### 2.1 Lenses

The lens is the most widely used optical device. A lens is a refracting device that reconfigures a transmitted energy distribution. That is true whether we are dealing with UV, lightwaves, IR, microwaves, radiowaves, or even sound waves. It's frequently necessary to collect incoming parallel rays and bring them together at a point, thereby focusing the energy, as is done with a burning-glass or a telescope lens.

### 2.1.1 Aspherical Surfaces

To see how a lens works, imagine that a transparent substance interpose in the path of a wave in which the wave's speed is different than it was initially. Figure 2.1a presents a cross-sectional view of a diverging spherical wave traveling in an incident medium of index $n_{i}$ impinging on the curved interface of a transmitting medium of index $n_{t}$. When $n_{t}$ is greater than $n_{i}$, the central area of the wavefront travels more slowly than its outer extremities, which are still moving quickly through the incident medium. These extremities overtake the mid region, continuously flattening the wavefront. If the interface is properly configured, the spherical wavefront bends into a plane wave. The alternative ray representation is shown in Fig. 2.1b; the rays simply bend toward the local normal upon entering the more dense medium, and if the surface configuration is just right, the rays emerge parallel. To find the required shape of the interface, refer to Fig. 2.1c, wherein point-A can lie anywhere on the boundary. A little spherical surface of constant phase emitted from $S$ must evolve into a flat surface of constant phase at $\overline{D D}$. Whatever path the light takes from $S$ to $\overline{D D}$, it must always be the same number of wavelengths long, so that the disturbance begins and ends in-phase.


Figure 2. 1.A hyperbolic interface between air and glass. (a) The wavefronts bend and straighten out. (b) The rays become parallel. (c) The hyperbola is such that the optical path from $S$ to $A$ to $D$ is the same no matter where $A$ is.

Radiant energy leaving $S$ as a single wavefront must arrive at the plane $\overline{D D}$, having traveled for the same amount of time to get there, no matter what the actual route taken by any particular ray. In other words, , $\overline{F_{1} A} / \lambda_{i}$ (the number of wavelengths along the arbitrary ray from $F_{1}$ to $A$ plus $\overline{A D} / \lambda_{t}$ (the number of wavelengths along the ray from $A$ to $D$ ) must be constant regardless of where on then interface $A$ happens to be. Now, adding these and multiplying by $\lambda_{0}$, yields

$$
\begin{equation*}
n_{i}\left(\overline{F_{1} A}\right)+n_{t}(\overline{A D})=\mathrm{constant} \tag{2.1}
\end{equation*}
$$

Each term on the left is the length traveled in a medium multiplied by the index of that medium, and, of course, each represents the optical path length, $O P L$, traversed. The optical path lengths from $S$ to $\overline{D D}$, are all equal. If Eq. (2.1) is divided by $c$, the first term becomes the time it takes light to travel from $S$ to $A$ and the second term, the time from $A$ to $D$; the right side remains constant (not the same constant, but constant). Equation (2.1) is equivalent to saying that all paths from $S$ to $\overline{D \dot{D}}$ must take the same amount of time to traverse. Let's return to finding the shape of the interface. Divide Eq. (2.1) by $n_{i}$, and it becomes

$$
\begin{equation*}
\overline{F_{1} A}+\left(\frac{n_{t}}{n_{i}}\right) \overline{A D}=\text { constant } \tag{2.2}
\end{equation*}
$$

This is the equation of a hyperbola in which the eccentricity (e), which measures the bending of the curve, is given by $\left(n_{t} / n_{i}>1\right)$; that is, $e=n_{t i}>1$. The greater the eccentricity, the flatter the hyperbola (the larger the difference in the indices, the less the surface need be curved).

## Example 2.1

Use the figure to show that if a point source is placed at the focus F1 of the ellipsoid, plane waves will emerge from the far side. Remember that the defining requirement for an ellipse is that the net distance from one focus to the curve and back to the other focus is constant.


The $O P L$ from F1 to D on $\Sigma$ must be constant:
$n_{2}\left(\overline{F_{1} A}\right)+n_{1}(\overline{A D})=C$ and $\left(\overline{F_{1} A}\right)+(\overline{A D}) n_{12}=C / n_{2}=\dot{C}$
if $\Sigma$ corresponding to the directrix of ellipse,
$\left(\overline{F_{2} A}\right)=e(\overline{A D})$ where $e$ is the eccentricity;
if $n_{12}=e$
we get $\left(\overline{F_{1} A}\right)+\left(\overline{F_{2} A}\right)=C ́$
One of the first people to suggest using conic sections as surfaces for lenses and mirrors was Johann Kepler (1611), but he wasn't able to go very far with the idea without Snell's Law. Once that relationship was discovered, Descartes (1637), using his invention of Analytic Geometry, could develop the theoretical foundations of the optics of aspherical surfaces.

