

This is negative, so  $\pi$  has a maximum at  $Q = 8$ . In fact, the corresponding maximum profit is

$$\pi = -\frac{3}{2}(8)^2 + 24(8) - 7 = 89$$

Finally, we are invited to calculate the marginal revenue and marginal cost at this particular value of  $Q$ . To find expressions for MR and MC we need only differentiate TR and TC, respectively. If

$$TR = 30Q - Q^2$$

then

$$\begin{aligned} MR &= \frac{d(TR)}{dQ} \\ &= 30 - 2Q \end{aligned}$$

so when  $Q = 8$

$$MR = 30 - 2(8) = 14$$

If

$$TC = \frac{1}{2}Q^2 + 6Q + 7$$

then

$$\begin{aligned} MC &= \frac{d(TC)}{dQ} \\ &= Q + 6 \end{aligned}$$

so when  $Q = 8$

$$MC = 8 + 6 = 14$$

We observe that at  $Q = 8$ , the values of MR and MC are equal.

In this particular example we discovered that at the point of maximum profit,

$$\boxed{\begin{array}{c} \text{marginal} \\ \text{revenue} \end{array}} = \boxed{\begin{array}{c} \text{marginal} \\ \text{cost} \end{array}}$$

There is nothing special about this example and in the next section we show that this result holds for any profit function.

### Practice Problem

- 3** The demand equation of a good is given by

$$P + 2Q = 20$$

and the total cost function is

$$Q^3 - 8Q^2 + 20Q + 2$$

- (a) Find the level of output that maximizes total revenue.  
 (b) Find the maximum profit and the value of  $Q$  at which it is achieved. Verify that, at this value of  $Q$ ,  $MR = MC$ .

### Example

The cost of building an office block,  $x$  floors high, is made up of three components:

- (1) \$10 million for the land
- (2)  $\frac{1}{4}$  million per floor
- (3) specialized costs of  $\$10\,000x$  per floor.

How many floors should the block contain if the average cost per floor is to be minimized?

### Solution

The \$10 million for the land is a fixed cost because it is independent of the number of floors. Each floor costs  $\frac{1}{4}$  million, so if the building has  $x$  floors altogether then the cost will be  $250\,000x$ .

In addition there are specialized costs of  $10\,000x$  per floor, so if there are  $x$  floors this will be

$$(10\,000x)x = 10\,000x^2$$

Notice the square term here, which means that the specialized costs rise dramatically with increasing  $x$ . This is to be expected, since a tall building requires a more complicated design. It may also be necessary to use more expensive materials.



The total cost, TC, is the sum of the three components: that is,

$$TC = 10\,000\,000 + 250\,000x + 10\,000x^2$$

The average cost per floor, AC, is found by dividing the total cost by the number of floors: that is,

$$\begin{aligned} AC &= \frac{TC}{x} \\ &= \frac{10\,000\,000 + 250\,000x + 10\,000x^2}{x} \\ &= \frac{10\,000\,000}{x} + 250\,000 + 10\,000x \\ &= 10\,000\,000x^{-1} + 250\,000 + 10\,000x \end{aligned}$$

### Step 1

At a stationary point

$$\frac{d(AC)}{dx} = 0$$

In this case

$$\frac{d(AC)}{dx} = -10\,000\,000x^{-2} + 10\,000 = \frac{-10\,000\,000}{x^2} + 10\,000$$

so we need to solve

$$10\,000 = \frac{10\,000\,000}{x^2} \text{ or equivalently } 10\,000x^2 = 10\,000\,000$$

Hence

$$x^2 = \frac{10\,000\,000}{10\,000} = 1000$$

This has solution

$$x = \pm\sqrt{1000} = \pm31.6$$

We can obviously ignore the negative value because it does not make sense to build an office block with a negative number of floors, so we can deduce that  $x = 31.6$ .

