section 4.5

Elasticity

Objectives

At the end of this section you should be able to:

- Calculate price elasticity averaged along an arc.
- Calculate price elasticity evaluated at a point.
- Decide whether supply and demand are inelastic, unit elastic or elastic.
- Understand the relationship between price elasticity of demand and revenue.
- Determine the price elasticity for general linear demand functions.

One important problem in business is to determine the effect on revenue of a change in the price of a good. Let us suppose that a firm's demand curve is downward-sloping. If the firm lowers the price then it will receive less for each item, but the number of items sold increases. The formula for total revenue, TR, is

$$\mathrm{TR} = PQ$$

and it is not immediately obvious what the net effect on TR will be as P decreases and Q increases. The crucial factor here is not the absolute changes in P and Q but rather the proportional or percentage changes. Intuitively, we expect that if the percentage rise in Q is greater than the percentage fall in P then the firm experiences an increase in revenue. Under these circumstances we say that demand is ***elastic***, since the demand is relatively sensitive to changes in price. Similarly, demand is said to be ***inelastic*** if demand is relatively insensitive to price changes. In this case, the percentage change in quantity is less than the percentage change in price. A firm can then increase revenue by raising the price of the good. Although demand falls as a result, the increase in price more than compensates for the reduced volume of sales and revenue rises. Of course, it could happen that the percentage changes in price and quantity are equal, leaving revenue unchanged. We use the term ***unit elastic*** to describe this situation.

We quantify the responsiveness of demand to price change by defining the ***price elasticity of demand*** to be

$$E = \frac{\text{percentage change in demand}}{\text{percentage change in price}}$$

Notice that because the demand curve slopes downwards, a positive change in price leads to a negative change in quantity and vice versa. Consequently, the value of *E* is always negative. It is conventional to avoid this by deliberately changing the sign and taking

$$E = -\frac{\text{percentage change in demand}}{\text{percentage change in price}}$$

which makes *E* positive. The previous classification of demand functions can now be restated more succinctly in terms of *E*.

Demand is said to be

- **inelastic if $E < 1$**
- **unit elastic if $E = 1$**
- **elastic if $E > 1$.**

Advice

You should note that not all economists adopt the convention of ignoring the sign to make E positive. If the negative sign is left in, the demand will be inelastic if $E > -1$, unit elastic if $E = -1$ and elastic if $E < -1$. You should check with your lecturer the particular convention that you need to adopt.

As usual, we denote the changes in P and Q by ΔP and ΔQ respectively, and seek a formula for E in terms of these symbols. To motivate this, suppose that the price of a good is \$12 and that it rises to \$18. A moment's thought should convince you that the percentage change in price is then 50%. You can probably work this out in your head without thinking too hard. However, it is worthwhile identifying the mathematical process involved. To obtain this figure we first express the change

$$18 - 12 = 6$$

as a fraction of the original to get

$$\frac{6}{12} = 0.5$$

and then multiply by 100 to express it as a percentage. This simple example gives us a clue as to how we might find a formula for E. In general, the percentage change in price is

change in price expressed as a fraction of the original price

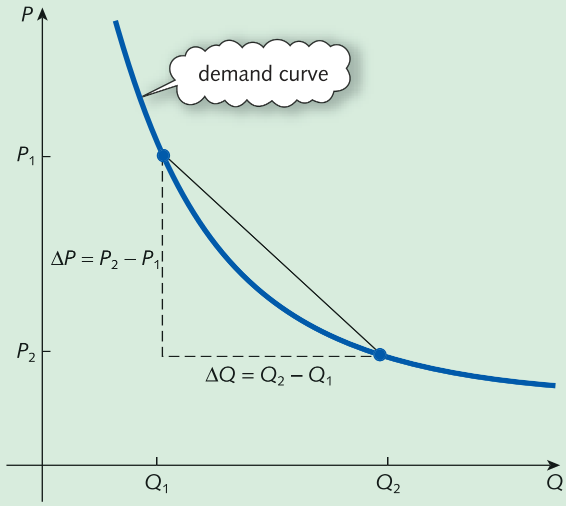
$$\frac{\Delta P}{P} \times 100$$

multiply by 100 to convert fractions into percentages

Similarly, the percentage change in quantity is

$$\frac{\Delta Q}{Q} \times 100$$

Figure 4.19



Hence

$$E = -\left(\frac{\Delta Q}{Q} \times 100\right) \div \left(\frac{\Delta P}{P} \times 100\right)$$

