University of Anbar

College of Engineering

Civil Engineering Department

LECTURE NOTE

COURSE CODE- CE 2208

CALCULUS III

By

Dr. Maher Shakir Mahmood

Course Description:

This course is the third part of our standard four-semester calculus sequence. It includes vector calculus; functions of several variables; differentials and applications; and double and triple integrals.

Course Objectives/Goals:

The goals of this course are to enable students to:

1. Learn the basics of the calculus of functions of two and three variables.
2. Study vectors in three-dimensional space, derivatives, and integrals.
3. Apply these ideas to a wide range of problems like motion in space, optimization, arc length, etc.

Course Learning Outcomes:

By the end of successful completion of this course, the student will be able to:

1. Visualize geometry in three-dimensional space;
2. Perform the calculus of scalar functions of several variables and the calculus of vector functions;
3. Do calculus operations on multivariable functions, including partial derivatives, directional derivatives, and multiple integrals;
4. Apply concepts of multivariable calculus to real world problems.

Text Book(s):

- Anton, Howard, Irl C. Bivens, and Stephen Davis. Calculus Single Variable. John Wiley & Sons, 2012.

Recommended readings:

- Any materials on Calculus III like lecture notes or books that are available online.

Weekly Distribution of Course Topics/Contents

Week	Topic
1.	Rectangular Coordinate systems in 3-space. Vectors
2.	Dot product, projections. Cross product
3.	Parametric equations of a line. Planes in 3-space
4.	Introduction to vector-valued functions. Calculus of vector-valued functions
5.	Change of parameters, Arc Length. Unit Tangent, Normal and Binormal vectors
6.	Curvature
7.	Quadric Surfaces. Functions of two or more variables
8.	**Mid-term Exam**
9.	Limits and continuity. Partial derivatives
10.	Differentiability, Local Linearity. The Chain rule
11.	Directional derivatives and gradients. Tangent planes and normal vectors
12.	Maxima and minima of functions of two variables. Lagrange multipliers
13.	Double integrals. Double integrals over non rectangular regions
14.	Double integrals in polar coordinates. Triple integrals
15.	Cylindrical and spherical coordinates, Triple integrals in cylindrical and Spherical coordinates

Students' Assessment:

Students are assessed as follows:

Assessment Tool(s)	Date	Weight (%)
Semester activities. These include quizzes, homework, and classroom interactions	Week-15	10%
Mid semester exam	Week-7	20%
Progress exam	Week-4 and week-11	10%
Final Exam	Week-16	60%
Total		**100%**

CHAPTER ONE
RECTANGULAR COORDINATE SYSTEMS IN 3-SPACE AND VECTORS

1.1 RECTANGULAR COORDINATE SYSTEMS IN 3-SPACE

- It will be called three-dimensional space ***3-space***, two-dimensional space (a plane) ***2-space***, and one-dimensional space (a line) ***1-space***.

- To locate a point in a plane, this point has 2 dimensional coordinates (a, b). a is called x-coordinate and b is called y-coordinate.

- To locate a point in a space, three coordinates are required. This point has 3 dimensional coordinates (a, b, c). Points in 3-space can be placed in one-to-one correspondence with triples of real numbers by using three mutually perpendicular coordinate lines, called the ***x-axis***, the ***y-axis***, and the ***z-axis***, positioned so that their origins coincide (Figure 1-1).

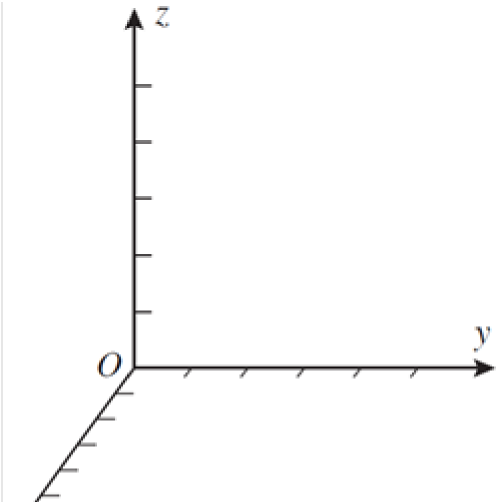


Figure 1-1

- The three coordinate axes form a three dimensional ***rectangular coordinate system*** (or ***Cartesian coordinate system***).