In Fig. 2.2a diverging incident spherical waves are made into plane waves at the first interface. These plane waves within the lens strike the back face perpendicularly and emerge unaltered: $\theta_{i}=0$ and $\theta_{t}=0$. Because the rays are reversible, plane waves incoming from the right will converge to point-F1, which is known as the focal point of the lens. Exposed on its flat face to the parallel rays from the Sun, our rather sophisticated lens would serve nicely as a burning-glass. In Fig. 2.2b, the plane waves within the lens are made to converge toward the axis by bending at the second interface. Both of these lenses are thicker at their midpoints than at their edges and are therefore said to be convex (from the Latin convexus, meaning arched). Each lens causes the incoming beam to converge somewhat, to bend a bit more toward the central axis; therefore, they are referred to as converging lenses. In contrast, a concave lens (from the Latin concavus, meaning hollow-and most easily remembered because it contains the word cave) is thinner in the middle than at the edges, as is evident in Fig. 2.2c. It causes the rays that enter as a parallel bundle to diverge. All such devices that turn rays outward away from the central axis (and in so doing add divergence to the beam) are called diverging lenses. In Fig. 2.2c, parallel rays enter from the left and, on emerging, seem to diverge from $F_{2}$; still, that point is taken as a focal point. When a parallel bundle of rays passes through a converging lens, the point to which it converges (or when passing through a diverging lens, the point from which it diverges) is a focal point of the lens.

(d)


Figure 2. 2.(a), (b), and (c) Several hyperbolic lenses seen in cross section. (d) A selection of aspherical lenses.

If a point source is positioned on the central or optical axis at the point- $F_{1}$ in front of the lens in Fig. 2.2b, rays will converge to the conjugate point- $F_{2}$. the conjugate point- $F_{2}$. A luminous image of the source would appear on a screen placed at $F_{2}$, an image that is therefore said to be real. On the other hand, in Fig. 2.2c the point source is at infinity, and the rays emerging from the system this time are diverging. They appear to come from a point- $F_{2}$, but no actual luminous image would appear on a screen at that location. The image here is spoken of as virtual, as is the familiar image generated by a plane mirror.

### 2.1.2 Refraction at Spherical Surfaces

The vast majority of quality lenses in use today have surfaces that are segments of spheres. Our intent here is to establish techniques for using such surfaces to simultaneously image a great many object points in light composed of a broad range of frequencies. Image errors, known as aberrations, will occur, but it is possible with the present technology to construct high-quality spherical lens systems whose aberrations are so well controlled that image fidelity is limited only by diffraction.

Figure 2.3 depicts a wave from the point source $S$ impinging on a spherical interface of radius $R$ Rcentered at $C$. The point- $V$ is called the vertex of the surface. The length $s_{0}=\overline{S V}$ is known as the object distance. The ray $\overline{S A}$ will be refracted at the interface toward the local normal ( $n_{2}>n_{1}$ ) and therefore toward the central or optical axis. Assume that at some point- $P$ the ray will cross the axis, as will all other rays incident at the same angle $\theta_{i}$ (Fig. 2.4). The length $s_{i}=\overline{V P}$ is the image distance. Fermat's Principle maintains that the optical path length OPL will be stationary; that is, its derivative with respect to the position variable will be zero. For the ray in question,

$$
\begin{equation*}
O P L=n_{1} l_{0}+n_{2} l_{i} \tag{2.3}
\end{equation*}
$$

Using the law of cosines in triangles $S A C$ and $A C P$ along with the fact that

$$
\begin{aligned}
& \cos \varphi=-\cos (180-\varphi) \text {, we get } \\
& \qquad l_{0}=\left[R^{2}+\left(s_{0}+R\right)^{2}-2 R\left(s_{0}+R\right) \cos \varphi\right]^{1 / 2}
\end{aligned}
$$

and

$$
l_{i}=\left[R^{2}+\left(s_{i}+R\right)^{2}-2 R\left(s_{i}+R\right) \cos \varphi\right]^{1 / 2}
$$

The OPL can be rewritten as
OPL $=n_{1}\left[R^{2}+\left(s_{0}+R\right)^{2}-2 R\left(s_{0}+R\right) \cos \varphi\right]^{1 / 2}+n_{2}\left[R^{2}+\left(s_{i}+R\right)^{2}-2 R\left(s_{i}+R\right) \cos \varphi\right]^{1 / 2}$


Figure 2. 3. Refraction at a spherical interface.


Figure 2. 4. Rays incident at the same angle.

All the quantities in the diagram ( $s_{i}, s_{0}, R$, etc.) are positive numbers, and these form the basis of a sign convention that is gradually unfolding and to which we shall return time and again. Inasmuch as the pointA moves at the end of a fixed radius (i.e., $R=$ constant), $\varphi$ is the position variable, and thus setting $d(O P L) / d \varphi=0$, via Fermat's Principle we have

$$
\begin{equation*}
\frac{n_{1} R\left(s_{0}+R\right) \sin \varphi}{2 l_{0}}-\frac{n_{2} R\left(s_{i}+R\right) \sin \varphi}{2 l_{i}}=0 \tag{2.4}
\end{equation*}
$$

From which it follow that

$$
\begin{equation*}
\frac{n_{1}}{l_{0}}+\frac{n_{2}}{l_{i}}=\frac{1}{R}\left(\frac{n_{2} s_{i}}{l_{i}}-\frac{n_{1} s_{0}}{l_{i 0}}\right) \tag{2.5}
\end{equation*}
$$

