section 4.6

Optimization of economic functions

Objectives

At the end of this section you should be able to:

- Use the first-order derivative to find the stationary points of a function.
- Use the second-order derivative to classify the stationary points of a function.
- Find the maximum and minimum points of an economic function.
- Use stationary points to sketch graphs of economic functions.

In Section 2.1 a simple three-step strategy was described for sketching graphs of quadratic functions of the form

$$f(x) = ax^2 + bx + c$$

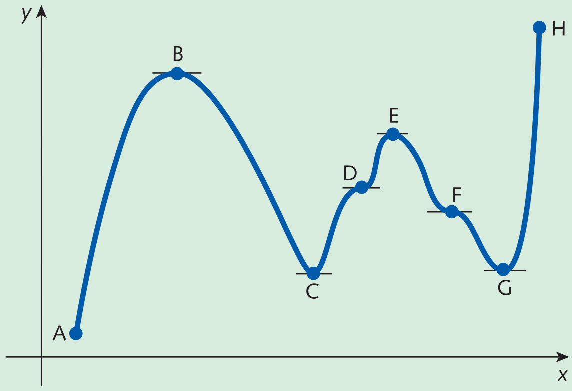
The basic idea is to solve the corresponding equation

$$ax^2 + bx + c = 0$$

to find where the graph crosses the x axis. Provided that the quadratic equation has at least one solution, it is then possible to deduce the coordinates of the maximum or minimum point of the parabola. For example, if there are two solutions, then by symmetry the graph turns round at the point exactly halfway between these solutions. Unfortunately, if the quadratic equation has no solution then only a limited sketch can be obtained using this approach.

In this section we show how the techniques of calculus can be used to find the coordinates of the turning point of a parabola. The beauty of this approach is that it can be used to locate the maximum and minimum points of any economic function, not just those represented by quadratics. Look at the graph in Figure 4.23 (overleaf). Points B, C, D, E, F and G are referred to as the ***stationary points*** (sometimes called ***critical points***, ***turning points*** or ***extrema***) of the function. At a stationary point the tangent to the graph is horizontal and so has zero slope.

Figure 4.23



Consequently, at a stationary point of a function $f(x)$,

$$f'(x) = 0$$

The reason for using the word 'stationary' is historical. Calculus was originally used by astronomers to predict planetary motion. If a graph of the distance travelled by an object is sketched against time then the speed of the object is given by the slope, since this represents the rate of change of distance with respect to time. It follows that if the graph is horizontal at some point then the speed is zero and the object is instantaneously at rest: that is, stationary.

Stationary points are classified into one of three types: local maxima, local minima and stationary points of inflection.

At a ***local maximum*** (sometimes called a ***relative maximum***) the graph falls away on both sides. Points B and E are the local maxima for the function sketched in Figure 4.23. The word 'local' is used to highlight the fact that, although these are the maximum points relative to their locality or neighbourhood, they may not be the overall or global maximum. In Figure 4.23 the highest point on the graph actually occurs at the right-hand end, H, which is not a stationary point, since the slope is not zero at H.

At a ***local minimum*** (sometimes called a ***relative minimum***) the graph rises on both sides. Points C and G are the local minima in Figure 4.23. Again, it is not necessary for the global minimum to be one of the local minima. In Figure 4.23 the lowest point on the graph occurs at the left-hand end, A, which is not a stationary point.

At a ***stationary point of inflection*** the graph rises on one side and falls on the other. The stationary points of inflection in Figure 4.23 are labelled D and F. These points are of little value in economics, although they do sometimes assist in sketching graphs of economic functions. Maxima and minima, on the other hand, are important. The calculation of the maximum points of the revenue and profit functions is clearly worthwhile. Likewise, it is useful to be able to find the minimum points of average cost functions.

For most examples in economics, the local maximum and minimum points coincide with the global maximum and minimum. For this reason we shall drop the word 'local' when describing stationary points. However, it should always be borne in mind that the global maximum and minimum could actually be attained at an end point and this possibility may need to be checked. This can be done by comparing the function values at the end points with those of the stationary points and then deciding which of them gives rise to the largest or smallest values.