- The point of intersection of the coordinate axes is called the ***origin*** of the coordinate system.

- The coordinate axes, taken in pairs, determine three ***coordinate planes***: the ***xy-plane***, the ***xz-plane***, and the ***yz-plane*** (Figure 1-2).

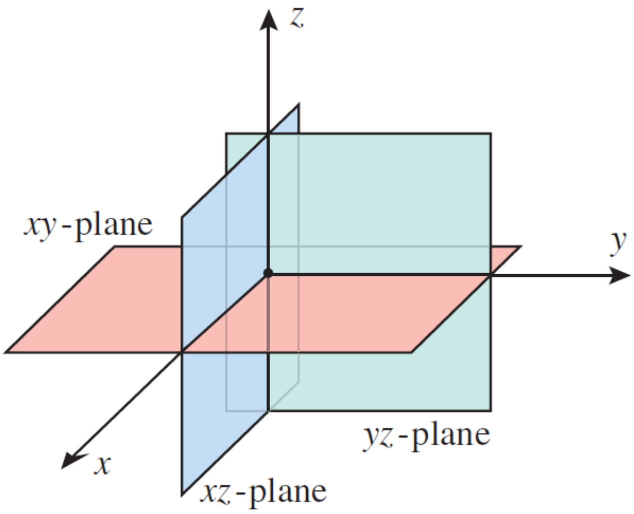


Figure 1-2

- To each point P in 3-space, we can assign a triple of real numbers by passing three planes through P parallel to the coordinate planes and letting a, b, and c be the coordinates of the intersections of those planes with the x-axis, y-axis, and z-axis, respectively (Figure 1-3). We call a, b, and c the ***x-coordinate***, ***y-coordinate***, and ***z-coordinate*** of P, respectively, and we denote the point P by (a, b, c).

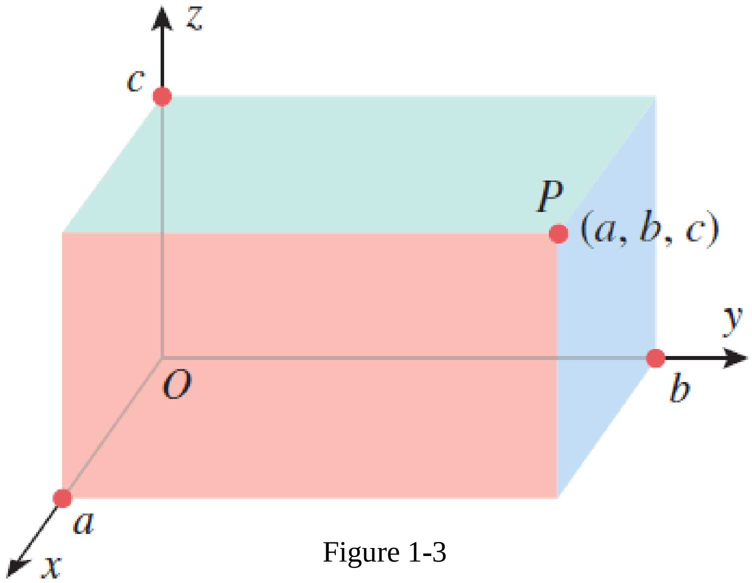


Figure 1-3

-Just as the coordinate axes in a two-dimensional coordinate system divide 2-space into four ***quadrants***, so the coordinate planes of a three-dimensional coordinate system divide 3-space into eight parts, called ***octants***. The set of points with three positive coordinates forms the ***first octant***; the remaining octants have no standard numbering.

-You should be able to visualize the following facts about three-dimensional rectangular co-ordinate systems:

REGION	DESCRIPTION
xy-plane	Consists of all points of the form $(x, y, 0)$
xz-plane	Consists of all points of the form $(x, 0, z)$
yz-plane	Consists of all points of the form $(0, y, z)$
x-axis	Consists of all points of the form $(x, 0, 0)$
y-axis	Consists of all points of the form $(0, y, 0)$
z-axis	Consists of all points of the form $(0, 0, z)$

1.1.1 Distance in 3-Space

In 2-space, the distance d between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance formula in 3-space has the same form, but it has a third term to account for the added dimension. The distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 1.1 Find the distance *d* between the points (2, 3, −1) and (4, −1, 3).