Although this expression is exact, it is rather complicated. If $A$ is moved to a new location by changing $\varphi$, the new ray will not intercept the optical axis at $P$. The approximations that are used to represent $l_{0}$ and $l_{i}$ and thereby simplify Eq. (2.5), are crucial in all that is to follow. Recall that
and

$$
\begin{gather*}
\cos \varphi=1-\frac{\varphi^{2}}{2!}+\frac{\varphi^{4}}{4!}+\frac{\varphi^{6}}{6!}+\cdots  \tag{2.6}\\
\sin \varphi=\varphi-\frac{\varphi^{3}}{3!}+\frac{\varphi^{5}}{5!}+\frac{\varphi^{7}}{7!}+\cdots \tag{2.7}
\end{gather*}
$$

If we assume small values of $\varphi$ (i.e., $A$ close to $V$ ), $\cos \varphi \approx 1$. Consequently, the expressions for $l_{0}$ and $l_{i}$ yield $l_{0} \approx s_{0}$ and $l_{i} \approx s_{i}$ and to that approximation.

$$
\begin{equation*}
\frac{n_{1}}{s_{0}}+\frac{n_{2}}{s_{i}}=\frac{n_{2}-n_{1}}{R} \tag{2.8}
\end{equation*}
$$

This approximation delineates the domain of what is called first-order theory; we'll examine third-order theory. Rays that arrive at shallow angles with respect to the optical axis (such that $\varphi$ and $h$ are appropriately small) are known as paraxial rays. The emerging wavefront segment corresponding to these paraxial rays is essentially spherical and will form a "perfect" image at its centre $P$ located at $s_{i}$. Notice that Eq. (2.8) is independent of the location of $A$ over $a$ small area about the symmetry axis, namely, the paraxial region.

Gauss, in 1841, was the first to give a systematic exposition of the formation of images under the above approximation, and the result is variously known as first-order, paraxial, or Gaussian Optics. It soon became the basic theoretical tool by which lenses would be designed for several decades to come. If the optical system is well corrected, an incident spherical wave will emerge in a form very closely resembling a spherical wave. Consequently, as the perfection of the system increases, it more closely approaches first-order theory. Deviations from that of paraxial analysis will provide a convenient measure of the quality of an actual optical device.

If point-Fo in Fig. 2.5 is imaged at infinity $\left(s_{i}=\infty\right)$, we have

$$
\frac{n_{1}}{s_{0}}+\frac{n_{2}}{\infty}=\frac{n_{2}-n_{1}}{R}
$$

That special object distance is defined as the first focal length or the object focal length, $s_{0}=f_{0}$, so that

$$
\begin{equation*}
f_{0}=\frac{n_{1}}{n_{2}-n_{1}} R \tag{2.9}
\end{equation*}
$$

Point- $F_{0}$ is known as the first or object focus. Similarly, the second or image focus is the axial point- $F_{i}$, where the image is formed when $s_{0}=\infty$; that is,


Figure 2. 5. Plane waves propagating beyond a spherical interface-the object focus.

$$
\frac{n_{1}}{\infty}+\frac{n_{2}}{s_{i}}=\frac{n_{2}-n_{1}}{R}
$$

Defining the second or image focal length $F_{i}$ as equal to $s_{i}$ in this special case (Fig.2.5), we have

$$
\begin{equation*}
f_{i}=\frac{n_{2}}{n_{2}-n_{1}} R \tag{2.10}
\end{equation*}
$$

Recall that an image is virtual when the rays diverge from it (Fig. 2.6). Analogously, an object is virtual when the rays converge toward it (Fig. 2.7). Observe that the virtual object is now on the right-hand side of the vertex, and therefore $s_{0}$ will be a negative quantity. Moreover, the surface is concave, and its radius will also be negative, as required by Eq. (2.9), since $f_{0}$ would be negative. In the same way, the virtual image distance appearing to the left of $V$ is negative.


Figure 2. 7. A virtual image point.


Figure 2. 6. A virtual object point

## Example 2.2

Making use of Fig. P.5.5, Snell's Law, and the fact that in the paraxial region $\alpha=h / s_{0}, \varphi \approx$ $h / R$, and $\beta \approx h / s_{i}$, derive Eq. (2.8).


## Solution

$$
\begin{gathered}
\theta_{2}+\left(180^{\circ}-\varphi\right)+\beta=180^{\circ} \\
\theta_{2}=\varphi-\beta \\
\sin \theta_{2}=\sin (\varphi-\beta) \\
=\sin \varphi \cos (-\beta)+\cos \varphi \sin (-\beta) \\
\approx \sin \varphi-\sin \beta \\
h / R-h / s_{i} \\
\left(180^{\circ}-\theta_{1}\right)+\varphi-\alpha=180^{\circ} \\
\theta_{1}=\varphi+\alpha \\
\sin \theta_{1}=\sin (\varphi+\alpha) \\
=\sin \varphi \cos \alpha+\cos \varphi \sin \alpha \\
\approx h / R+h / s \\
n_{1} \sin \theta_{1}=n_{2} \sin \theta_{2} ; n_{1}\left(h / R+h / s_{0}\right)=n_{2}\left(h / R+h / s_{i}\right) \\
n_{1} / s_{0}+n_{2} / s_{i}=\left(n_{2}-n_{1}\right) / R
\end{gathered}
$$

### 2.1.3 Thin Lenses

Lenses are made in a wide range of forms; for example, there are acoustic and microwave lenses. Some are made of glass or wax in easily recognizable shapes, whereas others are far more subtle in appearance. Most often a lens has two or more refracting interfaces, and at least one of these is curved. Generally, the
nonplanar surfaces are centred on a common axis. These surfaces are most frequently spherical segments and are often coated with thin dielectric films to control their transmission properties.

