Department of physics

Second class

Mathematics

Triple Integrals

The notation for the general triple integrals is,

$$\iiint_E f(x,y,z)\,dV$$

Let's start simple by integrating over the box,

$$B = [a,b]\times[c,d]\times[r,s]$$

Note that when using this notation we list the *x*'s first, the *y*'s second and the *z*'s third.

The triple integral in this case is,

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

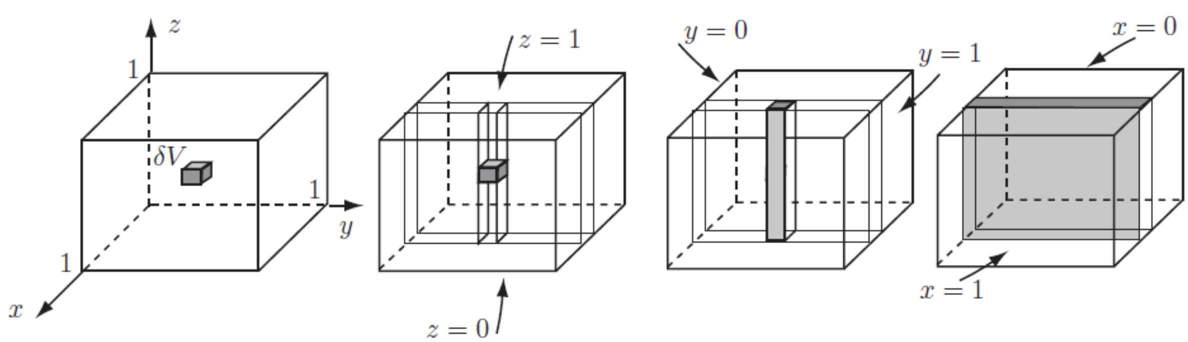
Note that we integrated with respect to *x* first, then *y*, and finally *z*

Example: 1

Consider a cube V of side 1.

(a) Express the integral $\int_V f\,dV$ (where f is any function of x, y and z) as a triple integral.

(b) Hence evaluate $\int_V (y^2+z^2)\,dV$



Solution

(a) Consider a little element of length dx, width dy and height dz. Then δV (the volume of the small element) is the product of these lengths $dxdydz$. The function is integrated three times. The first integration represents the integral over the vertical strip from $z = 0$ to $z = 1$. The second integration represents this strip sweeping across from $y = 0$ to $y = 1$ and is the integration over the slice that is swept out by the strip. Finally the integration with respect to x represents this slice sweeping from $x = 0$ to $x = 1$ and is the integration over the entire cube. The integral therefore becomes

$$\int_0^1 \int_0^1 \int_0^1 f(x, y, z)\, dzdydx$$

(b) In the particular case where the function is $f(x, y, z) = y^2 + z^2$, the integral becomes

$$\int_0^1 \int_0^1 \int_0^1 (y^2 + z^2)\, dzdydx$$

The inner integral is

$$\int_0^1 (y^2 + z^2)\, dz = \left[y^2 z + \frac{1}{3}z^3\right]_{z=0}^{1} = y^2 \times 1 + \frac{1}{3} \times 1 - y^2 \times 0 - \frac{1}{3} \times 0 = y^2 + \frac{1}{3}$$

This inner integral is now placed into the intermediate integral to give

$$\int_0^1 (y^2 + \frac{1}{3})\, dy = \left[\frac{1}{3}y^3 + \frac{1}{3}y\right]_{y=0}^{1} = \frac{1}{3} \times 1^3 + \frac{1}{3} \times 1 - \frac{1}{3} \times 0^3 - \frac{1}{3} \times 0 = \frac{2}{3}$$

Finally, this intermediate integral can be placed into the outer integral to give

$$\int_0^1 \frac{2}{3}\, dx = \left[\frac{2}{3}x\right]_0^1 = \frac{2}{3} \times 1 - \frac{2}{3} \times 0 = \frac{2}{3}$$

Example:2

Evaluate the following integral.