Solution:

$$d = \sqrt{(4-2)^2 + (-1-3)^2 + (3+1)^2} = \sqrt{36} = 6$$

- The standard equation of the circle in 2-space that has centre (x_0, y_0) and radius r is

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

- The ***standard equation of the sphere*** in 3-space that has centre (x_0, y_0, z_0) and radius r is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Example 1.2:

EQUATION	GRAPH
$(x-3)^2 + (y-2)^2 + (z-1)^2 = 9$	Sphere with center (3, 2, 1) and radius 3
$(x+1)^2 + y^2 + (z+4)^2 = 5$	Sphere with center (−1, 0, −4) and radius $\sqrt{5}$
$x^2 + y^2 + z^2 = 1$	Sphere with center (0, 0, 0) and radius 1

Example 1.3 Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 - 2x - 4y + 8z + 17 = 0$$

$$(x^2 - 2x) + (y^2 - 4y) + (z^2 + 8z) = -17$$

$$(x^2 - 2x + 1) + (y^2 - 4y + 4) + (z^2 + 8z + 16) = -17 + 21$$

$$(x - 1)^2 + (y - 2)^2) + (z + 4)^2 = 4$$

From the equation, the centre of the sphere with (1, 2, −4) and radius 2.

- In general, completing the squares in the previous equation produces an equation of the form

$$\boldsymbol{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = k}$$

If $k > 0$, then the graph of this equation is a sphere with centre (x_0, y_0, z_0) and radius $\sqrt{k}$.

If $k = 0$, then the sphere has radius zero, so the graph is the single point (x_0, y_0, z_0).

If $k < 0$, the equation is not satisfied by any values of x, y, and z (why?), so it has no graph.

1.1.2 CYLINDRICAL SURFACES

- Although it is natural to graph equations in two variables in 2-space and equations in three variables in 3-space, it is also possible to graph equations in two variables in 3-space.
- For example, the graph of the equation $y = x^2$ in an *xy*-coordinate system is a parabola; however, there is nothing to prevent us from inquiring about its graph in an *xyz*-coordinate system. To obtain this graph we need only observe that the equation $y = x^2$ does not impose any restrictions on z. Thus, if we find values of x and y that satisfy this equation, then the coordinates of the point (x, y, z) will also satisfy the equation for *arbitrary* values of z.
- Geometrically, the point (x, y, z) lies on the vertical line through the point $(x, y, 0)$ in the *xy*-plane, which means that we can obtain the graph of $y = x^2$ in an *xyz*-coordinate system by first graphing the equation in the *xy*-plane and then translating that graph parallel to the *z*-axis to generate the entire graph (Figure 1-4).

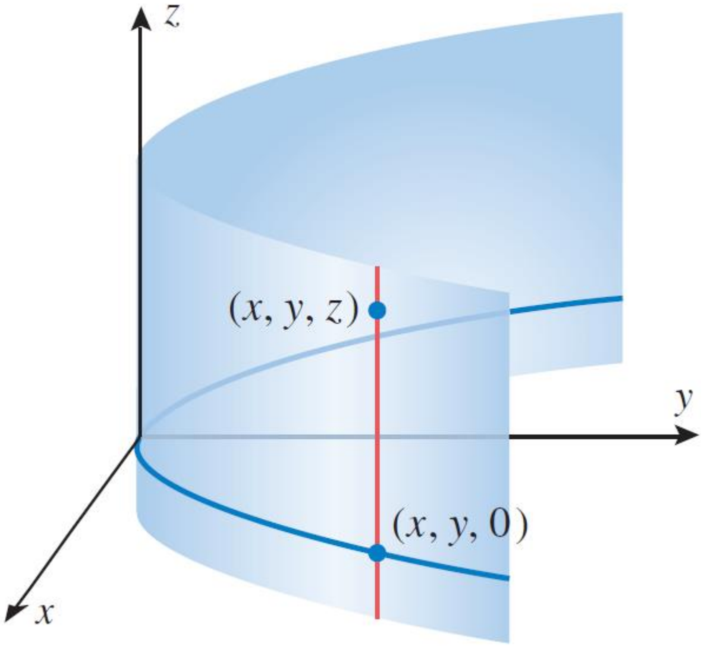
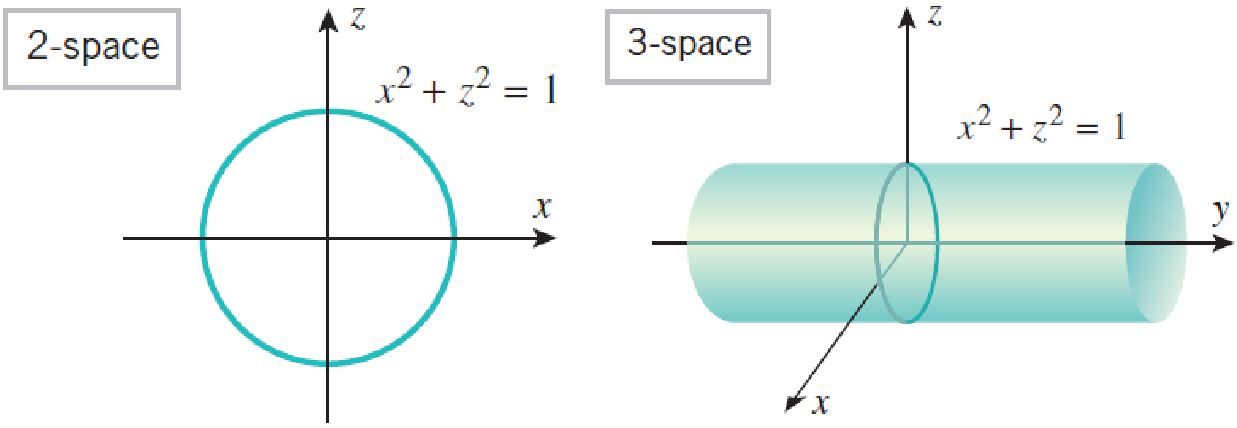