A lens that consists of one element (i.e., it has only two refracting surfaces) is a simple lens. The presence of more than one element makes it a compound lens. A lens is also classified as to whether it is thin or thick-that is, whether or not its thickness is effectively negligible. The simple lens can take the forms shown in Fig. 2.7.


Figure 2. 8. Cross sections of various centred spherical simple lenses. The surface on the left is $\neq 1$, since it is encountered first. Its radius is R1.

Lenses that are variously known as convex, converging, or positive are thicker at the centre and so tend to decrease the radius of curvature of the wavefronts. In other words, the incident wave converges more as it traverses the lens, assuming, of course, that the index of the lens is greater than that of the media in which it is immersed. Concave, diverging, or negative lenses, on the other hand, are thinner at the centre and tend to advance that portion of the incident wavefront, causing it to diverge more than it did prior to entry.

## Thin-Lens Equations

Return to the discussion of refraction at a single spherical interface, where the location of the conjugate points-S and -P is given by Eq. (2.8)

$$
\frac{n_{1}}{s_{o}}+\frac{n_{2}}{s_{i}}=\frac{n_{2}-n_{1}}{R}
$$

When $s_{o}$ is large for a fixed $n_{2}-n_{1} / R, s_{i}$ is relatively small. The cone of rays from $S$ has a small central angle, the rays do not diverge very much, and the refraction at the interface can cause them all to converge at P . As $s_{o}$ decreases, the ray-cone angle increases, the divergence of the rays increases, and $s_{i}$ moves away from the vertex; that is, both $\theta_{i}$ and $\theta_{t}$ increase until finally $s_{o}=f_{o}$ and $s_{i}=\infty$. At that point, $n_{1} / s_{o}=\left(n_{2}-n_{1}\right) / R$, so that if $s_{o}$ gets any smaller, $s_{i}$ will have to be negative, if Eq. (2.8) is to hold. In other words, the image becomes virtual (Fig. 2.9).


Figure 2. 9. Refraction at a spherical interface between two transparent media shown in cross section.

Let's now locate the conjugate points for a lens of index $n_{l}$ surrounded by a medium of index $n_{m}$, as in Fig. 2.10, where another end has simply been ground onto the piece in Fig. 2.9c. This certainly isn't the most general set of circumstances, but it is the most common, it is the simplest. We know from Eq. (2.8) that the paraxial rays issuing from S at $S_{o 1}$ will appear to meet at P , a distance, which we now call $s_{o 1}$, from $V_{1}$, given by

$$
\begin{equation*}
\frac{n_{m}}{s_{o 1}}+\frac{n_{l}}{s_{i 1}}=\frac{n_{l}-n_{m}}{R_{1}} \tag{2.11}
\end{equation*}
$$

Thus, as far as the second surface is concerned, it rays coming toward it from P , which serves as its object point a distance $S_{o 2}$ away. Furthermore, the rays arriving at that second surface are in the medium of index $n_{l}$. The object space for the second interface that contains $P$ therefore has an index $n_{l}$. Note that the rays from $\dot{P}$ to that surface are indeed straight lines. Considering the fact that

$$
\left|s_{o 2}\right|=\left|s_{i 1}\right|+d
$$

since $s_{o 2}$ is on the left and therefore positive, $s_{o 2}=\left|s_{o 2}\right|$, and $s_{i 1}$ is also on the left and therefore negative, $-s_{i 1}=\left|s_{i 1}\right|$, we have

$$
\begin{equation*}
s_{o 2}=-s_{i 1}+d \tag{2.12}
\end{equation*}
$$

At the second surface Eq. (2.8) yields

$$
\begin{equation*}
\frac{n_{m}}{-s_{i 1}+d}+\frac{n_{l}}{s_{i 2}}=\frac{n_{l}-n_{m}}{R_{1}} \tag{2.13}
\end{equation*}
$$

Here $n_{l}>n_{m}$ and $R_{2}>0$, so that the righthand side is positive. Adding Eqs. (2.11) and (2.13), we have


Figure 2. 10. A spherical lens. (a) Rays in a vertical plane passing through a lens. Conjugate foci. (b) Refraction at the interfaces where the lens is immersed in air and $\mathrm{n}_{\mathrm{m}}=\mathrm{n}_{\mathrm{a}}$. The radius drawn from C1 is normal to the first surface, and as the ray enters the lens it bends down toward that normal. The radius from C2 is normal to the second surface; and as the ray emerges, since $n_{1}>$

$$
\begin{equation*}
\frac{n_{m}}{s_{o 1}}+\frac{n_{m}}{s_{i 2}}=\left(n_{l}-n_{m}\right)\left(\frac{1}{R_{1}}-\frac{1}{R_{2}}\right)+\frac{n_{l} d}{\left(s_{i 1}-d\right) s_{i 1}} \tag{2.14}
\end{equation*}
$$

