### 1.1 General Principles

We now examine the general case of the motion of a particle in three dimensions. The vector form of the equation of motion for such a particle is

$$
\begin{equation*}
F=\frac{d p}{d t} \tag{1.1}
\end{equation*}
$$

in which $p=m v$ is the linear momentum of the particle. This vector equation is equivalent to three scalar equations in Cartesian coordinates.

$$
\begin{align*}
F_{x} & =m \ddot{x}  \tag{1.2}\\
F_{y} & =m \ddot{y} \\
F_{z} & =m \ddot{z}
\end{align*}
$$

There is no general method for obtaining an analytic solution to the above equations of motion. It is rare that one knows the direct way in which $F$ depends on time. The simplest situation is one in which $F$ is known to be a function of spatial coordinates only. There are many slightly more complex situations, in which $F$ is a known function of coordinate derivatives as well. Such cases include projectile motion with air resistance and the motion of a charged particle in a static electromagnetic field Finally, $F$ may be an implicit function of time. A prime example of such a situation involves the motion of a charged particle in a time-varying electromagnetic field.

## The Work Principle

Work done on a particle causes it to gain or lose kinetic energy. We would like to generalize the case of motion of a particle in one dimension to the case of three-dimensional motion. To do so, we first take the dot product of both sides of Equation 1.1 with the velocity $v$ :

$$
\begin{equation*}
F . v=\frac{d p}{d t} \cdot v=\frac{d(m v)}{d t} \cdot v \tag{1.3}
\end{equation*}
$$

Because $d(v \cdot v) / d t=2 v \cdot \dot{v}$, and assuming that the mass is constant, independent of the velocity of the particle, we may write Equation 1.3 as

$$
\begin{equation*}
F . v=\frac{d}{d t}\left(\frac{1}{2} m v \cdot v\right)=\frac{d T}{d t} \tag{1.4}
\end{equation*}
$$

in which $T$ is the kinetic energy, $m v^{2} / 2$. Because $v=d r / d t$, we can rewrite Equation 1.4 and then integrate the result to obtain

$$
\begin{gather*}
F \cdot \frac{d r}{d t}=\frac{d T}{d t}  \tag{1.5a}\\
\int F \cdot d r=\int d T=T_{f}-T_{i}=\Delta T \tag{1.5b}
\end{gather*}
$$



Figure 1. 1. The work done by a force $F$ is the line integral $\int_{A}^{B} F \cdot d r$.

The left-hand side of this equation is a line integral, or the integral of $F_{r} d r$, the component of $F$ parallel to the particle's displacement vector $d r$. The integral is carried out along the path of the particle from some initial point in space $A$ to some final point $B$. This situation is pictured in Figure1.1. The line integral represents the work done on the particle by the force $F$ as the particle moves along its trajectory from $A$ to $B$.

## Conservative Forces and Force Fields

The force acting on a particle were conservative, it could be derived as the derivative of a scalar potential energy function, $F_{x}=-d V(x) / d x$. This condition led us to the notion that the work done by such a force in moving a particle from point A to point B along the x -axis was $\int F_{x} d x=-\Delta V=V(A)-V(B)$, or equal to minus the change in the potential energy of the particle. The work done depended only upon the potential energy function evaluated at the endpoints of the motion. Moreover, because the work done was also equal to the change in kinetic energy of the particle, $\Delta T=T(B)-T(A)$, we were able to establish a general conservation of total energy principle, namely, $E_{\text {tot }}=V(A)+T(A)=V(A)+$ $T(B)=$ constant throughout the motion of the particle. This principle was based on the condition that the force acting on the particle was conservative. We would like to generalize this concept for a particle moving in three dimensions, and, more importantly, we would like to define just what is meant by the word conservative. The only way in which we could specify a unique value to the potential energy would be if the closed-loop work integral vanished. In such cases, the work done along a path from $A$ to $B$ would be path-independent and would equal both the potential energy loss and the kinetic energy gain. The total energy of the particle would be a constant, independent of its location in such a force field, therefore, we must find the constraint that a particular force must obey if its closed-loop work integral is to vanish.

To find the desired constraint, let us calculate the work done in taking a test particle counterclockwise around the rectangular loop of area $\Delta x \Delta y$ from the point $(x, y)$ and back again, as indicated in Figure1.2. We get the following result:


Figure 1. 2. A nonconservative force field whose force components are $F_{x}=-b y$ and $F_{y}=+b x$

$$
\begin{align*}
W & =\oint F \cdot d r \\
= & \int_{x}^{x+\Delta x} F_{x}(y) d x+\int_{y}^{y+\Delta y} F_{y}(x+\Delta x) d y+\int_{x+\Delta x}^{x} F_{x}(y+\Delta y) d x+\int_{y+\Delta y}^{y} F_{y}(x) d y \\
= & \int_{y}^{y+\Delta y}\left(F_{y}(x+\Delta x)-F_{y}(x)\right) d y+\int_{x}^{x+\Delta x}\left(F_{x}(y)-F_{x}(y+\Delta y)\right) d x  \tag{1.6}\\
= & (b(x+\Delta x)-b x) \Delta y+(b(y+\Delta y)-b y) \Delta x \\
= & 2 b \Delta x \Delta y
\end{align*}
$$