Now, when we divide two fractions we turn the denominator upside down and multiply, so

$$E = -\left(\frac{\Delta Q}{Q} \times \cancel{100}\right) \times \left(\frac{P}{\cancel{100} \times \Delta P}\right)$$

$$= -\frac{P}{Q} \times \frac{\Delta Q}{\Delta P}$$

A typical demand curve is illustrated in Figure 4.19, in which a price fall from P_1 to P_2 causes an increase in demand from Q_1 to Q_2.

Example

Determine the elasticity of demand when the price falls from 136 to 119, given the demand function

$$P = 200 - Q^2$$

Solution

In the notation of Figure 4.19 we are given that

$$P_1 = 136 \quad \text{and} \quad P_2 = 119$$

The corresponding values of Q_1 and Q_2 are obtained from the demand equation

$$P = 200 - Q^2$$

by substituting $P = 136$ and 119 respectively and solving for Q. For example, if $P = 136$ then

$$136 = 200 - Q^2$$

which rearranges to give

$$Q^2 = 200 - 136 = 64$$

This has solution $Q = \pm 8$ and, since we can obviously ignore the negative quantity, we have $Q_1 = 8$. Similarly, setting $P = 119$ gives $Q_2 = 9$. The elasticity formula is

$$E = -\frac{P}{Q} \times \frac{\Delta Q}{\Delta P}$$

and the values of ΔP and ΔQ are easily worked out to be

$$\Delta P = 119 - 136 = -17$$

$$\Delta Q = 9 - 8 = 1$$

However, it is not at all clear what to take for P and Q. Do we take P to be 136 or 119? Clearly we are going to get two different answers depending on our choice. A sensible compromise is to use their average and take

$$P = {}^1\!/\!_2(136 + 119) = 127.5$$

Similarly, averaging the Q values gives

$$Q = {}^1\!/\!_2(8 + 9) = 8.5$$

Hence

$$E = -\frac{127.5}{8.5} \times \left(\frac{1}{-17}\right) = 0.88$$

The particular application of the general formula considered in the previous example provides an estimate of elasticity averaged over a section of the demand curve between (Q_1, P_1) and (Q_2, P_2). For this reason it is called ***arc elasticity*** and is obtained by replacing P by ${}^1\!/\!_2(P_1 + P_2)$ and Q by ${}^1\!/\!_2(Q_1 + Q_2)$ in the general formula.

Practice Problem

1 Given the demand function

$$P = 1000 - 2Q$$

calculate the arc elasticity as P falls from 210 to 200.

A disappointing feature of the previous example is the need to compromise and calculate the elasticity averaged along an arc rather than calculate the exact value at a point. A formula for the latter can easily be deduced from

$$E = -\frac{P}{Q} \times \frac{\Delta Q}{\Delta P}$$

by considering the limit as ΔQ and ΔP tend to zero in Figure 4.19. All that happens is that the arc shrinks to a point and the ratio $\Delta Q/\Delta P$ tends to $\mathrm{d}Q/\mathrm{d}P$. The price elasticity at a point may therefore be found from

$$E = -\frac{P}{Q} \times \frac{\mathrm{d}Q}{\mathrm{d}P}$$

Example

Given the demand function

$$P = 50 - 2Q$$

find the elasticity when the price is 30. Is demand inelastic, unit elastic or elastic at this price?