If the lens is thin enough ( $d \rightarrow 0$ ), the last term on the right is effectively zero. As a further simplification, assume the surrounding medium to be air (i.e., $n_{m} \approx 1$ ). Accordingly, we have the very useful Thin-Lens Equation, often referred to as the Lensmaker's Formula:

$$
\begin{equation*}
\frac{1}{s_{o}}+\frac{1}{s_{i}}=\left(n_{l}-1\right)\left(\frac{1}{R_{1}}-\frac{1}{R_{2}}\right) \tag{2.15}
\end{equation*}
$$

where we let $s_{o 1}=s_{o}$ and $s_{i 1}=s_{i}$. The points $-V_{1}$ and $-V_{2}$ tend to coalesce as $(d \rightarrow 0)$, so that $s_{o}$ and $s_{i}$ can be measured from either the vertices or the lens centre. Just as in the case of the single spherical surface, if $s_{o}$ is moved out to infinity, the image distance becomes the focal length $f_{i}$, or symbolically,

$$
\lim _{s_{o} \rightarrow \infty} s_{i}=f_{i}
$$

Similarly

$$
\lim _{s_{i} \rightarrow \infty} s_{o}=f_{o}
$$

It is evident from Eq. (2.15) that for a thin lens $f_{i}=f_{o}$, and consequently we drop the subscripts altogether. Thus

$$
\begin{gather*}
\frac{1}{f}=\left(n_{l}-1\right)\left(\frac{1}{R_{1}}-\frac{1}{R_{2}}\right)  \tag{2.16}\\
\frac{1}{s_{o}}+\frac{1}{s_{i}}=\frac{1}{f} \tag{2.17}
\end{gather*}
$$

which is the famous Gaussian Lens Formula (see photo).


The actual wavefronts of a diverging lightwave partially focused by a lens. The picture was made using a holographic technique.

As an example of how these expressions might be used, let's compute the focal length in air of a thin planar-convex lens having a radius of curvature of 50 mm and an index of 1.5. With light entering on the planar surface ( $R_{1}=\infty, R_{2}=-50$ )

$$
\frac{1}{f}=(1.5-1)\left(\frac{1}{\infty}-\frac{1}{-50}\right)
$$

whereas if instead it arrives at the curved surface ( $R_{1}=+50, R_{2}=\infty$ ),

$$
\frac{1}{f}=(1.5-1)\left(\frac{1}{+50}-\frac{1}{\infty}\right)
$$

and in either case $f=100 \mathrm{~mm}$. If an object is alternately placed at distances $600 \mathrm{~mm}, 200 \mathrm{~mm}, 150 \mathrm{~mm}$, 100 mm , and 50 mm from the lens on either side, we can find the image points from Eq. (2.17). First, with $s_{o}=600 \mathrm{~mm}$

$$
s_{i}=\frac{s_{o} f}{s_{o}-f}=\frac{(600)(100)}{600-100}
$$

and $s_{i}=120 \mathrm{~mm}$. Similarly, the other image distances are $200 \mathrm{~mm}, 300 \mathrm{~mm}, \infty$, and -100 mm , respectively.

## Focal Points and Planes

Figure 2.11 summarizes some of the situations described analytically by Eq. 2.16. Observe that if a lens of index $n_{l}$ is immersed in a medium of index $n_{m}$,

$$
\begin{equation*}
\frac{1}{f}=\left(n_{l m}-1\right)\left(\frac{1}{R_{1}}-\frac{1}{R_{2}}\right) \tag{2.18}
\end{equation*}
$$



Figure 2. 11. Focal lengths for converging and diverging lenses.

The focal lengths in (a) and (b) of Fig. 2.11 are equal, because the same medium exists on either side of the lens. Since $n_{l}>n_{m}$, it follows that $n_{l m}>1$. In both cases $R_{1}>0$ and $R_{2}<0$, so that each focal length is positive. We have a real object in (a) and a real image in (b). In (c), $n_{l}<n_{m}$, and consequently f is negative. $\operatorname{In}$ (d) and (e), $n_{l m}>1$ but $R_{1}<0$, whereas $R_{2}>0$, so $f$ is again negative, and the object in one case and the image in the other are virtual. $\ln (f), n_{l m}<1$, yielding an $f>0$.