$$\iiint_B 8xyz\, dV, \quad B = [2,3] \times [1,2] \times [0,1]$$

Solution

Just to make the point that order doesn't matter let's use a different order from that listed above. We'll do the integral in the following order.

$$\iiint_B 8xyz\, dV = \int_1^2 \int_2^3 \int_0^1 8xyz\, dz\, dx\, dy$$

$$= \int_1^2 \int_2^3 4xyz^2 \Big|_0^1 \, dx\, dy$$

$$= \int_1^2 \int_2^3 4xy \, dx\, dy$$

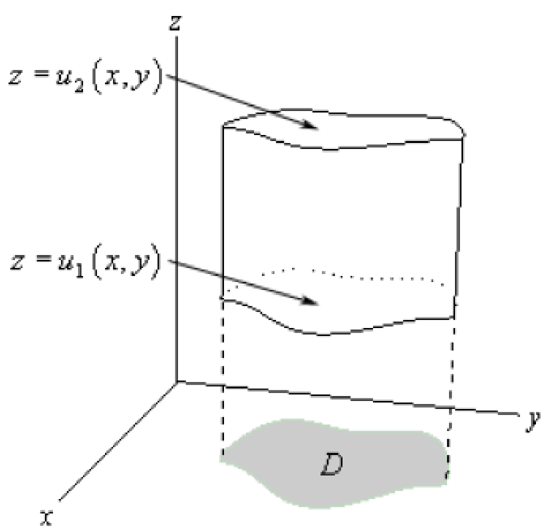
$$= \int_1^2 2x^2 y \Big|_2^3 \, dy$$

$$= \int_1^2 10y \, dy = 15$$

The volume of the three-dimensional region E is given by the integral,

$$V = \iiint\limits_E dV$$

Let's now move on the more general three-dimensional regions. We have three different possibilities for a general region. Here is a sketch of the first possibility.



In this case we define the region *E* as follows,

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$$

where $(x, y) \in D$ is the notation that means that the point (x, y) lies in the region *D* from the *xy*-plane. In this case we will evaluate the triple integral as follows,

$$\iiint\limits_E f(x, y, z)\,dV = \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz\right] dA$$

where the double integral can be evaluated in any of the methods that we saw in the previous couple of sections. In other words, we can integrate first with respect to *x*, we can integrate first with respect to *y*, or we can use polar coordinates as needed.

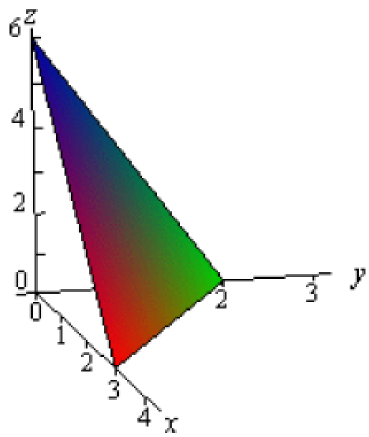
Example:3

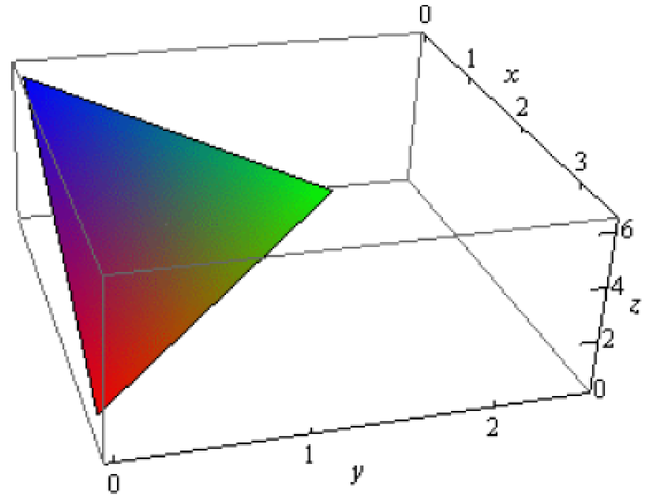
Evaluate $\iiint\limits_E 2x\,dV$ where *E* is the region under the plane $2x + 3y + z = 6$ that lies in the first octant.