The work done is nonzero and is proportional to the area of the loop, $\Delta \mathrm{A}=\Delta x \cdot \Delta y$, which was chosen in an arbitrary fashion. If we divide the work done by the area of the loop and take limits as $\Delta \mathrm{A} \rightarrow 0$, we obtain the value $2 b$. The result is dependent on the precise nature of this particular nonconservative force field. If we reverse the direction of one of the force components—say, let $F_{x}=+b y$ (thus "destroying" the circulation of the force field but everywhere preserving its magnitude) - then the work done per unit area in traversing the closed loop vanishes. The resulting force field is conservative and is shown in Figure 1.3. Clearly, the value of the closed-loop integral depends upon the precise way in which the vector force $F$ changes its direction as well as its magnitude as we move around on the $x y$ plane.


Figure 1. 3. A conservative force field whose components are $F_{x}=$ by and $F_{y}=b x$

There is obviously some sort of constraint that $F$ must obey if the closed-loop work integral is to vanish. We can derive this condition of constraint by evaluating the forces at $x+\Delta x$ and $y+\Delta y$ using a Taylor expansion and then inserting the resultant expansion into the closed-loop work integral of Equation 1.6. The result follows:

$$
\begin{gather*}
F_{x}(y+\Delta y)=F_{x}(y)+\frac{\partial F_{x}}{\partial y} \Delta y \\
F_{y}(x+\Delta x)=F_{y}(x)+\frac{\partial F_{y}}{\partial x} \Delta x  \tag{1.7}\\
\oint F \cdot d r=\int_{y}^{y+\Delta y}\left(\frac{\partial F_{y}}{\partial x} \Delta x\right) d y-\int_{x}^{x+\Delta x}\left(\frac{\partial F_{x}}{\partial y} \Delta y\right) d x \\
=\left(\frac{\partial F_{y}}{\partial x}-\frac{\partial F_{x}}{\partial y}\right) \Delta x \Delta y=2 b \Delta x \Delta y \tag{1.8}
\end{gather*}
$$

This last equation contains the term $\left(\partial F_{y} / \partial x-\partial F_{x} / \partial y\right)$ whose zero or nonzero value represents the test we are looking for. If this term were identically equal to zero instead of $2 b$, then the closed-loop work integral would vanish, which would ensure the existence of a potential energy function from which the force could be derived.

This condition is a rather simplified version of a very general mathematical theorem called Stokes' theorem. It is written as

$$
\begin{gather*}
\oint F \cdot d r=\int_{s} F \cdot \hat{n} d a \\
\operatorname{curl} F=i\left(\frac{\partial F_{z}}{\partial y}-\frac{\partial F_{y}}{\partial z}\right)+j\left(\frac{\partial F_{x}}{\partial z}-\frac{\partial F_{z}}{\partial x}\right)+k\left(\frac{\partial F_{y}}{\partial x}-\frac{\partial F_{x}}{\partial y}\right) \tag{1.9}
\end{gather*}
$$

The theoretical states that the closed-loop line integral of any vector function $F$ is equal to crul $F \cdot n d a$ integrated over a surface $S$ surrounded by the closed loop. The vector $n$ is a unit vector normal to the surface-area integration element $d a$. Its direction is that of the advance of a right-hand screw turned in the same rotational sense as the direction of traversal around the closed loop. In Figure.1.2, $n$ would be directed out of the paper. The surface would be the rectangular area enclosed by the dashed rectangular loop. Thus, a vanishing crul $F$ ensures that the line integral of $F$ around a closed path is zero and, thus, that $F$ is a conservative force.

### 1.1.2 The Potential Energy Function in Three-Dimensional Motion: The Del Operator

Assume that we have a test particle subject to some force whose curl vanishes. Then all the components of crul $F$ in Equation1. 9 vanish. We can make certain that the curl vanishes if we derive $F$ from a potential energy function $V(x, y, z)$ according to

$$
\begin{equation*}
F_{x}=-\frac{\partial V}{\partial x} \quad F_{y}=-\frac{\partial V}{\partial y} \quad F_{z}=-\frac{\partial V}{\partial z} \tag{1.10}
\end{equation*}
$$

For example, the $z$ component of crul $F$ becomes

$$
\begin{equation*}
\frac{\partial F_{x}}{\partial y}=-\frac{\partial^{2} V}{\partial y \partial x} \quad \frac{\partial F_{y}}{\partial x}=-\frac{\partial^{2} V}{\partial x \partial y}=-\frac{\partial^{2} V}{\partial y \partial x} \quad \therefore \frac{\partial F_{y}}{\partial x}-\frac{\partial F_{x}}{\partial y}=0 \tag{1.11}
\end{equation*}
$$

This last step follows if we assume that $V$ is everywhere continuous and differentiable. We reach the same conclusion for the other components of crul F. One might wonder whether there are other reasons why crul F might vanish, besides its being derivable from a potential energy function. However, crul $F=0$ is a necessary and sufficient condition for the existence of $V(x, y, z)$ such that Equation 1.10 holds.