The transverse distances above the optical axis are taken as positive quantities, and those below the axis are given negative numerical values. Therefore in Fig. $2.12 y_{o}>0$ and $y_{i}<0$. Here the image is said to be inverted, whereas if $y_{i}>0$ when $y_{o}<0$, it is right-side-up or erect. Observe that triangles $A O F_{i}$ and $P_{2} P_{1} F_{i}$ are similar. Ergo

$$
\begin{equation*}
\frac{y_{o}}{\left|y_{i}\right|}=\frac{f}{\left(s_{i}-f\right)} \tag{2.19}
\end{equation*}
$$

In the same way, triangles $S_{2} S_{1} O$ and $P_{2} P_{1} O$ are similar, and

$$
\begin{equation*}
\frac{y_{o}}{\left|y_{i}\right|}=\frac{s_{o}}{s_{i}} \tag{2.20}
\end{equation*}
$$

where all quantities other than $y_{i}$ are positive. Hence

$$
\begin{equation*}
\frac{S_{o}}{s_{i}}=\frac{f}{\left(s_{i}-f\right)} \tag{2.21}
\end{equation*}
$$

and

$$
\frac{1}{f}=\frac{1}{S_{o}}+\frac{1}{S_{i}}
$$



Figure 2. 12. Object and image location for a thin lens..
which is, of course, the Gaussian Lens Equation [Eq. (2.17)]. Furthermore, triangles $S_{2} S_{1} F_{o}$ and $B O F_{o}$ are similar and

$$
\begin{equation*}
\frac{f}{\left(s_{o}-f\right)}=\frac{\left|y_{i}\right|}{y_{o}} \tag{2.22}
\end{equation*}
$$

Using the distances measured from the focal points and combining this information with Eq. (2.19) leads to

$$
\begin{equation*}
x_{o} x_{i}=f^{2} \tag{2.23}
\end{equation*}
$$

This is the Newtonian form of the lens equation, the first statement of which appeared in Newton's Opticks in 1704. The signs of $x_{o}$ and $x_{i}$ are consedered with respect to their concomitant foci. By convention, $x_{o}$ is taken to be positive left of $F_{o}$, whereas $x_{i}$ is positive on the right of $F_{i}$. It is evident from Eq. (2.23) that $x_{o}$ and $x_{i}$ have like signs, which means that the object and image must be on opposite sides of their respective focal points. The ratio of the transverse dimensions of the final image formed by any optical system to the corresponding dimension of the object is defined as the lateral or transverse magnification, $M_{T}$, that is,

$$
\begin{equation*}
M_{T} \equiv \frac{y_{i}}{y_{o}} \tag{2.24}
\end{equation*}
$$

Or from Eq. (2.20)

$$
\begin{equation*}
M_{T}=-\frac{s_{i}}{s_{o}} \tag{2.25}
\end{equation*}
$$

A positive $\mathrm{M}_{\mathrm{T}}$ connotes an erect image, while a negative value means the image is inverted (see Table 2.1).

TABLE 2.1 Meanings Associated with the Signs of Various Thin Lens and Spherical Interface Parameters

| Quantity | Sign |  |
| :---: | :---: | :---: |
|  | + | - |
| $s_{o}$ | Real object | Virtual object |
| $s_{i}$ | Real image | Virtual image |
| $f$ | Converging lens | Diverging lens |
| $y_{o}$ | Erect object | Inverted object |
| $y_{i}$ | Erect image | Inverted image |
| $M_{T}$ | Erect image | Inverted image |

Bear in mind that $s_{i}$ and $s_{o}$ are both positive for real objects and images. Clearly, then, all real images formed by a single thin lens will be inverted. The Newtonian expression for the magnification follows from Eqs. (2.19) and (2.22) and Fig. 2.12:

$$
\begin{equation*}
M_{T}=-\frac{x_{i}}{f}=-\frac{f}{x_{o}} \tag{2.26}
\end{equation*}
$$

The term magnification is a bit of a misnomer, since the magnitude of $M_{T}$ can certainly be less than 1 , in which case the image is smaller than the object. We have $M_{T}=-1$ when the object and image distances are positive and equal, and that happens [Eq. (2.17)] only when $s_{o}=s_{i}=2 f$. This turns out to be the configuration in which the object and image are as close together as they can possibly get.

## Example 2.3

A biconvex (also called a double convex) thin spherical lens has radii of 100 cm and 20.0 cm . The lens is made of glass with an index of 1.54 and is immersed in air. (a) If an object is placed 70.0 cm in front of the $100-\mathrm{cm}$ surface, locate the resulting image and describe it in detail. (b) Determine the transverse magnification of the image. (c) Draw a ray diagram.

## SOLUTION

(a) We don't have the focal length, but we do know all the physical parameters, so Eq. (2.16) comes to mind:

$$
\frac{1}{f}=\left(n_{l}-1\right)\left(\frac{1}{R_{1}}-\frac{1}{R_{2}}\right)
$$

Leaving everything in centimetres

$$
\begin{gathered}
\frac{1}{f}=(1.54-1)\left(\frac{1}{100}-\frac{1}{-20}\right) \\
\frac{1}{f}=(0.54)\left(\frac{1}{100}+\frac{1}{20}\right) \\
\frac{1}{f}=(0.54)\left(\frac{6}{100}\right) \\
f=30.86 \mathrm{~cm} \approx 30.9 \mathrm{~cm}
\end{gathered}
$$

Now we can find the image. Since $s_{o}=70.0 \mathrm{~cm}$, that's greater than $2 f$-hence, even before we calculate $s_{i}$, we know that the image will be real, inverted, located between $f$ and $2 f$, and minified. To find $s_{i}$, having $f$ we use Gauss's Equation:

$$
\begin{gathered}
\frac{1}{30.86}=\frac{1}{70}+\frac{1}{S_{i}} \\
\frac{1}{S_{i}}=\frac{1}{30.86}-\frac{1}{70}=0.01812 \\
S_{i}=55.19=55.2 \mathrm{~cm}
\end{gathered}
$$