Solution

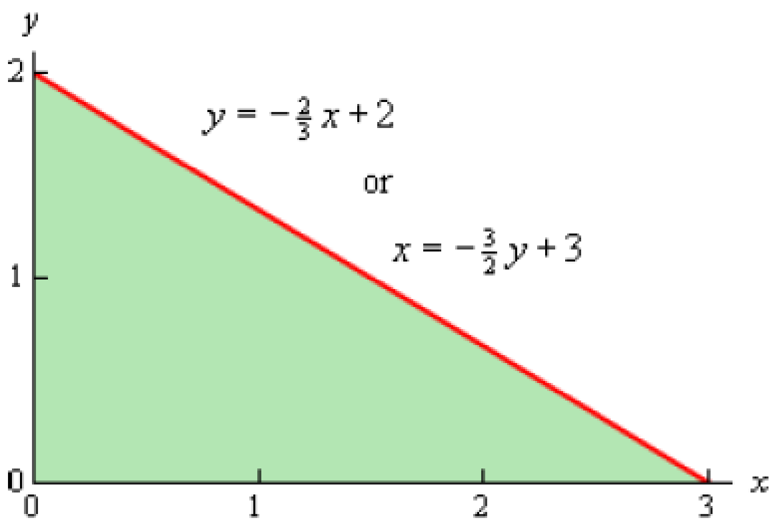
We should first define *octant*. Just as the two-dimensional coordinates system can be divided into four quadrants the three-dimensional coordinate system can be divided into eight octants. The first octant is the octant in which all three of the coordinates are positive.

Here is a sketch of the plane in the first octant.





We now need to determine the region D in the xy-plane. We can get a visualization of the region b pretending to look straight down on the object from above. What we see will be the region D in th xy-plane. So D will be the triangle with vertices at $(0,0)$, $(3,0)$, and $(0,2)$. Here is a sketch of D.



Now we need the limits of integration. Since we are under the plane and in the first octant (so we're above the plane $z = 0$) we have the following limits for *z*.

$$0 \le z \le 6 - 2x - 3y$$

We can integrate the double integral over *D* using either of the following two sets of inequalities.

$$0 \le x \le 3 \qquad\qquad 0 \le x \le -\frac{3}{2}y + 3$$

$$0 \le y \le -\frac{2}{3}x + 2 \qquad\qquad 0 \le y \le 2$$

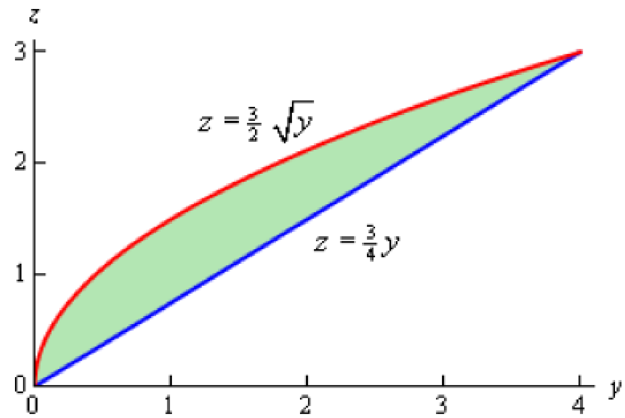
Since neither really holds an advantage over the other we'll use the first one. The integral is then,