We can now express a conservative force $F$ vectorially as

$$
\begin{equation*}
F=-i \frac{\partial V}{\partial x}-j \frac{\partial V}{\partial y}-k \frac{\partial V}{\partial z} \tag{1.12}
\end{equation*}
$$

This equation can be written more succinctly as

$$
\begin{equation*}
F=-\nabla V \tag{1.13}
\end{equation*}
$$

where we have introduced the vector operator del:

$$
\begin{equation*}
\nabla V=i \frac{\partial}{\partial x}+j \frac{\partial}{\partial y}+k \frac{\partial}{\partial z} \tag{1.14}
\end{equation*}
$$

The expression $\nabla V$ is also called the gradient of $V$ and is sometimes written grad $V$. Mathematically, the gradient of a function is a vector that represents the maximum spatial derivative of the function in direction and magnitude. Physically, the negative gradient of the potential energy function gives the
direction and magnitude of the force that acts on a particle located in a field created by other particles. The meaning of the negative sign is that the particle is urged to move in the direction of decreasing potential energy rather than in the opposite direction. This is illustrated in Figure 1.4. Here the potential energy function is plotted out in the form of contour lines representing the curves of constant potential The force at any point is always normal to the equipotential curve or surface passing through the point in question.


Figure 1. 4. A force field represented by equipotential contour curves.
We can express crul $F$ using the del operator. Look at the components of crul $F$ in Equation 1.9. They are the components of the vector $\nabla \times F$. Thus, $\nabla \times F=\operatorname{crul} F$. The condition that a force be conservative can be written compactly as

$$
\begin{equation*}
\nabla \times F=i\left(\frac{\partial F_{z}}{\partial y}-\frac{\partial F_{y}}{\partial z}\right)+j\left(\frac{\partial F_{x}}{\partial z}-\frac{\partial F_{z}}{\partial x}\right)+k\left(\frac{\partial F_{y}}{\partial x}-\frac{\partial F_{x}}{\partial y}\right)=0 \tag{1.15}
\end{equation*}
$$

Furthermore, if $\nabla \times F=0$, then $F$ can be derived from a scalar function $V$ by the operation $F=-\nabla V$, since $\nabla \times \nabla \mathrm{V} \equiv 0$, or the crul of any gradient is identically 0 .

We are now able to generalize the conservation of energy principle to three dimensions. The work done by a conservative force in moving a particle from point A to point B can be written as

$$
\begin{align*}
\int_{A}^{B} F \cdot d r & =\int_{A}^{B} \nabla V(r) \cdot d r=-\int_{A_{x}}^{B_{x}} \frac{\partial V}{\partial x} d x-\int_{A_{x}}^{B_{x}} \frac{\partial V}{\partial y} d y-\int_{A_{x}}^{B_{x}} \frac{\partial V}{\partial z} d z \\
& =-\int_{A}^{B} d V(r)=-\Delta V=V(A)-V(B) \tag{1.16}
\end{align*}
$$

The last step illustrates the fact that $\nabla \mathrm{V} \cdot \mathrm{dr}$ is an exact differential equal to dV . The work done by any net force is always equal to the change in kinetic energy, so

$$
\begin{gather*}
\int_{A}^{B} F \cdot d r=\Delta T=-\Delta V \\
\therefore \Delta(T+V)=0 \tag{1.17}
\end{gather*}
$$

$$
\therefore T(A)+V(A)=T(B)+V(B)=E=\text { constant }
$$

And we arrived at the desired law of conservation of total energy. If $F$ is a nonconservative force, it cannot be set equal to $-\nabla \mathrm{V}$. The work increase $F \cdot d r$ is not an exact differential and cannot be equated to $-d V$. In those cases where both conservative forces F and nonconservative forces $F$ are present, the total work increment is $(F+\dot{F}) \cdot d r=-d V+\dot{F} \cdot d r=d T$, and the generalized form of the work energy theorem becomes

$$
\begin{equation*}
\int_{A}^{B} \dot{F} \cdot d r=\Delta(T+V)=\Delta E \tag{1.18}
\end{equation*}
$$

The total energy $F$ does not remain a constant throughout the motion of the particle but increases or decreases depending upon the nature of the nonconservative force $\hat{F}$. In the case of dissipative forces such as friction and air resistance, the direction of. $\dot{F}$ is always opposite the motion; hence, $\hat{F} \cdot d r$ is negative, and the total energy of the particle decreases as it moves through space.

## Example 1

Given the two-dimensional potential energy function $V(r)=V_{o}-\frac{1}{2} k \delta^{2} e^{-r^{2} / \delta^{2}}$ where $r=i x+j y$ and $V_{o}, k$, and $\delta$ are constants, find the force function.