### Step 2

To confirm that this is a minimum we need to differentiate a second time. Now

$$\frac{d(AC)}{dx} = -10\,000\,000x^{-2} + 10\,000$$

so

$$\frac{d^2(AC)}{dx^2} = -2(-10\,000\,000)x^{-3} = \frac{20\,000\,000}{x^3}$$

When  $x = 31.6$  we see that

$$\frac{d^2(AC)}{dx^2} = \frac{20\,000\,000}{(31.6)^3} = 633.8$$

It follows that  $x = 31.6$  is indeed a minimum because the second-order derivative is a positive number. 

At this stage it is tempting to state that the answer is 31.6. This is mathematically correct but is a physical impossibility since  $x$  must be a whole number. To decide whether to take  $x$  to be 31 or 32 we simply evaluate AC for these two values of  $x$  and choose the one that produces the lower average cost.

When  $x = 31$ ,

$$AC = \frac{10\,000\,000}{31} + 250\,000 + 10\,000(31) = \$882\,581$$

When  $x = 32$ ,

$$AC = \frac{10\,000\,000}{32} + 250\,000 + 10\,000(32) = \$882\,500$$

Therefore an office block 32 floors high produces the lowest average cost per floor.

### Practice Problem

4 The total cost function of a good is given by

$$TC = Q^2 + 3Q + 36$$

Calculate the level of output that minimizes average cost. Find AC and MC at this value of  $Q$ . What do you observe?

### Example

The supply and demand equations of a good are given by

$$P = Q_s + 8$$

and

$$P = -3Q_d + 80$$

respectively.

The government decides to impose a tax,  $t$ , per unit. Find the value of  $t$  which maximizes the government's total tax revenue on the assumption that equilibrium conditions prevail in the market.

### Solution

The idea of taxation was first introduced in Chapter 1. In Section 1.3 the equilibrium price and quantity were calculated from a given value of  $t$ . In this example  $t$  is unknown but the analysis is exactly the same. All we need to do is to carry the letter  $t$  through the usual calculations and then to choose  $t$  at the end so as to maximize the total tax revenue.

To take account of the tax we replace  $P$  by  $P - t$  in the supply equation. This is because the price that the supplier actually receives is the price,  $P$ , that the consumer pays less the tax,  $t$ , deducted by the government. The new supply equation is then

$$P - t = Q_s + 8$$

so that

$$P = Q_s + 8 + t$$

In equilibrium

$$Q_s = Q_d$$

If this common value is denoted by  $Q$  then the supply and demand equations become

$$P = Q + 8 + t$$

$$P = -3Q + 80$$

Hence

$$Q + 8 + t = -3Q + 80$$

since both sides are equal to  $P$ . This can be rearranged to give

$$Q = -3Q + 72 - t \quad (\text{subtract } 8 + t \text{ from both sides})$$

$$4Q = 72 - t \quad (\text{add } 3Q \text{ to both sides})$$

$$Q = 18 - \frac{1}{4}t \quad (\text{divide both sides by } 4)$$

Now, if the number of goods sold is  $Q$  and the government raises  $t$  per good then the total tax revenue,  $T$ , is given by

$$\begin{aligned} T &= tQ \\ &= t(18 - \frac{1}{4}t) \\ &= 18t - \frac{1}{4}t^2 \end{aligned}$$

This then is the expression that we wish to maximize.

### Step 1

At a stationary point

$$\frac{dT}{dt} = 0$$

so

$$18 - \frac{1}{2}t = 0$$

which has solution

$$t = 36$$

### Step 2

To classify this point we differentiate a second time to get

$$\frac{d^2T}{dt^2} = \frac{1}{2} < 0$$

which confirms that it is a maximum.

Hence the government should impose a tax of \$36 on each good.

### Practice Problem

- 5 The supply and demand equations of a good are given by

$$P = \frac{1}{2}Q_S + 25$$

and

$$P = -2Q_D + 50$$

respectively.

The government decides to impose a tax,  $t$ , per unit. Find the value of  $t$  which maximizes the government's total tax revenue on the assumption that equilibrium conditions prevail in the market.

