac{600}{600-f_y}\right)$$

$$\rho \geq \bar{\rho}_{cy}$$

where

$$\bar{\rho}_{cy} = \frac{0.85 f_c' d'}{d f_y}\beta_1\left(\frac{600}{600-f_y}\right) + \rho' \qquad (*)$$

$\bar{\rho}_{cy}$ − minimum tensile reinforcement ratio that will ensure yielding of compression steel at failure.

In the previous equation of $\bar{\rho}_{cy}$ was ignored that part of the compression zone is occupied by the compression reinforcement, the value of ignored compressive force is $A_s'(0.85f_c')$. So the depth of stress block can be expressed

$$a = \frac{A_s f_y - A_s'(f_y - 0.85f_c')}{0.85f_c' b},$$

and

$$\bar{\rho}_{cy} = \frac{0.85f_c' d'}{d f_y}\beta_1\left(\frac{600}{600 - f_y}\right) + \rho'(1 - \frac{0.85f_c'}{f_y})$$

In all calculations, the equation $(*)$ for $\bar{\rho}_{cy}$ will be used.

$T = A_s f_y = C_c + C_s = T_1 + T_2$

$C_c = 0.85f_c' ab,\qquad C_s = A_s'(f_y - 0.85f_c')$

$A_s f_y = 0.85f_c' ab + A_s'(f_y - 0.85f_c')\qquad from\ where\qquad a = \dfrac{A_s f_y - A_s'(f_y - 0.85f_c')}{0.85f_c' b}$

The nominal moment strength for rectangular section with tension and **compression steel is yielded**

$$M_n = \left(A_s f_y - A_s'(f_y - 0.85f_c')\right)\left(d - \frac{a}{2}\right) + A_s'(f_y - 0.85f_c')(d - d'),$$

$$or\qquad M_n = 0.85f_c' ab\left(d - \frac{a}{2}\right) + A_s'(f_y - 0.85f_c')(d - d').$$

For simplicity, $A_s'(0.85f_c')$ can be ignored and then:

$$T = A_s f_y = C_c + C_s = T_1 + T_2,\qquad C_c = 0.85f_c' ab,\qquad C_s = A_s' f_y\qquad a = \frac{(A_s - A_s')f_y}{0.85f_c' b}$$

$$M_n = (A_s - A_s')f_y\left(d - \frac{a}{2}\right) + A_s' f_y(d - d') = 0.85f_c' ab\left(d - \frac{a}{2}\right) + A_s' f_y(d - d')$$

- **Compression steel is NOT yielded**

Compression steel is NOT yielded when

$$\varepsilon_s' < \varepsilon_y = \frac{f_y}{E_s}\qquad \text{or}\qquad f_s' = \varepsilon_s' E_s < f_y\qquad \text{or}\qquad \rho < \bar{\rho}_{cy}$$

$$f_s' = \varepsilon_s' E_s = 0.003\left(\frac{c - d'}{c}\right)200\,000 = 600\left(\frac{c - d'}{c}\right)$$

$T = A_s f_y = C_c + C_s = T_1 + T_2$

$C_c = 0.85f_c' ab,\qquad C_s = A_s'(f_s' - 0.85f_c')$

$$A_s f_y = 0.85 f_c' ab + A_s'(f_s' - 0.85 f_c') \quad from\ where \quad a = \frac{A_s f_y - A_s'(f_s' - 0.85 f_c')}{0.85 f_c' b} = \beta_1 c.$$

Note that in the above equation two unknowns "c" and "f_s'". Substitiuting $f_s' = 600\left(\frac{c-d'}{c}\right)$ in "a" we get an quadratic equation in "c", the only unknown, which is easily solved for "c".

The nominal moment strength for rectangular section with tension and compression **steel is NOT yielded**

$$M_n = \left(A_s f_y - A_s'(f_s' - 0.85 f_c')\right)\left(d - \frac{a}{2}\right) + A_s'(f_s' - 0.85 f_c')(d - d'),$$

or

$$M_n = 0.85 f_c' ab\left(d - \frac{a}{2}\right) + A_s'(f_s' - 0.85 f_c')(d - d').$$

For simplicity, $A_s'(0.85 f_c')$ can be ignored and then:

$$T = A_s f_y = C_c + C_s = T_1 + T_2, \qquad C_c = 0.85 f_c' ab, \qquad C_s = A_s' f_s' \qquad a = \frac{A_s f_y - A_s' f_s'}{0.85 f_c' b}$$