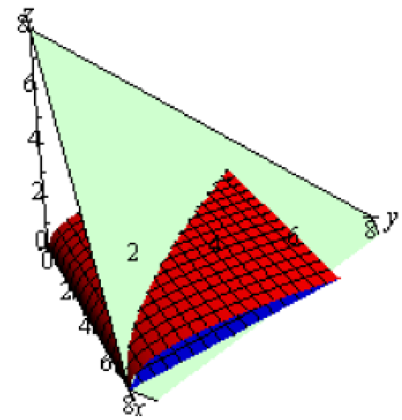
$$\begin{aligned}
\iiint\limits_E 2x\,dV &= \iint\limits_D \left[\int_0^{6-2x-3y} 2x\,dz\right] dA \\
&= \iint\limits_D 2xz\Big|_0^{6-2x-3y}\,dA \\
&= \int_0^3 \int_0^{-\frac{2}{3}x+2} 2x\left(6-2x-3y\right)dy\,dx \\
&= \int_0^3 \left(12xy - 4x^2y - 3xy^2\right)\Big|_0^{-\frac{2}{3}x+2}\,dx \\
&= \int_0^3 \frac{4}{3}x^3 - 8x^2 + 12x\,dx \\
&= \left(\frac{1}{3}x^4 - \frac{8}{3}x^3 + 6x^2\right)\Big|_0^3 \\
&= 9
\end{aligned}$$

Example 3 Determine the volume of the region that lies behind the plane $x+y+z=8$ and in front of the region in the *yz*-plane that is bounded by $z=\frac{3}{2}\sqrt{y}$ and $z=\frac{3}{4}y$.

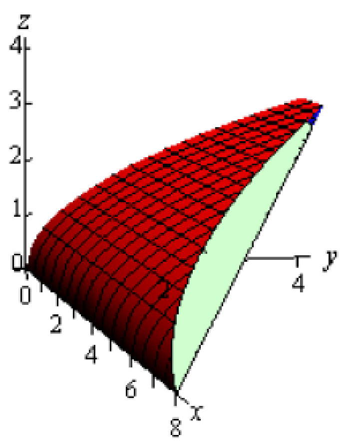
Solution

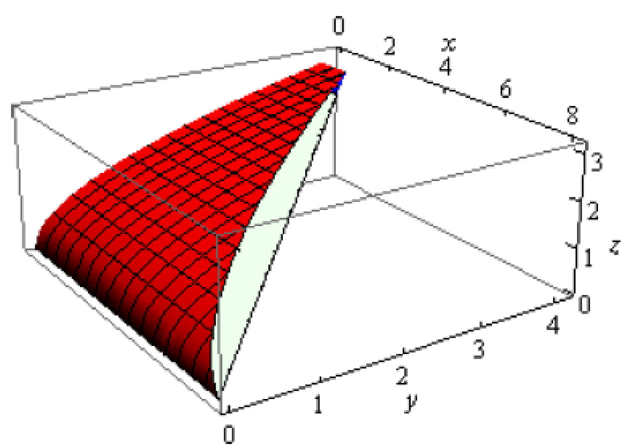
In this case we've been given *D* and so we won't have to really work to find that. Here is a sketch of the region *D* as well as a quick sketch of the plane and the curves defining *D* projected out past the plane so we can get an idea of what the region we're dealing with looks like.





Now, the graph of the region above is all okay, but it doesn't really show us what the region is. So, here is a sketch of the region itself.





Here are the limits for each of the variables.

$$0 \le y \le 4$$

$$\frac{3}{4}y \le z \le \frac{3}{2}\sqrt{y}$$

$$0 \le x \le 8-y-z$$

The volume is then,

$$V = \iiint\limits_E dV = \iint\limits_D \left[\int_0^{8-y-z} dx \right] dA$$

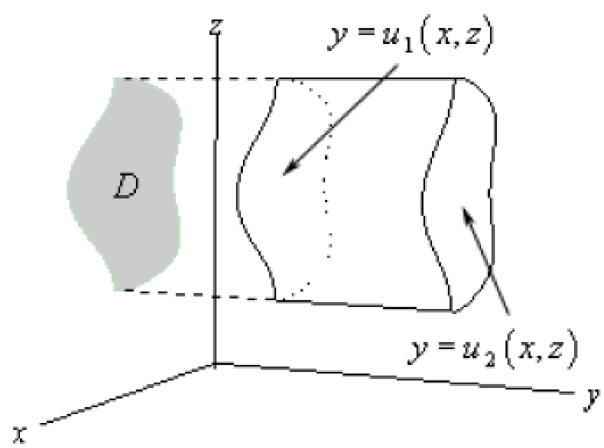
$$= \int_0^4 \int_{3y/4}^{3\sqrt{y}/2} 8 - y - z \, dz \, dy$$