Two obvious questions remain. How do we find the stationary points of any given function and how do we classify them? The first question is easily answered. As we mentioned earlier, stationary points satisfy the equation

$$f'(x) = 0$$

so all we need do is to differentiate the function, to equate to zero and to solve the resulting algebraic equation. The classification is equally straightforward. It can be shown that if a function has a stationary point at $x = a$ then

- if $f''(a) > 0$ then $f(x)$ has a minimum at $x = a$
- if $f''(a) < 0$ then $f(x)$ has a maximum at $x = a$

Therefore, all we need do is to differentiate the function a second time and to evaluate this second-order derivative at each point. A point is a minimum if this value is positive and a maximum if this value is negative. These facts are consistent with our interpretation of the second-order derivative in Section 4.2. If $f''(a) > 0$ the graph bends upwards at $x = a$ (points C and G in Figure 4.23). If $f''(a) < 0$ the graph bends downwards at $x = a$ (points B and E in Figure 4.23). There is, of course, a third possibility, namely $f''(a) = 0$. Sadly, when this happens it provides no information whatsoever about the stationary point. The point $x = a$ could be a maximum, minimum or inflection. This situation is illustrated in Practice Problem 7 at the end of this section.

Advice

If you are unlucky enough to encounter this case, you can always classify the point by tabulating the function values in the vicinity and use these to produce a local sketch.

To summarize, the method for finding and classifying stationary points of a function, $f(x)$, is as follows:

Step 1

Solve the equation $f'(x) = 0$ to find the stationary points, $x = a$.

Step 2

If

- $f''(a) > 0$ then the function has a minimum at $x = a$
- $f''(a) < 0$ then the function has a maximum at $x = a$
- $f''(a) = 0$ then the point cannot be classified using the available information

Example

Find and classify the stationary points of the following functions. Hence sketch their graphs.

(a) $f(x) = x^2 - 4x + 5$ **(b)** $f(x) = 2x^3 + 3x^2 - 12x + 4$

Solution

(a) In order to use steps 1 and 2 we need to find the first- and second-order derivatives of the function

$$f(x) = x^2 - 4x + 5$$

Differentiating once gives

$$f'(x) = 2x - 4$$

and differentiating a second time gives

$$f''(x) = 2$$

Step 1

The stationary points are the solutions of the equation

$$f'(x) = 0$$

so we need to solve

$$2x - 4 = 0$$

This is a linear equation so has just one solution. Adding 4 to both sides gives

$$2x = 4$$

and dividing through by 2 shows that the stationary point occurs at

$$x = 2$$

Step 2

To classify this point we need to evaluate

$$f''(2)$$

In this case

$$f''(x) = 2$$

for all values of x, so in particular

$$f''(2) = 2$$

This number is positive, so the function has a minimum at $x = 2$.

We have shown that the minimum point occurs at $x = 2$. The corresponding value of y is easily found by substituting this number into the function to get

$$y = (2)^2 - 4(2) + 5 = 1$$

so the minimum point has coordinates (2, 1). A graph of $f(x)$ is shown in Figure 4.24 (overleaf).

(**b**) In order to use steps 1 and 2 we need to find the first- and second-order derivatives of the function

$$f(x) = 2x^3 + 3x^2 - 12x + 4$$

Differentiating once gives

$$f'(x) = 6x^2 + 6x - 12$$

and differentiating a second time gives

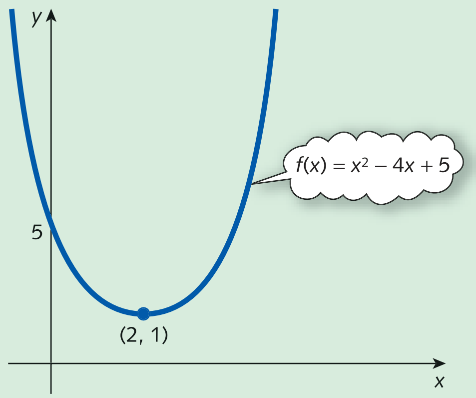
$$f''(x) = 12x + 6$$

Step 1

The stationary points are the solutions of the equation

$$f'(x) = 0$$

Figure 4.24



so we need to solve

$$6x^2 + 6x - 12 = 0$$

This is a quadratic equation and so can be solved using 'the formula'. However, before doing so, it is a good idea to divide both sides by 6 to avoid large numbers. The resulting equation

$$x^2 + x - 2 = 0$$

has solution

$$x = \frac{-1 \pm \sqrt{(1^2 - 4(1)(-2))}}{2(1)} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2} = -2, 1$$

In general, whenever $f(x)$ is a cubic function the stationary points are the solutions of a quadratic equation, $f'(x) = 0$. Moreover, we know from Section 2.1 that such an equation can have two, one or no solutions. It follows that a cubic equation can have two, one or no stationary points. In this particular example we have seen that there are two stationary points, at $x = -2$ and $x = 1$.