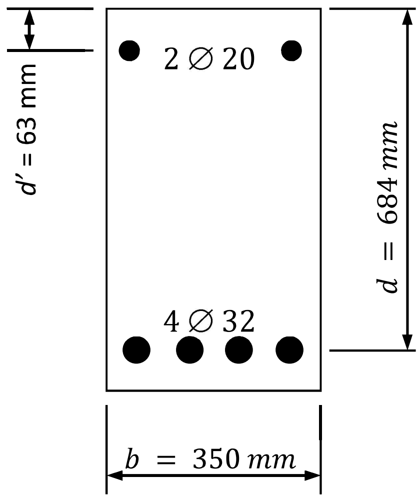
$$M_n = \left(A_s f_y - A_s' f_s'\right)\left(d - \frac{a}{2}\right) + A_s' f_s'(d - d') = 0.85 f_c' ab\left(d - \frac{a}{2}\right) + A_s' f_s'(d - d')$$

For both cases (compression steel is yielded and is NOT yielded) $\varepsilon_s \geq 0.005$ (tension-controlled section).

Example:

Determine the nominal positive moment strength of the section of rectangular cross sectional beam. The beam is reinforced with $4 \varnothing 32$ in the tension zone and $2 \varnothing 20$ in the compression zone.

Take $f_c' = 20\ MPa, \quad f_y = 400\ MPa.$



__**Solution:**__

$A_s(4 \varnothing 32) = 32.17\ cm^2$

$A_s'(2 \varnothing 20) = 6.28\ cm^2$

$$\rho = \frac{A_s}{bd} = \frac{3217}{350 \cdot 684} = 0.0134$$

$$\rho' = \frac{A_s'}{bd} = \frac{628}{350 \cdot 684} = 0.0026, \qquad \beta_1 = 0.85,$$

$$\bar{\rho}_{cy} = \frac{0.85 f_c' d'}{d f_y}\beta_1\left(\frac{600}{600 - f_y}\right) + \rho' = \frac{0.85 \cdot 20 \cdot 63}{684 \cdot 400} 0.85\left(\frac{600}{600 - 400}\right) + 0.0026 = 0.01258$$

$\rho = 0.0134 > \bar{\rho}_{cy} = 0.01258$ compression steel is yielded ($\varepsilon_s' \geq \varepsilon_y$)

$T = A_s f_y = C_c + C_s$

$A_s f_y = 0.85 f_c' ab + A_s'(f_y - 0.85 f_c')$ from where

$$a = \frac{A_s f_y - A_s'(f_y - 0.85 f_c')}{0.85 f_c' b} = \frac{3217 \cdot 400 - 628 \cdot (400 - 0.85 \cdot 20)}{0.85 \cdot 20 \cdot 350} = 175.84\ mm,$$

$$c = \frac{a}{\beta_1} = \frac{175.84}{0.85} = 206.88\ mm,$$

$$M_n = 0.85 f_c' ab \left(d - \frac{a}{2}\right) + A_s'(f_y - 0.85 f_c')(d - d') =$$

$$= \left[0.85 \cdot 20 \cdot 175.84 \cdot 350 \left(684 - \frac{175.84}{2}\right) + 628(400 - 0.85 \cdot 20)(684 - 63)\right] \times 10^{-6} =$$

$$= 773.01\ KN \cdot m$$

Check for $\varepsilon_s \geq 0.005$:

$$\varepsilon_s = 0.003 \left(\frac{d - c}{c}\right) = 0.003 \left(\frac{684 - 206.88}{206.88}\right) = 0.00691 > 0.005 \qquad OK$$

Take $\phi = 0.9$ for flexure as tension-controlled section.

$$\phi M_n = 0.9 \cdot 773.01 = 695.71\ KN \cdot m$$

Example:

Repeat the previous example using $f_c' = 30\ MPa$.