## Solution:

We first write the potential energy function as a function of $x$ and $y$,

$$
V(x, y)=V_{o}-\frac{1}{2} k \delta^{2} e^{-\left(x^{2}+y^{2}\right) / \delta^{2}}
$$

and then apply the gradient operator:

$$
\begin{aligned}
F & =-\nabla V=-\left(i \frac{\partial}{\partial x}+j \frac{\partial}{\partial y}\right) V(x, y) \\
& =-k(i x+j y) e^{-\left(x^{2}+y^{2}\right) / \delta^{2}} \\
& =-k e^{-r^{2} / \delta^{2}}
\end{aligned}
$$

## Example 2

Suppose a particle of mass $m$ is moving in the above force field, and at time $t=0$ the particle passes through the origin with speed $v_{o}$. What will the speed of the particle be at some small distance away from the origin given by $r=e_{r} \Delta$, where $\Delta \ll \delta$ ?

## Solution:

The force is conservative, because a potential energy function exists. Thus, the total energy $E=T+V=$ constant,

$$
E=\frac{1}{2} m v^{2}+V(r)=\frac{1}{2} m v_{o}^{2}+V(0)
$$

and solving for $v$, we obtain

$$
\begin{aligned}
& v^{2}=v_{o}^{2}+\frac{2}{m}[V(0)-V(r)] \\
& =v_{o}^{2}+\frac{2}{m}\left[\left(V_{o} \frac{1}{2} k \delta^{2}\right)-\left(V_{o}-\frac{1}{2} k \delta^{2} e^{-\Delta^{2} / \delta^{2}}\right)\right] \\
& \approx v_{o}^{2}-\frac{k \delta^{2}}{m}\left[1-e^{-\Delta^{2} / \delta^{2}}\right] \\
& =v_{o}^{2}-\frac{k \delta^{2}}{m}\left[1-\left(1-\Delta^{2} / \delta^{2}\right)\right]
\end{aligned}
$$

The potential energy is a quadratic function of the displacement from the origin for small displacements, so this solution reduces to the conservation of for the simple harmonic oscillator.

## Example 3

A particle of mass $m$ moving in three dimensions under the potential function $V(x, y, z)=a x+\beta y^{2}+\gamma z^{3}$ has speed $v_{o}$ when it passes through the origin.
(a) What will its speed be if and when it passes through the point $(1,1,1)$ ?
(b) If the point $(1,1,1)$ is a turning point in the motion $(v=0)$, what is $v_{0}$ ?
(c) What are the component differential equations of motion of the particle?
(Note: It is not necessary to solve the differential equations of motion in this problem.) H.W

## Example 4

Show that the variation of gravity with height can be accounted for approximately by the following potential energy function:

$$
V=m g z\left(1-\frac{Z}{r_{e}}\right)
$$

in which re is the radius of the Earth. Find the force given by the above potential function. From this find the component differential equations of motion of a projectile under such a force. If the vertical component of the initial velocity is how high does the projectile go? H.W

### 1.2 Forces of the Separable Type: Projectile Motion

A Cartesian coordinate system can be frequently chosen such that the components of a force field involve the respective coordinates alone, that is,

$$
\begin{equation*}
F=i F_{x}(x)+j F_{y}(y)+k F_{z}(z) \tag{1.19}
\end{equation*}
$$

Forces of this type are separable. The crul of such a force is identically zero:

$$
\nabla \times F=\left|\begin{array}{ccc}
i & j & k  \tag{1.20}\\
\partial / \partial x & \partial / \partial y & \partial / \partial z \\
F_{x}(x) & F_{y}(y) & F_{z}(z)
\end{array}\right|
$$

## No Air Resistance

For simplicity, we first consider the case of a projectile moving with no air resistance. Only one force, gravity, acts on the projectile, and, consistent with Galileo's observations as we shall see, it affects only its vertical motion. Choosing the $z$-axis to be vertical, we have the following equation of motion:

$$
\begin{equation*}
m=\frac{d^{2} r}{d t^{2}}=-k m g \tag{1.21}
\end{equation*}
$$

In the case of projectiles that don't rise too high or travel too far, we can take the acceleration of gravity, g , to be constant. Then the force function is conservative and of the separable type, because it is a special case of Equation 1.19. $v_{o}$ is the initial speed of the projectile, and the origin of the coordinate system is its initial position. Because there are no horizontally directed forces acting on the projectile, the motion occurs solely in the $x z$ vertical plane. Thus, the position of the projectile at any time is (see Figure1.5)


Figure 1. 5. The parabolic path of a projectile.

$$
\begin{equation*}
r=i x+k z \tag{1.22}
\end{equation*}
$$

The speed of the projectile can be calculated as a function of its height, $z$, using the energy equation (Equation 1.17)

$$
\begin{equation*}
\frac{1}{2} m\left(\dot{x}^{2}+\dot{z}^{2}\right)+m g z=\frac{1}{2} m v_{o}^{2} \tag{1.23}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
v^{2}=v_{o}^{2}-2 g z \tag{1.23b}
\end{equation*}
$$

We can calculate the velocity of the projectile at any instant of time by integrating Equation 1.21

$$
\begin{equation*}
v=\frac{d r}{d t}=-k g t+v_{o} \tag{1.24a}
\end{equation*}
$$