We conclude this section by describing the use of a computer package to solve optimization problems. Although a spreadsheet could be used to do this, by tabulating the values of a function, it cannot handle the associated mathematics. A symbolic computation system such as Maple, Matlab, Mathcad or Derive can not only sketch the graphs of functions, but also differentiate and solve algebraic equations. Consequently, it is possible to obtain the exact solution using one of these packages. In this book we have chosen to use Maple.

### Advice

A simple introduction to this package is described in the Getting Started section at the very beginning of this book. If you have not used Maple before, go back and read through this section now.

The following example makes use of three basic Maple instructions: `plot`, `diff` and `solve`. As the name suggests, `plot` produces a graph of a function by joining together points which are accurately plotted over a specified range of values. The instruction `diff`, not surprisingly, differentiates a given expression with respect to any stated variable, and `solve` finds the exact solution of an equation.

### Example

MAPLE

The price,  $P$ , of a good varies over time,  $t$ , during a 15-year period according to

$$P = 0.064t^3 - 1.44t^2 + 9.6t + 10 \quad (0 \leq t \leq 15)$$

- Sketch a graph of this function and use it to estimate the local maximum and minimum points.
- Find the exact coordinates of these points using calculus.

### Solution

It is convenient to give the cubic expression the name `price`, and to do this in Maple, we type

```
>price:=0.064*t^3-1.44*t^2+9.6*t+10;
```

- (a) To plot a graph of this function for values of  $t$  between 0 and 15 we type

```
>plot(price, t=0..15);
```

Maple responds by producing a graph of price over the specified range (see Figure 4.28). The graph shows that there is one local maximum and one local minimum. (It also shows very clearly that the overall, or global, minimum and maximum occur at the ends, 0 and 15 respectively.) If you now move the cursor to some point on the plot and click, you will discover that two things happen. You will first notice that the graph is now surrounded by a box. More significantly, if you look carefully at the top of the screen, you will see that a graphics toolbar has appeared. In the left-hand corner of this is a small window containing the coordinates of the position of the cursor. To estimate the local maximum and minimum all you need do is to move the cursor to the relevant points, click, and read off the answer from the screen. Looking carefully at Figure 4.28, in which the cursor is positioned over the local maximum, we see that the coordinates of this point are approximately (5.02, 30.01). A similar estimate could be found for the local minimum point.

- (b) To find the exact coordinates we need to use calculus. The simple instruction

```
>diff(price, t);
```

will produce the first derivative of price with respect to  $t$ . However, since we want to equate this to zero and solve the associated equation, it makes sense to give this a name. You can use whatever combination of symbols you like for a name in Maple, provided it does not begin with a number and it has not already been reserved by Maple. So, you are not allowed to use `1deriv`, say (because it starts with the digit 1), or `subs` (which Maple recognizes as one of its own in-house instructions for substituting numbers for letters in an expression). If we choose to call it `deriv1` we type:

```
>deriv1:=diff(price, t);
```

and Maple responds with

```
deriv1:= .192t2-2.88t+9.6
```

To find the stationary points, we need to equate this to zero and solve for  $t$ . This is achieved in Maple by typing:

```
>solve(deriv1=0, t);
```

and Maple responds with:

```
5. , 10.
```

These are the values of  $t$  at the stationary points. It is clear from the graph in Figure 4.28 that  $t = 5$  is a local maximum and  $t = 10$  is a local minimum. To find the price at the maximum we substitute  $t = 5$  into the expression for price, so we type:

```
>subs(t=5, price);
```

and Maple responds with

```
30.000
```

To find the price at the local minimum we edit the instruction to create

```
>subs(t=10, price);
```