Figure 1-4

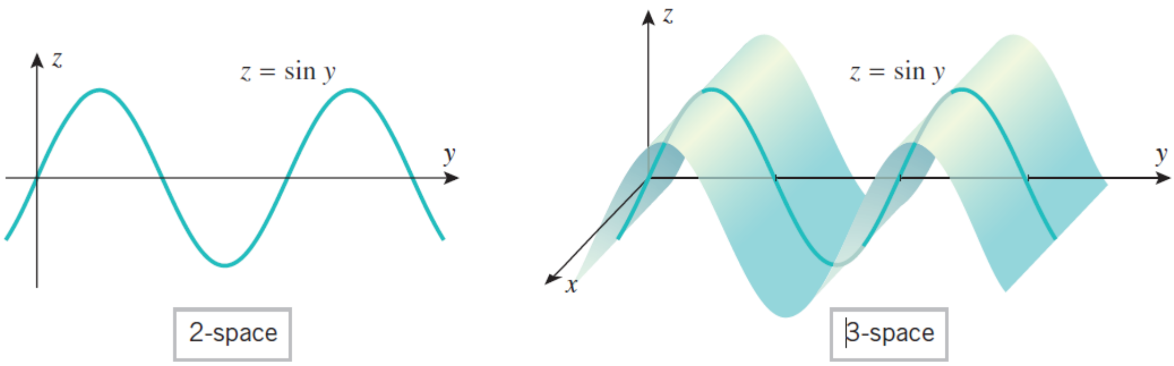
- The process of generating a surface by translating a plane curve parallel to some line is called ***extrusion***, and surfaces that are generated by extrusion are called ***cylindrical surfaces***.
- A familiar example is the surface of a right circular cylinder, which can be generated by translating a circle parallel to the axis of the cylinder.

Theorem: *An equation that contains only two of the variables x, y, and z represents a cylindrical surface in an xyz-coordinate system. The surface can be obtained by graphing the equation in the coordinate plane of the two variables that appear in the equation and then translating that graph parallel to the axis of the missing variable.*

Example 1.4 Sketch the graph of $x^2 + z^2 = 1$ in 3-space.



Example 1.5 Sketch the graph of z = sin(y) in 3-space.