Step 2

To classify these points we need to evaluate $f''(-2)$ and $f''(1)$. Now

$$f''(-2) = 12(-2) + 6 = -18$$

This is negative, so there is a maximum at $x = -2$. When $x = -2$,

$$y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 4 = 24$$

so the maximum point has coordinates (–2, 24). Now

$$f''(1) = 12(1) + 6 = 18$$

This is positive, so there is a minimum at $x = 1$. When $x = 1$,

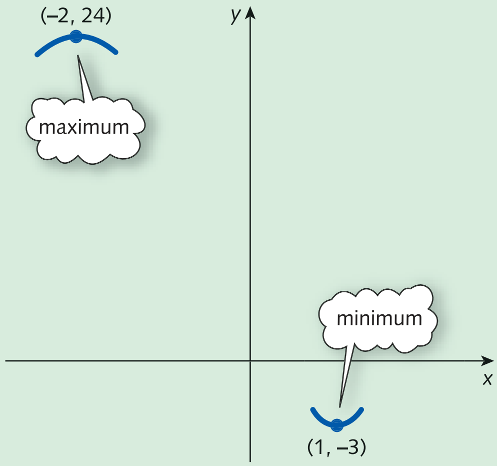
$$y = 2(1)^3 + 3(1)^2 - 12(1) + 4 = -3$$

so the minimum point has coordinates (1, –3).

This information enables a partial sketch to be drawn as shown in Figure 4.25. Before we can be confident about the complete picture it is useful to plot a few more points such as those below.

x	–10	0	10
y	–1816	4	2184

Figure 4.25



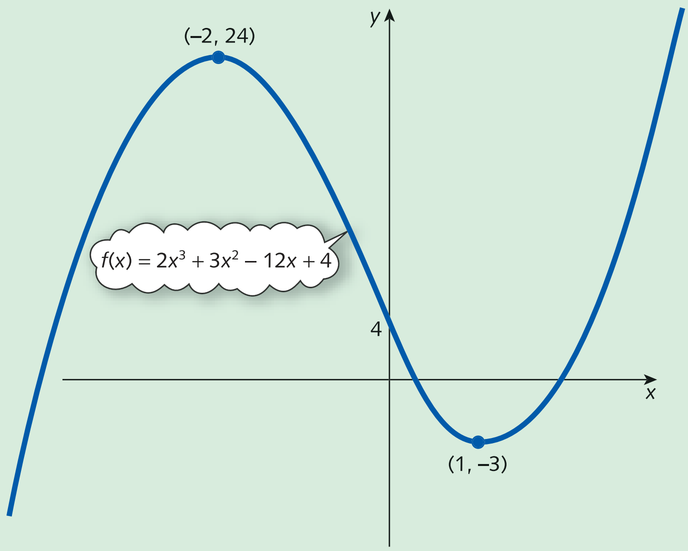
This table indicates that when x is positive the graph falls steeply downwards from a great height. Similarly, when x is negative the graph quickly disappears off the bottom of the page. The curve cannot wiggle and turn round except at the two stationary points already plotted (otherwise it would have more stationary points, which we know is not the case). We now have enough information to join up the pieces and so sketch a complete picture as shown in Figure 4.26.

In an ideal world it would be nice to calculate the three points at which the graph crosses the x axis. These are the solutions of

$$2x^3 + 3x^2 - 12x + 4 = 0$$

There is a formula for solving cubic equations, just as there is for quadratic equations, but it is extremely complicated and is beyond the scope of this book.

Figure 4.26



Practice Problem

1 Find and classify the stationary points of the following functions. Hence sketch their graphs.

(a) $y = 3x^2 + 12x - 35$ **(b)** $y = -2x^3 + 15x^2 - 36x + 27$

The task of finding the maximum and minimum values of a function is referred to as ***optimization***. This is an important topic in mathematical economics. It provides a rich source of examination questions and we devote the remaining part of this section and the whole of the next to applications of it. In this section we demonstrate the use of stationary points by working through four 'examination-type' problems in detail. These problems involve the optimization of specific revenue, cost, profit and production functions. They are not intended to exhaust all possibilities, although they are fairly typical. The next section describes how the mathematics of optimization can be used to derive general theoretical results.

Example

A firm's short-run production function is given by

$$Q = 6L^2 - 0.2L^3$$

where L denotes the number of workers.

(a) Find the size of the workforce that maximizes output and hence sketch a graph of this production function.