The constant of integration is the initial velocity $v_{0}$. In terms of unit vectors, the velocity is $v$

$$
\begin{equation*}
v=i v_{o} \cos \alpha+k\left(v_{o} \sin \alpha-g t\right) \tag{1.24b}
\end{equation*}
$$

Integrating once more yields the position vector

$$
\begin{equation*}
r=-k \frac{1}{2} g t^{2}+v_{o} t+r_{o} \tag{1.25a}
\end{equation*}
$$

The constant of integration is the initial position of the projectile, $r_{o}$, which is equal to zero; therefore, in terms of unit vectors, Equation 1.25a becomes

$$
\begin{equation*}
r=i\left(v_{o} \cos \alpha\right) t+k\left(\left(v_{o} \sin \alpha\right) t-\frac{1}{2} g t^{2}\right) \tag{1.25b}
\end{equation*}
$$

In terms of components, the position of the projectile at any instant of time is

$$
\begin{align*}
& x=\dot{x}_{o} t=\left(v_{o} \cos \alpha\right) t \\
& y=\dot{y}_{o} t \equiv 0 \tag{1.25c}
\end{align*}
$$

$$
z=\dot{z}_{o} t-\frac{1}{2} g t^{2}=\left(v_{o} \sin \alpha\right) t \frac{1}{2} g t^{2}
$$

$\dot{x}_{o}=v_{0} \cos \alpha, \dot{y}_{0}=0$, and $\dot{z}_{o}=v_{0} \sin \alpha$ are the components of the initial velocity $v_{0}$.
We can now show, as Galileo did in 1609, that the path of the projectile is a parabola. We find $z(x)$ by using the first of Equations 1.25 c to solve fort as a function of $x$ and then substitute the resulting expression in the third of Equations 1.25 c

$$
\begin{gather*}
t=\frac{x}{v_{o} \cos \alpha}  \tag{1.26}\\
z=(\tan \alpha) x-\left(\frac{g}{2 v_{o}^{2} \cos ^{2} \alpha}\right) x^{2} \tag{1.27}
\end{gather*}
$$

Equation 1.27 is the equation of a parabola and is shown in Figure 1.5

## The maximum height obtained by the projectile

We can calculate the maximum height of the projectile using Equation 1.23b and noting that at maximum height the vertical component of the velocity of the projectile is zero so that its velocity is in the horizontal direction and equal to the constant horizontal component, $v_{o} \cos \alpha$. Thus

$$
\begin{equation*}
v_{o}^{2} \cos ^{2} \alpha=v_{o}^{2} 2 g z_{\max } \tag{1.28}
\end{equation*}
$$

We solve this to obtain

$$
\begin{equation*}
z_{\max }=\frac{v_{0} \sin ^{2} \alpha}{2 g} \tag{1.29}
\end{equation*}
$$

The time it takes to reach maximum height can be obtained from Equation 1.24 b where we again make use of the fact that at maximum height, the vertical component of the velocity vanishes, so

Or

$$
\begin{gather*}
v_{o} \sin \alpha-g t_{\max }=0 \\
t_{\max }=\frac{v_{o} \sin \alpha}{g} \tag{1.30}
\end{gather*}
$$

We can obtain the total time of flight $T$ of the projectile by setting $z=0$ in the last of Equations 1.25 c , which yields

$$
\begin{equation*}
T=\frac{2 v_{o} \sin \alpha}{g} \tag{1.31}
\end{equation*}
$$

This is twice the time it takes the projectile to reach maximum height. This indicates that the upward flight of the projectile to the apex of its trajectory is symmetrical to its downward flight away from it.

Finally, we calculate the range of the projectile by substituting the total time of flight, $T$, into the first of Equations $1.25 c$, obtaining

$$
\begin{equation*}
R=x=\frac{v_{o}^{2} \sin ^{2} 2 \alpha}{g} \tag{1.32}
\end{equation*}
$$

$R$ has its maximum value $R_{\max }=v_{o}^{2} / g$ at $\alpha=45^{\circ}$

## Linear Air Resistance

In this case, the motion does not conserve total energy, which continually diminishes during the flight of the projectile. To solve the problem analytically, we assume that the resisting force varies linearly with the velocity. To simplify the resulting equation of motions, we take the constant of proportionality to be $m \gamma$ where $m$ is the mass of the projectile. The equation of motion is then

$$
\begin{equation*}
m \frac{d^{2} r}{d t^{2}}=-m \gamma v-k m g \tag{1.33}
\end{equation*}
$$

Upon cancelling $m^{\prime} s$, the equation simplifies to

$$
\begin{equation*}
\frac{d^{2} r}{d t^{2}}=-\gamma v-k g \tag{1.34}
\end{equation*}
$$

Before integrating, we write Equation 1.34 in component form

$$
\begin{align*}
& \ddot{\ddot{ }}=-\gamma \dot{x} \\
& \ddot{y}=-\gamma \dot{y}  \tag{1.35}\\
& \ddot{z}=-\gamma \dot{z}-g
\end{align*}
$$