Solution

To find dQ/dP we need to differentiate Q with respect to P. However, we are actually given a formula for P in terms of Q, so we need to transpose

$$P = 50 - 2Q$$

for Q. Adding $2Q$ to both sides gives

$$P + 2Q = 50$$

and if we subtract P then

$$2Q = 50 - P$$

Finally, dividing through by 2 gives

$$Q = 25 - {}^1\!/\!_2 P$$

Hence

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = -{}^1\!/\!_2$$

We are given that $P = 30$ so, at this price, demand is

$$Q = 25 - {}^1\!/\!_2(30) = 10$$

These values can now be substituted into

$$E = -\frac{P}{Q} \times \frac{\mathrm{d}Q}{\mathrm{d}P}$$

to get

$$E = -\frac{30}{10} \times \left(-\frac{1}{2}\right) = 1.5$$

Moreover, since $1.5 > 1$, demand is elastic at this price.

Practice Problem

2 Given the demand function

$$P = 100 - Q$$

calculate the price elasticity of demand when the price is

(a) 10 **(b)** 50 **(c)** 90

Is the demand inelastic, unit elastic or elastic at these prices?

It is quite common in economics to be given the demand function in the form

$$P = f(Q)$$

where P is a function of Q. In order to evaluate elasticity it is necessary to find

$$\frac{dQ}{dP}$$

which assumes that Q is actually given as a function of P. Consequently, we may have to transpose the demand equation and find an expression for Q in terms of P before we perform the differentiation. This was the approach taken in the previous example. Unfortunately, if $f(Q)$ is a complicated expression, it may be difficult, if not impossible, to carry out the initial rearrangement to extract Q. An alternative approach is based on the fact that

$$\frac{dQ}{dP} = \frac{1}{dP/dQ}$$

A proof of this can be obtained via the chain rule, although we omit the details. This result shows that we can find dQ/dP by just differentiating the original demand function to get dP/dQ and reciprocating.

Example

Given the demand function

$$P = -Q^2 - 4Q + 96$$

find the price elasticity of demand when $P = 51$. If this price rises by 2%, calculate the corresponding percentage change in demand.

Solution

We are given that $P = 51$, so to find the corresponding demand we need to solve the quadratic equation

$$-Q^2 - 4Q + 96 = 51$$

that is,

$$-Q^2 - 4Q + 45 = 0$$

To do this we use the standard formula

$$\frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

discussed in Section 2.1, which gives

$$Q = \frac{-(-4) \pm \sqrt{((-4)^2 - 4(-1)(45))}}{2(-1)}$$

$$= \frac{4 \pm \sqrt{196}}{-2}$$

$$= \frac{4 \pm 14}{-2}$$

The two solutions are −9 and 5. As usual, the negative value can be ignored, since it does not make sense to have a negative quantity, so $Q = 5$.

To find the value of E we also need to calculate

$$\frac{dQ}{dP}$$

from the demand equation, $P = -Q^2 - 4Q + 96$. It is not at all easy to transpose this for Q. Indeed, we would have to use the formula for solving a quadratic, as above, replacing the number 51 by the letter P. Unfortunately this expression involves square roots and the subsequent differentiation is quite messy. (You might like to have a go at this yourself!) However, it is easy to differentiate the given expression with respect to Q to get

$$\frac{dP}{dQ} = -2Q - 4$$

and so

$$\frac{dQ}{dP} = \frac{1}{dP/dQ} = \frac{1}{-2Q-4}$$

Finally, putting $Q = 5$ gives

$$\frac{dQ}{dP} = -\frac{1}{14}$$

The price elasticity of demand is given by

$$E = -\frac{P}{Q} \times \frac{dQ}{dP}$$

and if we substitute $P = 51$, $Q = 5$ and $dQ/dP = -1/14$ we get

$$E = -\frac{51}{5} \times \left(-\frac{1}{14}\right) = 0.73$$

To discover the effect on Q due to a 2% rise in P we return to the original definition

$$E = -\frac{\text{percentage change in demand}}{\text{percentage change in price}}$$

We know that $E = 0.73$ and that the percentage change in price is 2, so

$$0.73 = -\frac{\text{percentage change in demand}}{2}$$

which shows that demand changes by

$$-0.73 \times 2 = -1.46\%$$

A 2% rise in price therefore leads to a fall in demand of 1.46%.

Practice Problem

3 Given the demand equation

$$P = -Q^2 - 10Q + 150$$

find the price elasticity of demand when $Q = 4$. Estimate the percentage change in price needed to increase demand by 10%.