$$= \int_0^4 \left(8z - yz - \frac{1}{2}z^2 \right) \Bigg|_{\frac{3y}{4}}^{\frac{3\sqrt{y}}{2}} dy$$

$$= \int_0^4 12y^{\frac{1}{2}} - \frac{57}{8}y - \frac{3}{2}y^{\frac{3}{2}} + \frac{33}{32}y^2 \, dy$$

$$= \left(8y^{\frac{3}{2}} - \frac{57}{16}y^2 - \frac{3}{5}y^{\frac{5}{2}} + \frac{11}{32}y^3 \right) \Bigg|_0^4 = \frac{49}{5}$$

We now need to look at the third (and final) possible three-dimensional region we may run i
triple integrals. Here is a sketch of this region.



In this final case *E* is defined as,

$$E = \{(x, y, z) \mid (x, z) \in D, \ u_1(x, z) \le y \le u_2(x, z)\}$$

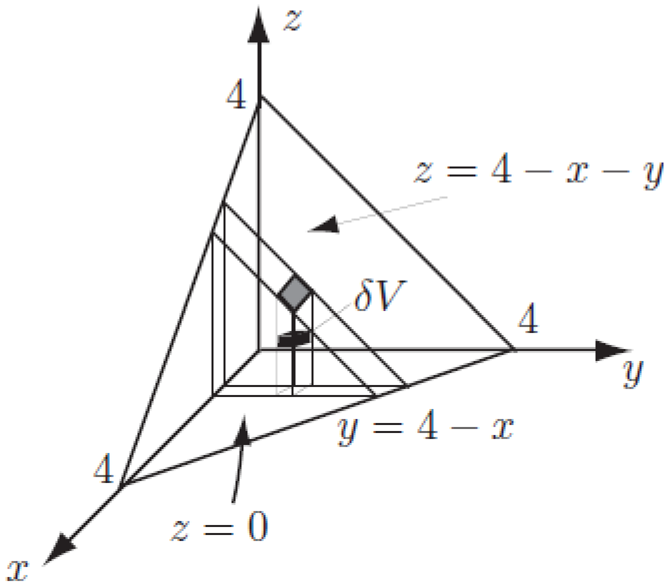
and here the region *D* will be a region in the *xz*-plane. Here is how we will evaluate these integrals.

$$\iiint\limits_E f(x, y, z) \, dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z) \, dy \right] dA$$

where we will can use either of the two possible orders for integrating *D* in the *xz*-plane or we can use polar coordinates if needed.

Example:4

V is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x+y+z = 4$.



(a) Express $\int_V f(x, y, z)\ dV$ (where f is a function of x, y and z) as a triple integral.

(b) Hence find $\int_V x\ dV$.

Solution

The tetrahedron is divided into a series of slices parallel to the yz-plane and each slice is divided into a series of vertical strips. For each strip, the bottom is at $z = 0$ and the top is on the plane $x + y + z = 4$ i.e. $z = 4 - x - y$. So the integral up each strip is given by $\int_{z=0}^{4-x-y} f(x, y, z)\ dz$ and this (inner) integral will be a function of x and y.

This, in turn, is integrated over all strips which form the slice. For each value of x, one end of the slice will be at $y = 0$ and the other end at $y = 4 - x$. So the integral over the slice is $\displaystyle\int_{y=0}^{4-x} \int_{z=0}^{4-x-y} f(x, y, z)\, dzdy$ and this (intermediate) integral will be a function of x.