We see that the equations are separated; therefore, each can be solved individually. We can write down the solutions immediately, noting that here $\gamma=c_{1} / m, c_{1}$ being the linear drag coefficient. The results are

$$
\begin{align*}
& \dot{x}=x_{o} e^{-\gamma t} \\
& \dot{y}=y_{o} e^{-\gamma t}  \tag{1.35}\\
& \dot{z}=z_{o} e^{-\gamma t}-\frac{g}{\gamma}\left(1-e^{-\gamma t}\right)
\end{align*}
$$

for the velocity components. As before, we orient the coordinate system such that the x -axis lies along the projection of the initial velocity onto the $x y$ horizontal plane. Then $\dot{y}=\dot{y}_{o}=0$ and the motion is confined to the $x z$ vertical plane. Integrating once more, we obtain the position coordinates

$$
\begin{align*}
& x=\frac{\dot{x}_{o}}{\gamma}\left(1-e^{-\gamma t}\right)  \tag{1.36}\\
& z=\left(\frac{\dot{z}_{o}}{\gamma}+\frac{g}{\gamma^{2}}\right)\left(1-e^{-\gamma t}\right)-\frac{g}{\gamma} t
\end{align*}
$$

We have taken the initial position of the projectile to be zero, the origin of the coordinate system. This solution can be written vectorially as

$$
\begin{equation*}
r=\left(\frac{v_{o}}{\gamma}+\frac{k g}{\gamma^{2}}\right)\left(1-e^{-\gamma t}\right)-k \frac{g t}{\gamma} \tag{1.37}
\end{equation*}
$$

Contrary to the case of zero air resistance the path of the projectile is not a parabola, but rather a curve that lies below the corresponding parabolic trajectory. This is illustrated in Figure 1.6. Inspection of the $x$ equation shows that, for large $t$, the value of $x$ approaches the limiting value

$$
\begin{equation*}
x \rightarrow \frac{\dot{x}_{o}}{\gamma} \tag{1.38}
\end{equation*}
$$



Figure 1. 6. Comparison of the paths of a projectile with and without air resistance.

## Horizontal Range

The horizontal range of a projectile with linear air drag is found by setting $z=0$ in the second of Eq. 1.36 and then eliminating t among the two equations. From the first of Eq. 1.36, we have $1-\gamma x / \dot{x_{o}}=e^{-\gamma t}$, so $t=-\gamma^{-1} \ln \left(1-\gamma x / \dot{x}_{0}\right)$, thus the horizontal range $x_{\max }$ is given by implicit expression

$$
\begin{equation*}
\left(\frac{\dot{z}_{o}}{\gamma}+\frac{g}{\gamma^{2}}\right) \frac{\gamma x_{\max }}{x_{o}}+\frac{g}{\gamma^{2}} \ln \left(1-\frac{\gamma x_{\max }}{x_{o}}\right)=0 \tag{1.39}
\end{equation*}
$$

This is a transcendental equation and must be solved by some approximation method to find $x_{h}$. We can expand the logarithmic term by use of the series

$$
\begin{equation*}
\ln (1-u)=-u-\frac{u^{2}}{2}-\frac{u^{3}}{3}-\cdots \tag{1.40}
\end{equation*}
$$

which is valid for $|u|<1$. With $u=\gamma x_{\max } / \dot{x}_{o}$ it is left as a problem to show that this leads to the following expression for the horizontal range:

$$
\begin{equation*}
x_{\max }=\frac{2 \dot{x}_{o} \dot{z}_{o}}{g}-\frac{8 \dot{x}_{o} \dot{z}_{o}^{2}}{3 g^{2}} \gamma+\cdots \tag{1.41a}
\end{equation*}
$$

If the projectile is fired at angle of elevation $\alpha$ with initial speed $v_{0}$, then $\dot{x}_{0}=v_{0} \cos \alpha, \dot{z}_{0}=v_{0} \sin \alpha$ and $2 \dot{x}_{o} \dot{z}_{o}=2 v_{0}^{2} \sin \alpha \cos \alpha=v_{0}^{2} \sin 2 \alpha$. An equivalent expression is then

$$
\begin{equation*}
x_{\max }=\frac{v_{o}^{2} \sin 2 \alpha}{g}-\frac{4 v_{o}^{3} \sin 2 \alpha \sin \alpha}{3 g^{2}} \gamma+\cdots \tag{1.41b}
\end{equation*}
$$

The first term on the right is the range in the absence of air resistance. The remainder is the decrease due to air resistance.

### 1.3 The Harmonic Oscillator in Two and Three Dimensions

Consider the motion of a particle subject to a linear restoring force that is always directed toward a fixed point, the origin of our coordinate system. Such a force can be represented by the expression

$$
\begin{equation*}
F=-k r \tag{1.42}
\end{equation*}
$$

Accordingly, the differential equation of motion is simply expressed as

$$
\begin{equation*}
m \frac{d^{2} r}{d t^{2}}=-k r \tag{1.42}
\end{equation*}
$$

The situation can be represented approximately by a particle attached to a set of elastic springs as shown in Figure1.7. This is the three-dimensional generalization of the linear oscillator. Equation 1.42 is the differential equation of the linear isotropic oscillator.