(b) Find the size of the workforce that maximizes the average product of labour. Calculate MP_L and AP_L at this value of L. What do you observe?

Solution

(a) In the first part of this example we want to find the value of L which maximizes

$$Q = 6L^2 - 0.2L^3$$

Step 1

At a stationary point

$$\frac{dQ}{dL} = 12L - 0.6L^2 = 0$$

This is a quadratic equation and so we could use 'the formula' to find L. However, this is not really necessary in this case because both terms have a common factor of L and the equation may be written as

$$L(12 - 0.6L) = 0$$

It follows that either

$$L = 0 \quad \text{or} \quad 12 - 0.6L = 0$$

that is, the equation has solutions

$$L = 0 \quad \text{and} \quad L = 12/0.6 = 20$$

Step 2

It is obvious on economic grounds that $L = 0$ is a minimum and presumably $L = 20$ is the maximum. We can, of course, check this by differentiating a second time to get

$$\frac{d^2Q}{dL^2} = 12 - 1.2L$$

When $L = 0$,

$$\frac{d^2Q}{dL^2} = 12 > 0$$

which confirms that $L = 0$ is a minimum. The corresponding output is given by

$$Q = 6(0)^2 - 0.2(0)^3 = 0$$

as expected. When $L = 20$,

$$\frac{d^2Q}{dL^2} = -12 < 0$$

which confirms that $L = 20$ is a maximum.

The firm should therefore employ 20 workers to achieve a maximum output

$$Q = 6(20)^2 - 0.2(20)^3 = 800$$

We have shown that the minimum point on the graph has coordinates (0, 0) and the maximum point has coordinates (20, 800). There are no further turning points, so the graph of the production function has the shape sketched in Figure 4.27.

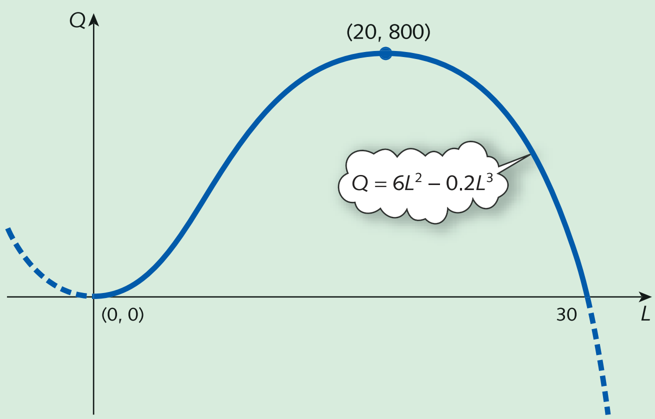
It is possible to find the precise values of L at which the graph crosses the horizontal axis. The production function is given by

$$Q = 6L^2 - 0.2L^3$$

so we need to solve

$$6L^2 - 0.2L^3 = 0$$

Figure 4.27



We can take out a factor of L^2 to get

$$L^2(6 - 0.2L) = 0$$

Hence, either

$$L^2 = 0 \quad \text{or} \quad 6 - 0.2L = 0$$

The first of these merely confirms the fact that the curve passes through the origin, whereas the second shows that the curve intersects the L axis at $L = 6/0.2 = 30$.

(b) In the second part of this example we want to find the value of L which maximizes the average product of labour. This is a concept that we have not met before in this book, although it is not difficult to guess how it might be defined.

The ***average product of labour***, AP_L, is taken to be total output divided by labour, so that in symbols

$$AP_L = \frac{Q}{L}$$

This is sometimes called ***labour productivity***, since it measures the average output per worker.

In this example,

$$AP_L = \frac{6L^2 - 0.2L^3}{L} = 6L - 0.2L^2$$

Step 1

At a stationary point

$$\frac{d(AP_L)}{dL} = 0$$

so

$$6 - 0.4L = 0$$

which has solution $L = 6/0.4 = 15$.

Step 2

To classify this stationary point we differentiate a second time to get

$$\frac{d^2(AP_L)}{dL^2} = -0.4 < 0$$

which shows that it is a maximum.

The labour productivity is therefore greatest when the firm employs 15 workers. In fact, the corresponding labour productivity, AP_L, is

$$6(15) - 0.2(15)^2 = 45$$

In other words, the largest number of goods produced per worker is 45.

Finally, we are invited to calculate the value of MP_L at this point. To find an expression for MP_L we need to differentiate Q with respect to L to get

$$MP_L = 12L - 0.6L^2$$

When $L = 15$,

$$MP_L = 12(15) - 0.6(15)^2 = 45$$

We observe that at $L = 15$ the values of MP_L and AP_L are equal.