Solution:

$$\bar{\rho}_{cy} = \frac{0.85 f_c' d'}{d f_y} \beta_1 \left(\frac{600}{600 - f_y}\right) + \rho' = \frac{0.85 \cdot 30 \cdot 63}{684 \cdot 400} 0.836 \left(\frac{600}{600 - 400}\right) + 0.0026 = 0.0173$$

$\rho = 0.0134 < \bar{\rho}_{cy} = 0.0173$ compression steel is NOT yielded ($\varepsilon_s' < \varepsilon_y$)

$$T = A_s f_y = C_c + C_s$$

$$f_s' = 600 \left(\frac{c - d'}{c}\right), \qquad \beta_1 = 0.85 - 0.007(f_c' - 28) = 0.85 - 0.007(30 - 28) = 0.836$$

$A_s f_y = 0.85 f_c' ab + A_s'(f_s' - 0.85 f_c')$ from where

$$a = \frac{A_s f_y - A_s'(f_s' - 0.85 f_c')}{0.85 f_c' b} = \beta_1 c$$

$$\frac{3217 \cdot 400 - 628 \cdot \left(600 \left(\frac{c - 63}{c}\right) - 0.85 \cdot 30\right)}{0.85 \cdot 30 \cdot 350} = 0.836\ c$$

$$103.755 + \frac{2659.764}{c} = 0.836\ c, \qquad \Longrightarrow \ 0.836\ c^2 - 103.755\ c - 2659.764 = 0,$$

solution of quadratic equation $\qquad x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$c^2 - 124.109\ c - 3181.536 = 0,$$

$$c_{1,2} = \frac{124.109 \pm \sqrt{124.109^2 - 4 \cdot 1 \cdot (-3181.536)}}{2} = \frac{124.109 \pm 167.718}{2}$$

Choose only $c > 0$, $\qquad c = 145.91\ mm$

$a = \beta_1 c = 0.836 \cdot 145.91 = 121.98\, mm,$

$$f_s' = 600\left(\frac{c - d'}{c}\right) = 600\left(\frac{145.91 - 63}{145.91}\right) = 340.94\, MPa \quad < f_y = 400\, MPa,$$

$$M_n = 0.85 f_c' ab\left(d - \frac{a}{2}\right) + A_s'(f_s' - 0.85 f_c{'})(d - d') =$$

$$= \left[0.85 \cdot 30 \cdot 121.98 \cdot 350\left(684 - \frac{121.98}{2}\right) + 628(340.94 - 0.85 \cdot 30)(684 - 63)\right] \times 10^{-6} =$$

$$= 801.27\, KN \cdot m$$

Check for $\varepsilon_s \geq 0.005$:

$$\varepsilon_s = 0.003\left(\frac{d - c}{c}\right) = 0.003\left(\frac{684 - 145.91}{145.91}\right) = 0.011 > 0.005 \qquad OK$$

Take $\phi = 0.9$ for flexure as tension-controlled section.

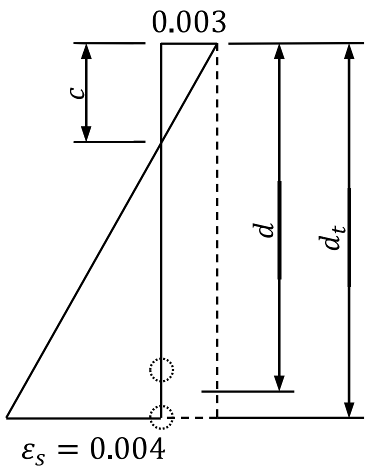
$\phi M_n = 0.9 \cdot 801.27 = 721.14\, KN \cdot m$

4.10.2 Design of doubly reinforced concrete sections.

When the factored moment M_u is greater than the design strength ϕM_n of the beam when it is reinforced with the maximum permissible amount of tension reinforcement, compression reinforcement becomes necessary.

The logical procedure for designning a doubly reinforced sections is to determine first whether compression steel is needed for strength. This may be done by comparing the required moment strength with the moment strength of a singly reinforced section with the maximum permissible amount of tension steel ρ_{max}.