Figure 1. 7. A model of a three-dimensional harmonic oscillator.

## The Two-Dimensional Isotropic Oscillator

In the case of motion in a single plane, Eq. 1.42 is equivalent to the two component equations

$$
\begin{align*}
& m \ddot{x}=-k x  \tag{1.43}\\
& m \ddot{y}=-k y
\end{align*}
$$

These are separated, and we can immediately write down the solutions in the form

$$
\begin{equation*}
x=A \cos (w t+\alpha) \quad y=B \cos (w t+\beta) \tag{1.44}
\end{equation*}
$$

In which

$$
\begin{equation*}
w=\left(\frac{k}{m}\right)^{1 / 2} \tag{1.45}
\end{equation*}
$$

The constants of integration $A, B, \alpha$ and $\beta$ are determined from the initial conditions in any given case. To find the equation of the path, we eliminate the time $t$ between the two equations. To do this, let us write the second equation in the form

$$
\begin{equation*}
y=B \cos (w t+\alpha+\Delta) \tag{1.46}
\end{equation*}
$$

Where

$$
\begin{equation*}
\Delta=\beta-\alpha \tag{1.47}
\end{equation*}
$$

Then $\quad y=B[\cos (w t+\alpha) \cos \Delta-\sin (w t+\alpha) \sin \Delta]$
Combining the above with the first of Eq. 1.44 , we then have

$$
\begin{equation*}
\frac{y}{B}=\frac{x}{A} \cos \Delta-\left[1-\frac{x^{2}}{y^{2}}\right]^{1 / 2} \sin \Delta \tag{1.47}
\end{equation*}
$$

and upon transposing and squaring terms, we obtain

$$
\begin{equation*}
\frac{x^{2}}{A^{2}}-x y \frac{2 \cos \Delta}{A B}+\frac{y^{2}}{B^{2}}=\sin ^{2} \Delta \tag{1.48}
\end{equation*}
$$

which is a quadratic equation in $x$ and $y$. Now the general quadratic

$$
\begin{equation*}
a x^{2}+b x y+c y^{2}+d x+e y=f \tag{1.49}
\end{equation*}
$$

represents an ellipse, a parabola, or a hyperbola, depending on whether the discriminant

$$
\begin{equation*}
b^{2}-4 a c \tag{1.50}
\end{equation*}
$$

is negative, zero, or positive, respectively. In our case the discriminant is equal to $-(2 \sin \Delta / A B)^{2}$, which is negative, so the path is an ellipse as shown in Figure 1.8.

In particular, if the phase difference $\Delta$ is equal to $\pi / 2$, then the equation of the path reduces to the equation

$$
\begin{equation*}
\frac{x^{2}}{A^{2}}+\frac{y^{2}}{B^{2}}=1 \tag{1.51}
\end{equation*}
$$

which is the equation of an ellipse whose axes coincide with the coordinate axes. On the other hand, if the phase difference is 0 or $\pi$ then the equation of the path reduces to that of a straight line, namely,

$$
\begin{equation*}
y= \pm \frac{B}{A} x \tag{1.52}
\end{equation*}
$$



Figure 1. 8. The elliptical path of a two-dimensional isotropic oscillator
The positive sign is taken if $\Delta=0$, and the negative sign, if $\Delta=\pi$ the general case it is possible to show that the axis of the elliptical path is inclined to the $x$-axis by the angle $\Psi$ where

$$
\begin{equation*}
\tan 2 \psi=\frac{2 A B \cos \Delta}{A^{2}-B^{2}} \tag{1.53}
\end{equation*}
$$

The derivation is left as an exercise.

## The Three-Dimensional Isotropic

## Harmonic Oscillator

In the case of three-dimensional motion, the differential equation of motion is equivalent to the three equations

$$
\begin{equation*}
m \ddot{x}=-k x \quad m \ddot{y}=-k y \quad m \ddot{z}=-k z \tag{1.54}
\end{equation*}
$$

which are separated. Hence, the solutions maybe written in the form of Eq. 1.44, or, alternatively, we may write

$$
\begin{align*}
& x=A_{1} \sin w t+B_{1} \cos w t \\
& y=A_{2} \sin w t+B_{2} \cos \omega t  \tag{1.55a}\\
& z=A_{3} \sin w t+B_{3} \cos w t
\end{align*}
$$

The six constants of integration are determined from the initial position and velocity of the particle. Now Eq. 1.54 can be expressed vectorially as

$$
\begin{equation*}
r=A \sin w t+B \cos w t \tag{1.55b}
\end{equation*}
$$

in which the components of $A$ are $A_{1}, A_{2}$, and $A_{3}$, and similarly for $B$. It is clear that the motion takes place entirely in a single plane, which is common to the two constant vectors $A$ and $B$, and that the path
of the particle in that plane is an ellipse, as in the two dimensional case. Hence, the analysis concerning the shape of the elliptical path under the two-dimensional case also applies to the three-dimensional case.