Finally, integration is carried out over x. The limits on x are $x = 0$ and $x = 4$. Thus the triple integral is $\displaystyle\int_{x=0}^{4} \int_{y=0}^{4-x} \int_{z=0}^{4-x-y} f(x, y, z)\, dzdydx$ and this (outer) integral will be a constant.

Hence $\displaystyle\int_V f(x, y, z)\, dV = \int_{x=0}^{4} \int_{y=0}^{4-x} \int_{z=0}^{4-x-y} f(x, y, z)\, dzdydx.$

In the case where $f(x, y, z) = x$, the integral becomes

$$\begin{aligned}
\int_V f(x,y,z)\, dV &= \int_{x=0}^{4} \int_{y=0}^{4-x} \int_{z=0}^{4-x-y} x\, dzdydx \\
&= \int_{x=0}^{4} \int_{y=0}^{4-x} \Big[\, xz \, \Big]_{z=0}^{4-x-y} dydx \\
&= \int_{x=0}^{4} \int_{y=0}^{4-x} [(4-x-y)x - 0]\ dydx \\
&= \int_{x=0}^{4} \int_{y=0}^{4-x} \left[4x - x^2 - xy\right]\ dydx \\
&= \int_{x=0}^{4} \left[4xy - x^2y - \frac{1}{2}xy^2\right]_{y=0}^{4-x} dx \\
&= \int_{x=0}^{4} \left[4x(4-x) - x^2(4-x) - \frac{1}{2}x(4-x)^2 - 0\right]_{y=0}^{4-x} dx \\
&= \int_{x=0}^{4} \left[16x - 4x^2 - 4x^2 + x^3 - 8x + 4x^2 - \frac{1}{2}x^3\right]\ dx \\
&= \int_{x=0}^{4} \left[8x - 4x^2 + \frac{1}{2}x^3\right]\ dx \\
&= \left[4x^2 - \frac{4}{3}x^3 + \frac{1}{8}x^4\right]_0^4 = 4 \times 4^2 - \frac{4}{3} \times 4^3 + \frac{1}{8} \times 4^4 - 0 \\
&= 64 - \frac{256}{3} + 32 \\
&= \frac{192 - 256 + 96}{3} \\
&= \frac{32}{3}
\end{aligned}$$

Example:5

Evaluate $\iiint\limits_E \sqrt{3x^2+3z^2}\, dV$ where E is the solid bounded by $y = 2x^2+2z^2$ and the

$$\iiint\limits_E \sqrt{3x^2+3z^2}\, dV = \iint\limits_D \left[\int_{2x^2+2z^2}^{8} \sqrt{3x^2+3z^2}\, dy\right] dA$$
$$= \iint\limits_D \left(y\sqrt{3x^2+3z^2}\right)\Big|_{2x^2+2z^2}^{8} dA$$
$$= \iint\limits_D \sqrt{3\left(x^2+z^2\right)}\left(8-\left(2x^2+2z^2\right)\right) dA$$

Now, since we are going to do the double integral in polar coordinates let's get everything converted over to polar coordinates. The integrand is,

$$\sqrt{3\left(x^2+z^2\right)}\left(8-\left(2x^2+2z^2\right)\right) = \sqrt{3r^2}\left(8-2r^2\right)$$
$$= \sqrt{3}\; r\left(8-2r^2\right)$$
$$= \sqrt{3}\left(8r-2r^3\right)$$

The integral is then,

$$\iiint\limits_E \sqrt{3x^2+3z^2}\, dV = \iint\limits_D \sqrt{3}\,\left(8r-2r^3\right) dA$$
$$= \sqrt{3}\int_0^{2\pi}\int_0^2 \left(8r-2r^3\right) r\, dr\, d\theta$$
$$= \sqrt{3}\int_0^{2\pi} \left(\frac{8}{3}r^3-\frac{2}{5}r^5\right)\Big|_0^2 d\theta$$
$$= \sqrt{3}\int_0^{2\pi} \frac{128}{15}\, d\theta$$
$$= \frac{256\sqrt{3}\,\pi}{15}$$