In this particular example we discovered that at the point of maximum average product of labour

marginal product of labour	=	average product of labour

There is nothing special about this example and in the next section we show that this result holds for any production function.

Practice Problem

2 A firm's short-run production function is given by

$$Q = 300L^2 - L^4$$

where L denotes the number of workers. Find the size of the workforce that maximizes the average product of labour and verify that at this value of L

$$\text{MP}_L = \text{AP}_L$$

Example

The demand equation of a good is

$$P + Q = 30$$

and the total cost function is

$$\text{TC} = \tfrac{1}{2}Q^2 + 6Q + 7$$

(a) Find the level of output that maximizes total revenue.

(b) Find the level of output that maximizes profit. Calculate MR and MC at this value of Q. What do you observe?

Solution

(a) In the first part of this example we want to find the value of Q which maximizes total revenue. To do this we use the given demand equation to find an expression for TR and then apply the theory of stationary points in the usual way.

The total revenue is defined by

$$\text{TR} = PQ$$

We seek the value of Q which maximizes TR, so we express TR in terms of the variable Q only. The demand equation

$$P + Q = 30$$

can be rearranged to get

$$P = 30 - Q$$

Hence

$$\begin{aligned}\text{TR} &= (30 - Q)Q \\ &= 30Q - Q^2\end{aligned}$$

Step 1

At a stationary point

$$\frac{d(TR)}{dQ} = 0$$

so

$$30 - 2Q = 0$$

which has solution $Q = 30/2 = 15$.

Step 2

To classify this point we differentiate a second time to get

$$\frac{d^2(TR)}{dQ^2} = -2$$

This is negative, so TR has a maximum at $Q = 15$.

(b) In the second part of this example we want to find the value of Q which maximizes profit. To do this we begin by determining an expression for profit in terms of Q. Once this has been done, it is then a simple matter to work out the first- and second-order derivatives and so to find and classify the stationary points of the profit function.

The profit function is defined by

$$\pi = TR - TC$$

From part (a)

$$TR = 30Q - Q^2$$

We are given the total cost function

$$TC = \tfrac{1}{2}Q^2 + 6Q + 7$$

Hence

$$\begin{aligned}\pi &= (30Q - Q^2) - (\tfrac{1}{2}Q^2 + 6Q + 7)\\ &= 30Q - Q^2 - \tfrac{1}{2}Q^2 - 6Q - 7\\ &= -\tfrac{3}{2}Q^2 + 24Q - 7\end{aligned}$$

Step 1

At a stationary point

$$\frac{d\pi}{dQ} = 0$$

so

$$-3Q + 24 = 0$$

which has solution $Q = 24/3 = 8$.

Step 2

To classify this point we differentiate a second time to get

$$\frac{d^2\pi}{dQ^2} = -3$$

This is negative, so π has a maximum at $Q = 8$. In fact, the corresponding maximum profit is

$$\pi = -\tfrac{3}{2}(8)^2 + 24(8) - 7 = 89$$

Finally, we are invited to calculate the marginal revenue and marginal cost at this particular value of Q. To find expressions for MR and MC we need only differentiate TR and TC, respectively. If

$$\text{TR} = 30Q - Q^2$$

then

$$\begin{aligned}\text{MR} &= \frac{\text{d(TR)}}{\text{d}Q}\\ &= 30 - 2Q\end{aligned}$$

so when $Q = 8$

$$\text{MR} = 30 - 2(8) = 14$$

If

$$\text{TC} = \tfrac{1}{2}Q^2 + 6Q + 7$$

then

$$\begin{aligned}\text{MC} &= \frac{\text{d(TC)}}{\text{d}Q}\\ &= Q + 6\end{aligned}$$

so when $Q = 8$

$$\text{MC} = 8 + 6 = 14$$

We observe that at $Q = 8$, the values of MR and MC are equal.

In this particular example we discovered that at the point of maximum profit,

$$\boxed{\text{marginal revenue}} = \boxed{\text{marginal cost}}$$

There is nothing special about this example and in the next section we show that this result holds for any profit function.

Practice Problem

3 The demand equation of a good is given by

$$P + 2Q = 20$$

and the total cost function is

$$Q^3 - 8Q^2 + 20Q + 2$$

(a) Find the level of output that maximizes total revenue.

(b) Find the maximum profit and the value of Q at which it is achieved. Verify that, at this value of Q, MR = MC.