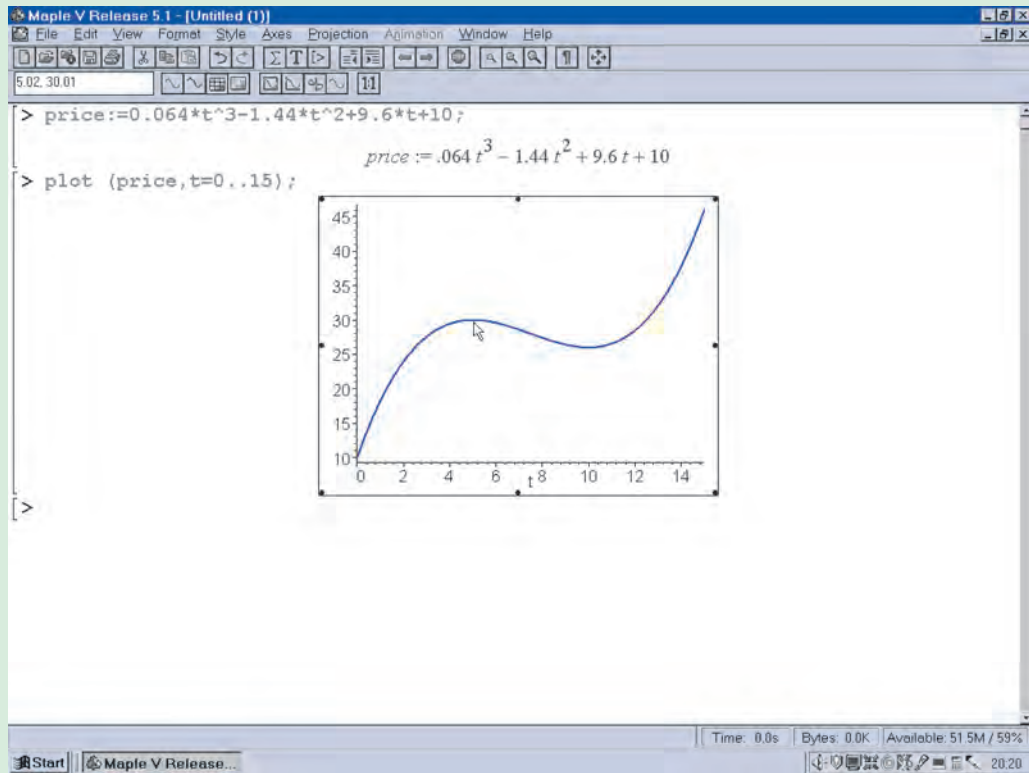
and Maple responds with

```
26.000
```

The local maximum and minimum have coordinates (5, 30) and (10, 26) respectively.



Figure 4.28



## Key Terms

**Average product of labour (or labour productivity)** Output per worker:  $AP_L = Q/L$ .

**Local (or relative) maximum** A point on a curve which has the highest function value in comparison with other values in its neighbourhood; at such a point the first-order derivative is zero and the second-order derivative is either zero or negative.

**Local (or relative) minimum** A point on a curve which has the lowest function value in comparison with other values in its neighbourhood; at such a point the first-order derivative is zero and the second-order derivative is either zero or positive.

**Optimization** The determination of the optimal (usually stationary) points of a function.

**Stationary points (critical points, turning points, extrema)** Points on a graph at which the tangent is horizontal; at a stationary point the first-order derivative is zero.

**Stationary point of inflection** A stationary point that is neither a maximum nor a minimum; at such a point both the first- and second-order derivatives are zero.



## Practice Problems

- 6 Find and classify the stationary points of the following functions. Hence give a rough sketch of their graphs.

(a)  $y = -x^2 + x + 1$

(b)  $y = x^2 - 4x + 4$

(c)  $y = x^2 - 20x + 105$

(d)  $y = -x^3 + 3x$

- 7 Show that all of the following functions have a stationary point at  $x = 0$ . Verify in each case that  $f''(0) = 0$ . Classify these points by producing a rough sketch of each function.

(a)  $f(x) = x^3$

(b)  $f(x) = x^4$

(c)  $f(x) = -x^6$

- 8 If the demand equation of a good is

$$P = 40 - 2Q$$

find the level of output that maximizes total revenue.

- 9 If fixed costs are 15 and the variable costs are  $2Q$  per unit, write down expressions for TC, AC and MC. Find the value of  $Q$  which minimizes AC and verify that  $AC = MC$  at this point.

- 10 A firm's short-run production function is given by

$$Q = 30L^2 - 0.5L^3$$

Find the value of  $L$  which maximizes  $AP_L$  and verify that  $MP_L = AP_L$  at this point.