## Non-isotropic Oscillator

The restoring force is independent of the direction of the displacement. If the magnitudes of the components of the restoring force depend on the direction of the displacement, we have the case of the non-isotropic oscillator. For a suitable choice of axes, the differential equations for the non-isotropic case can be written

$$
\begin{align*}
& m \ddot{x}=-k_{1} x \\
& m \ddot{y}=-k_{2} y  \tag{1.56}\\
& m \ddot{z}=-k_{3} y
\end{align*}
$$

Here we have a case of three different frequencies of oscillation, $w_{1}=\sqrt{k_{1} / m}, w_{2}=\sqrt{k_{2} / m}$, and $w_{3}=$ $\sqrt{k_{3} / m}$ and the motion is given by the solutions

$$
\begin{gather*}
x=A \cos \left(w_{1} t+\alpha\right) \\
y=B \cos \left(w_{2} t+\beta\right)  \tag{1.57}\\
z=\operatorname{Cosos}\left(w_{3} t+\gamma\right)
\end{gather*}
$$

Again, the six constants of integration in the above equations are determined from the initial conditions. The resulting oscillation of the particle lies entirely within a rectangular box (whose sides are $2 A, 2 B$, and 2C) centered on the origin. In the event that $w_{1}, w_{2}$, and $w_{3}$, are commensurate-that is, if

$$
\begin{equation*}
\frac{w_{1}}{n_{1}}=\frac{w_{2}}{n_{2}}=\frac{w_{3}}{n_{3}} \tag{1.58}
\end{equation*}
$$

where $n_{1}, n_{2}$, and $n_{3}$ are integers-the path, called a Lissajous figure, is closed, because after a time $2 \pi n_{1} / w_{1}=2 \pi n_{2} / w_{2}=2 \pi n_{3} / w_{3}$ the particle returns to its initial position and the motion is repeated

## Energy Considerations

we showed that the potential energy function of the one-dimensional harmonic oscillator is quadratic in the displacement, $V(x)=\frac{1}{2} k x^{2}$. For the general three-dimensional case, it is easy to verify that

$$
\begin{equation*}
\left.V(x, y, z)=\frac{1}{2} k_{1} x^{2}+\frac{1}{2} k_{2} y^{2}+\frac{1}{2} k_{3} z^{2}\right) \tag{1.59}
\end{equation*}
$$

Because $F_{x}=-\partial V / \partial x=-k_{1} x$ and similarly for $F_{y}$ and $F_{z}$. If $k_{1}=k_{2}=k_{3}=k$, we have the isotropic case and

$$
\begin{equation*}
V(x, y, z)=\frac{1}{2} k\left(x^{2}+y^{2}+z^{2}\right)=\frac{1}{2} k r^{2} \tag{1.60}
\end{equation*}
$$

The total energy in the isotropic case is then given by the simple expression

$$
\begin{equation*}
\frac{1}{2} m v^{2}+\frac{1}{2} k r^{2}=E \tag{1.61}
\end{equation*}
$$

## Example 5

A particle of mass $m$ moves in two dimensions under the following potential energy function: $V(r)=$ $\frac{1}{2} k\left(x^{2}+4 y^{2}\right)$ Find the resulting motion, given the initial condition at $t=0: x=a, y=0, \dot{x}=0, \dot{y}=v_{o}$

## Solution

This is a non-isotropic oscillator potential. The force function is

$$
F=-\nabla V=-i k x-j 4 k y=m \ddot{r}
$$

The component differential equations of motion are then

$$
m \ddot{x}+k x=0 \quad m \ddot{y}+4 k y=0
$$

The $x$-motion has angular frequency $w=(k / m)^{1 / 2}$ while the $y$-motion has angular frequency just twice that, namely, $w_{y}=(4 k / m)^{1 / 2}=2 w$. We shall write the general solution in the form

$$
\begin{gathered}
x=A_{1} \cos w t+B_{1} \sin w t \\
Y=A_{2} \cos 2 w t+B_{2} \sin 2 w t
\end{gathered}
$$

To use the initial condition we must first differentiate with respect to $t$ to find the general expression for the velocity components:

$$
\begin{gathered}
\dot{x}=-A_{1} w \sin w t+B_{1} w \cos w t \\
\dot{y}=-2 A_{2} w \sin 2 w t+2 B_{1} w \cos 2 w t
\end{gathered}
$$

Thus, at $t=0$, we see that the above equations for the components of position and velocity reduce to

$$
a=A_{1} \quad 0=A_{2} \quad 0=B_{1} w \quad v_{0}=2 B_{2} w
$$

These equations give directly the values of the amplitude coefficients $A_{1}=a, A_{2}=B_{1}=0$, and $B_{2}=$ $v_{o} / 2 w$, so the final equations for the motion are

$$
\begin{gathered}
x=a \cos w t \\
y=\frac{v_{o}}{2 w} \sin 2 w t
\end{gathered}
$$

The path is a Lissajous figure having the shape of a figure-eight as shown in Figure 1.9.

ANALYTICAL MECHANIC. 2

LECTURE 1

MOTION OF PARTICLES
IN THREE DIMENSIONS
Dr K.T. Hassan Physics Dep. $2^{\text {nd }}$ Stage


Figure 1. 9. A Lissajous figure.