Exercises

1. The distance between the points $(1, -2, 0)$ and $(4, 0, 5)$ is …………..

2. The graph of $(x - 3)2 + (y - 2)2 + (z + 1)2 = 16$ is a ……… of radius centered at ………....

3. The shortest distance from the point $(4, 0, 5)$ to the sphere $(x - 1)^2 + (y + 2)^2 + z^2 = 36$ is ……………….

4. Let S be the graph of $x^2 + z^2 + 6z = 16$ in 3-space.

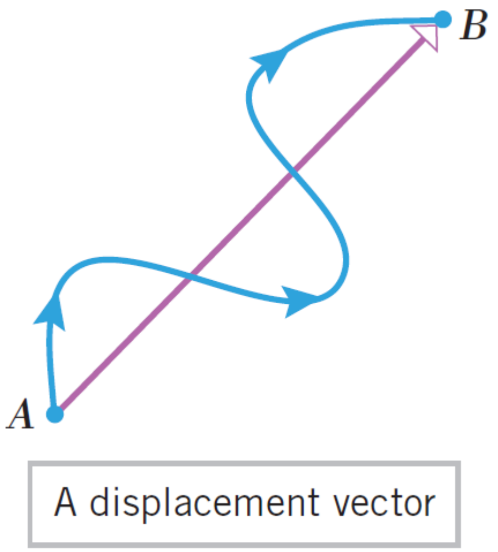
(a) The intersection of S with the xz-plane is a circle with centre ……… and radius ………. .

(b) The intersection of S with the xy-plane is two lines, x = ……….. and x = …………..

(c) The intersection of S with the yz-plane is two lines, z = ……….. and z = …………..

1.2 VECTOR

- Scalars are physical quantities such as area, length, mass, and temperature and completely described once the magnitude of the quantity is given.
- Other physical quantities, called "vectors," are not completely determined until both a magnitude and a direction are specified. There are many examples like force, velocity and displacement.
- A particle that moves along a line can move in only two directions, so its direction of motion can be described by taking one direction to be positive and the other negative. Thus, the displacement or change in position of the point can be described by a signed real number.
- For example, a displacement of +3 describes a position change of 3 units in the positive direction, and a displacement of −3 describes a position change of 3 units in the negative direction.
- However, for a particle that moves in two dimensions or three dimensions, a plus or minus sign is no longer sufficient to specify the direction of motion—other methods are required.
- One method is to use an arrow, called a ***vector***, that points in the direction of motion and whose length represents the distance from the starting point to the ending point; this is called the ***displacement vector*** for the motion. See Figure 1-5.



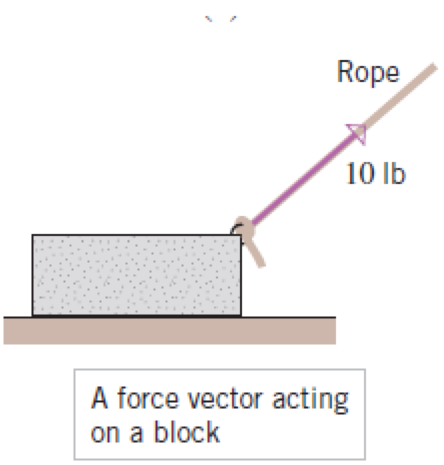


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Figure 1-5

1.2.1 Geometric vectors

- Vectors can be represented geometrically by arrows in 2-space or 3-space; the direction of the arrow specifies the direction of the vector, and the length of the arrow describes its magnitude.
- The tail of the arrow is called the ***initial point*** of the vector, and the tip of the arrow the ***terminal point***.
- We will denote vectors with lowercase boldface type such as **a**, **k**, **v**, **w**, and **x**. Two vectors, **v** and **w**, are considered to be ***equal*** (also called ***equivalent***) if they have the same length and same direction, in which case we write **v** = **w**.
- If the initial and terminal points of a vector coincide, then the vector has length zero; we call this the zero vector and denote it by 0. The zero vector does not have a specific direction.