The image is between $f$ and $2 f$ on the right of the lens. Note that $s_{i}>0$, which means the image is real.
(b) The magnification follows from

$$
M_{T}=-\frac{s_{i}}{s_{o}}=\frac{55.19}{70}=-0.788
$$

and the image is inverted ( $M_{T}<0$ ) and minified $\left(M_{T}<1\right)$.
(c) Draw the lens and mark out two focal lengths


We are now in a position to understand the entire range of behaviour of a single convex or concave lens. Figure 2.13 illustrates the behaviour pictorially. As the object approaches the lens, the real image moves away from it. When the object is very far away, the image (real, inverted, and minified $M_{T}<1$ ) is just to the right of the focal plane. As the object approaches the lens, the image (still real, inverted, and minified $M_{T}<1$ ) moves away from the focal plane, to the right, getting larger and larger. With the object between infinity and $2 f$ we have the arrangement for cameras and eyeballs, both of which require a minified, real image. By the way, it's the brain that flips the image so that you see things right-side-up. When the object is at two focal lengths, the image (real and inverted) is now life size, that is, $M_{T}=1$. This is the usual configuration of the photocopy machine. As the object comes closer to the lens (between $2 f$ and $f$ ), the image (real, inverted, and enlarged $M_{T}>1$ ) rapidly moves to the right and continues to increase in size. This configuration corresponds to the film projector where the crucial feature is that the image is real and enlarged. To compensate for the image being inverted, the film is simply put in upside-down.


Figure 2. 13. The image-forming behaviour of a thin positive lens.

## Longitudinal Magnification

The image of a three-dimensional object will itself occupy a three-dimensional region of space. The optical system can apparently affect both the transverse and longitudinal dimensions of the image. The longitudinal magnification, $M_{L}$, which relates to the axial direction, is defined as

$$
\begin{equation*}
M_{L} \equiv \frac{d x_{i}}{d x_{o}} \tag{2.27}
\end{equation*}
$$

This is the ratio of an infinitesimal axial length in the region of the image to the corresponding length in the region of the object. Differentiating Eq. (2.23) leads to

$$
\begin{equation*}
M_{L}=-\frac{f^{2}}{x_{o}^{2}}=-M_{T}^{2} \tag{2.28}
\end{equation*}
$$

for a thin lens in a single medium (Fig. 2.14). Evidently, $M_{L}<0$, which implies that a positive $d x_{o}$ corresponds to a negative $d x_{i}$ and vice versa. In other words, a finger pointing toward the lens is imaged pointing away from it (Fig. 2.15).


Figure 2. 14. The transverse magnification is different from the longitudinal magnification.


Figure 2. 15. Image orientation for a thin lens.

## Thin-Lens Combinations

We'll now derive expressions for parameters associated with thin-lens combinations. Consider two thin positive lenses $L_{1}$ and $L_{2}$ separated by a distance $d$, which is smaller than either focal length, as in Fig. 2.16. The resulting image can be located graphically as follows. Overlooking $L_{2}$ for a moment, construct the image formed exclusively by $L_{1}$ using rays- 2 and -3 . As usual, these pass through the lens object and image foci, $F_{o 1}$ and $F_{i 1}$, respectively. The object is in a normal plane, so that two rays determine the top of the image, and a perpendicular to the optical axis finds its bottom. Ray-4 is then constructed running backward from $\dot{P}_{1}$ through $O_{2}$. Insertion of $L_{2}$ has no effect on ray-4, whereas ray-3 is refracted through the image focus $F_{i 2}$ of $L_{2}$. The intersection of rays-4 and -3 fixes the image, which in this particular case is real, minified, and inverted.


Figure 2. 16. Two thin lenses separated by a distance smaller than either focal length.

A similar pair of lenses is illustrated in Fig. 2.17, in which the separation has been increased. Once again rays-2 and -3 through $F_{i 1}$ and $F_{o 1}$ fix the position of the intermediate image generated by $L_{1}$ alone. As before, ray- 4 is drawn backward from $O_{2}$ to $P_{1}$ to $S_{1}$. The intersection of rays-3 and -4 , as the former is refracted through $F_{i 2}$, locates the final image. This time it is real and erect. Notice that if the focal length of $L_{2}$ is increased with all else constant, the size of the image increases as well.