For example, for steel Grade 420 $\rho_{max} = 0.724\rho_b$ which defined from strain conditon $\varepsilon_t = 0.004$ for beams.



$$\frac{c}{0.003} = \frac{d_t}{0.003 + 0.004} \quad \Longrightarrow \quad c = \frac{3}{7} d_t, \qquad a = \beta_1 c$$

The maximum moment strength as a singly reinforced section

$$M_{n,max} = 0.85 f_c' ab\left(d - \frac{a}{2}\right),$$

If $M_u > \phi M_{n,max}$ Design the section as doubly reinforced section,

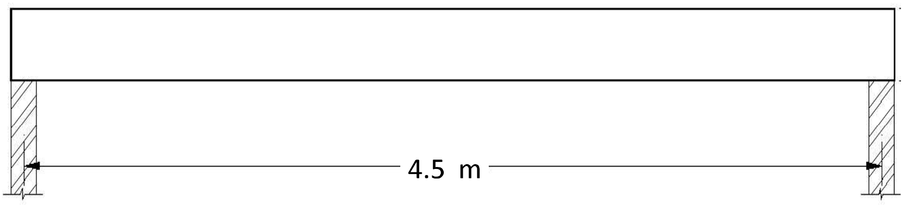
where $\quad \phi = 0.65 + (0.004 - 0.002)\dfrac{250}{3} = 0.817 \approx 0.82$

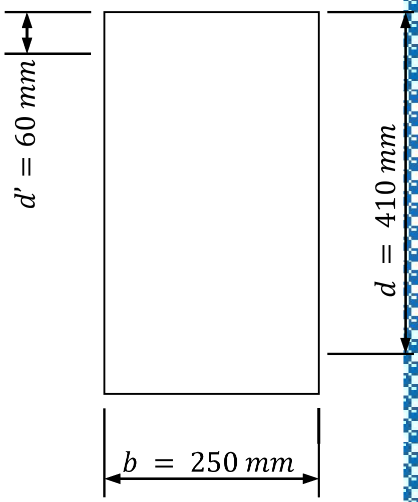
Example:

The beam is loaded by a uniform service $DL = 25\ KN/m$ and a uniform service $LL = 35\ KN/m$. Compute the area of steel reinforcement for the section.

Take $f_c' = 20\ MPa$, $f_y = 400\ MPa$.

Assume $d' = 60\ mm$, and one layer arrangement of tension steel.





<u>Solution:</u>

$$w_u = 1.2D + 1.6L = 1.2 \cdot 25 + 1.6 \cdot 35 = 86\ \ KN/m$$

$$M_u = M_{max} = \frac{w_u l^2}{8} = \frac{86 \cdot 4.5^2}{8} = 217.7\ KN \cdot m$$

Maximum nominal moment strength from strain condition $\varepsilon_s = 0.004$

$$c = \frac{3}{7}d = \frac{3}{7}410 = 175.7\ mm, \qquad \beta_1 = 0.85$$

$$a = \beta_1 c = 0.85 \cdot\ 175.7 = 149.4\ mm$$

$$M_{n,max} = 0.85 f_c' ab\left(d - \frac{a}{2}\right) = 0.85 \cdot 20 \cdot 149.4 \cdot 250\left(410 - \frac{149.4}{2}\right) \times 10^{-6} = 212.9\ KN \cdot m$$

$$\phi = 0.82$$

$$M_u = 217.7\ KN \cdot m > \phi M_n = 0.82 \cdot 212.9 = 174.6\ \ KN \cdot m$$

Design the section as doubly reinforced concrete section.

$$M_{ns} = \frac{M_u}{\phi} - M_{nc} = \frac{217.7}{0.82} -\ 212.9 = 52.59\ KN \cdot m$$

$$M_{ns} = C_s(d - d') = A_s'(f_s{}' - 0.85 f_c{}')(d - d') \quad \Rightarrow \quad A_s' = \frac{M_{ns}}{(f_s{}' - 0.85 f_c{}')(d - d')}$$

$$f_s' = 600\left(\frac{c - d'}{c}\right) = 600\left(\frac{175.7 - 60}{175.7}\right) = 395.1\ MPa\ < f_y = 400\ MPa,$$

Compression steel does NOT yield

$$A_s' = \frac{M_{ns}}{(f_s{}' - 0.85 f_c{}')(d - d')} = \frac{52.59 \cdot 10^6}{(395.1\ - 0.85 \cdot 20)(410 - 60)} = 397.4\ mm^2$$

$$T = C_c + C_s = 0.85 f_c' ab + A_s'(f_s' - 0.85 f_c{}') =$$

$$= [0.85 \cdot 20 \cdot 149.4 \cdot 250 + 397.4(395.1 - 0.85 \cdot 20)] \times 10^{-3} = 785.21\ KN$$

$$A_s = \frac{T}{f_y} = \frac{785.21 \cdot 10^3}{400} = 1963.02\ mm^2$$