- 11 If the fixed costs are 13 and the variable costs are  $Q + 2$  per unit, show that the average cost function is

$$AC = \frac{13}{Q} + Q + 2$$

- (a) Calculate the values of AC when  $Q = 1, 2, 3, \dots, 6$ . Plot these points on graph paper and hence produce an accurate graph of AC against  $Q$ .
- (b) Use your graph to estimate the minimum average cost.
- (c) Use differentiation to confirm your estimate obtained in part (b).
- 12 An electronic components firm launches a new product on 1 January. During the following year a rough estimate of the number of orders,  $S$ , received  $t$  days after the launch is given by

$$S = t^2 - 0.002t^3$$

- (a) What is the maximum number of orders received on any one day of the year?
- (b) After how many days does the firm experience the greatest increase in orders?
- 13 If the demand equation of a good is

$$P = \sqrt[3]{(1000 - 4Q)}$$

find the value of  $Q$  which maximizes total revenue.

- 14 The demand and total cost functions of a good are

$$4P + Q - 16 = 0$$

and

$$TC = 4 + 2Q - \frac{3Q^2}{10} + \frac{Q^3}{20}$$

respectively.



- (a) Find expressions for TR,  $\pi$ , MR and MC in terms of Q.  
 (b) Solve the equation

$$\frac{d\pi}{dQ} = 0$$

and hence determine the value of Q which maximizes profit.

- (c) Verify that, at the point of maximum profit,

$$MR = MC$$

- 15** The supply and demand equations of a good are given by

$$3P - Q_s = 3$$

and

$$2P + Q_d = 14$$

respectively.

The government decides to impose a tax,  $t$ , per unit. Find the value of  $t$  (in dollars) which maximizes the government's total tax revenue on the assumption that equilibrium conditions prevail in the market.

- 16 (Maple)** Plot a graph of each of the following functions over the specified range of values and use these graphs to estimate the coordinates of all of the stationary points. Use calculus to find the exact coordinates of these points.

(a)  $y = 3x^4 - 28x^3 + 84x^2 - 96x + 30 \quad (0 \leq x \leq 5)$

(b)  $y = x^4 - 8x^3 + 18x^2 - 10 \quad (-1 \leq x \leq 4)$

(c)  $y = \frac{x}{x^2 + 1} \quad (-4 \leq x \leq 4)$

- 17 (Maple)**

(a) Attempt to use Maple to plot a graph of the function  $y = 1/x$  over the range  $-4 \leq x \leq 4$ . What difficulty do you encounter? Explain briefly why this has occurred for this particular function.

(b) One way of avoiding the difficulty in part (a) is to restrict the range of the  $y$  values. Produce a plot by typing

```
plot(1/x, x=-4..4, y=-3..3);
```

(c) Use the approach suggested in part (b) to plot a graph of the curve

$$y = \frac{x - 3}{(x + 1)(x - 2)}$$

on the interval  $-2 \leq x \leq 6$ . Use calculus to find all of the stationary points.

- 18 (Maple)** The total cost, TC, and total revenue, TR, functions of a good are given by

$$TC = 80Q - \frac{15}{2}Q^2 + \frac{1}{3}Q^3 \quad \text{and} \quad TR = 50Q - Q^2$$

Obtain Maple expressions for  $\pi$ , MC and MR, naming them profit, MC and MR respectively. Plot all three functions on the same diagram using the instruction:

```
plot({profit, MC, MR}, Q=0..14);
```

Use this diagram to show that

(a) when the profit is a minimum,  $MR = MC$  and the  $MC$  curve cuts the  $MR$  curve from above

(b) when the profit is a maximum,  $MR = MC$  and the  $MC$  curve cuts the  $MR$  curve from below.

**19 (Maple)** A firm's short-run production function is given by

$$Q = 300L^{0.8}(240 - 5L)^{0.5} \quad (0 \leq L \leq 48)$$

where  $L$  is the size of the workforce. Plot a graph of this function and hence estimate the level of employment needed to maximize output. Confirm this by using differentiation.