Analytically, looking only at $L_{1}$ in Fig. 2.16,

$$
\begin{gather*}
\frac{1}{s_{i 1}}=\frac{1}{f_{1}}-\frac{1}{s_{o 1}}  \tag{2.29}\\
s_{i 1}=\frac{s_{o 1} f_{1}}{s_{o 1}-f_{1}} \tag{2.30}
\end{gather*}
$$

This is positive, and the intermediate image (at $P_{1}$ ) is to the right of $L_{1}$, when $s_{o 1}>f_{1}$ and $f_{1}>0$. Now considering the second lens $L_{2}$ with its object at $P_{1}$

$$
\begin{equation*}
s_{o 2}=d-s_{i 1} \tag{2.31}
\end{equation*}
$$



Figure 2. 17. Two thin lenses separated by a distance greater than the sum of their focal lengths. Because the intermediate image is real, you could start with point- $\mathrm{P}_{1}$ and treat it as if it were a real object point for $\mathrm{L}_{2}$. Thus a ray from $P_{1}$ through $F_{02}$ would arrive at $P_{1}$.
and if $d>s_{i 1}$, the object for $L_{2}$ is real (as in Fig. 2.17), whereas if $d>s_{i 1}$, it is virtual ( $s_{o 2}<0$, as in Fig. 2.16). In the former instance the rays approaching $L_{2}$ are diverging from $P_{1}$, whereas in the latter they are converging toward it. As drawn in Fig. 2.16a, the intermediate image formed by $L_{1}$ is the virtual object for $L_{2}$. Furthermore, for $L_{2}$
or

$$
\begin{gathered}
\frac{1}{s_{i 2}}=\frac{1}{f_{2}}-\frac{1}{s_{o 2}} \\
s_{i 2}=\frac{s_{o 2} f_{2}}{s_{o 2}-f_{2}}
\end{gathered}
$$

Using Eq. (2.31), we obtain

$$
\begin{equation*}
s_{i 2}=\frac{\left(d-s_{i 1}\right) f_{2}}{d-s_{i 1}-f_{2}} \tag{2.32}
\end{equation*}
$$

In this same way we could compute the response of any number of thin lenses. It will often be convenient to have a single expression, at least when dealing with only two lenses, so substituting for $s_{i 1}$ from Eq. (2.29),

$$
\begin{equation*}
s_{i 2}=\frac{f_{2} d-f_{2} s_{o 1} f_{1} /\left(s_{o 1}-f_{1}\right)}{d-f_{2}-s_{o 1} f_{1} /\left(s_{o 1}-f_{1}\right)} \tag{2.33}
\end{equation*}
$$

Here $s_{o 1}$ and $s_{i 2}$ are the object and image distances, respectively, of the compound lens.

Inasmuch as $L_{2}$ "magnifies" the intermediate image formed by $L_{1}$, the total transverse magnification of the compound lens is the product of the individual magnifications, that is,

$$
\begin{array}{r}
M_{T}=M_{T 1} M_{T 2} \\
M_{T}=\frac{f_{1} s_{i 2}}{d\left(s_{o 1}-f_{1}\right)-s_{o 1} f_{1}} \tag{2.34}
\end{array}
$$

## Example 2.4

Verify Eq. (2.34), which gives $M_{T}$ for a combination of two thin lenses. H.W

## Example 2.5

A thin biconvex lens having a focal length of +40.0 cm is located 30.0 cm in front (i.e., to the left) of a thin biconcave lens of focal length -40.0 cm . If a small object is situated 120 cm to the left of the positive lens (a) determine the location of its image by calculating the effect of each lens. (b) Compute the magnification. (c) Describe the image.

## SOLUTION

(a) The first lens forms an intermediate image at $s_{i 1}$, where

$$
\begin{aligned}
\frac{1}{f_{1}} & =\frac{1}{s_{01}}-\frac{1}{s_{i 1}} \\
\frac{1}{40} & =\frac{1}{120}-\frac{1}{s_{i 1}} \\
\frac{1}{s_{i 1}} & =\frac{1}{40}-\frac{1}{120}=\frac{2}{120} \\
s_{i 1} & =60 \mathrm{~cm}
\end{aligned}
$$

That's 30.0 cm to the right of the negative lens. Hence $s_{o 2}=-30 \mathrm{~cm}$ and

$$
\begin{aligned}
\frac{1}{f_{2}} & =\frac{1}{s_{02}}-\frac{1}{s_{i 2}} \\
\frac{1}{-40} & =\frac{1}{-30}-\frac{1}{s_{i 2}} \\
s_{i 2} & =+120 \mathrm{~cm}
\end{aligned}
$$

The image is formed 120 cm to the right of the negative lens.
(b) The magnification is

$$
\begin{gathered}
M_{T}=M_{T 1} M_{T 2}=\left(-\frac{s_{i 1}}{s_{o 1}}\right)\left(-\frac{s_{i 2}}{s_{o 2}}\right) \\
M_{T}=\left(-\frac{60}{120}\right)\left(-\frac{120}{-30}\right)=-2
\end{gathered}
$$

(c) The image is real, because $s_{i 2}>0$; inverted, because $M_{T}<0$; and magnified. We could check $M_{T}$ using Eq. (2.34)

$$
M_{T}=\frac{40(120)}{30(120-40)-120(40)}=\frac{40(120)}{-40(60)}
$$

$$
M_{T}=-2
$$

and $s_{i 2}$ using Eq. (2.33)

$$
\begin{gathered}
s_{i 2}=\frac{(-40)(30)-(-40)(120)(40) /(120-40)}{30-(-40)-120(40) /(120-40)} \\
s_{i 2}=\frac{-1200+(40)(60)}{70-60}=\frac{1200}{10}=120 \mathrm{~cm}
\end{gathered}
$$