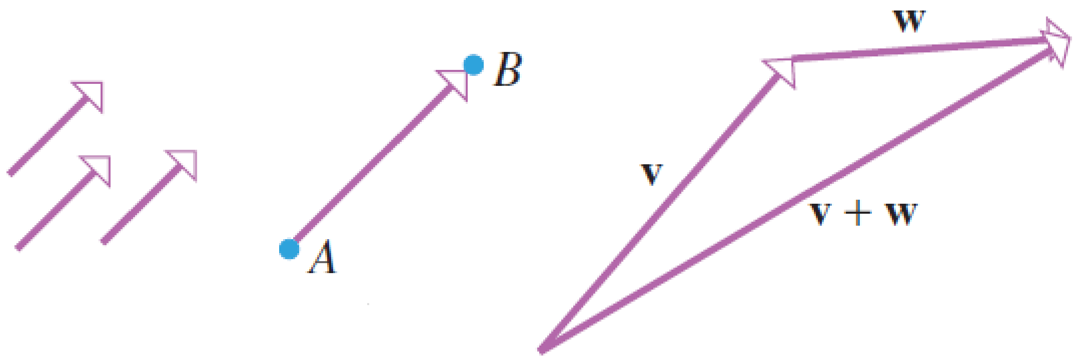


Figure 1-6

Definition If **v** and **w** are vectors, then the ***sum*** **v** + **w** is the vector from the initial point of **v** to the terminal point of **w** when the vectors are positioned so the initial point of **w** is at the terminal point of **v** (Figure 1-6).

- In Figure 1-7, we have constructed two sums, **v** + **w** (from purple arrows) and **w** + **v** (from green arrows). It is evident that

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

- The sum (gray arrow) coincides with the diagonal of the parallelogram determined by **v** and **w** when these vectors are positioned so they have the same initial point. Since the initial and terminal points of 0 coincide, it follows that

$$\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$$

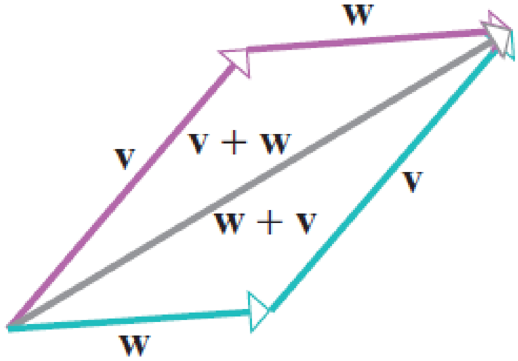
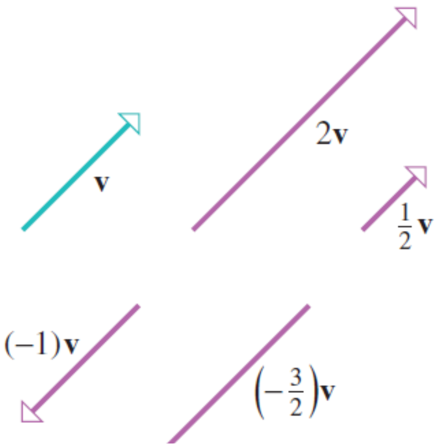


Figure 1-7

Definition If **v** is a nonzero vector and k is a nonzero real number (a scalar), then the ***scalar multiple*** k**v** is defined to be the vector whose length is $|k|$ times the length of **v** and whose direction is the same as that of **v** if $k > 0$ and opposite to that of **v** if $k < 0$. We define $k\mathbf{v} = \mathbf{0}$ if $k = 0$ or $\mathbf{v} = \mathbf{0}$.

Figure 1-8 shows the geometric relationship between a vector **v** and various scalar multiples of it.



- Observe that if k and **v** are nonzero, then the vectors **v** and k**v** lie on the same line if their initial points coincide and lie on parallel or coincident lines if they do not. Thus, we say that **v** and k**v** are ***parallel vectors***.
- Observe also that the vector $(-1)\mathbf{v}$ has the same length as **v** but is oppositely directed. We call $(-1)\mathbf{v}$ the ***negative*** of **v** and denote it by $-\mathbf{v}$ (Figure 1-9). In particular, $-\mathbf{0} = (-1)\mathbf{0} = \mathbf{0}$.

Vector subtraction is defined in terms of addition and scalar multiplication by

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$$

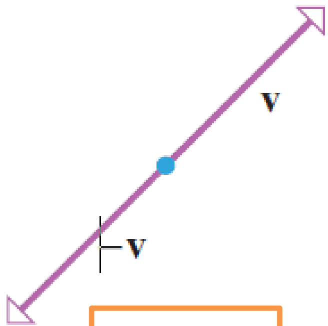


Figure 1-9

- The difference $\mathbf{v} - \mathbf{w}$ can be obtained geometrically by first constructing the vector $-\mathbf{w}$ and then adding **v** and $-\mathbf{w}$, say by the parallelogram method.
- However, if **v** and **w** are positioned so their initial points coincide, then $\mathbf{v} - \mathbf{w}$ can be formed more directly, as shown in Figure 1-10*b*, by drawing the vector from the terminal point of **w** (the second term) to the terminal point of **v** (the first term).

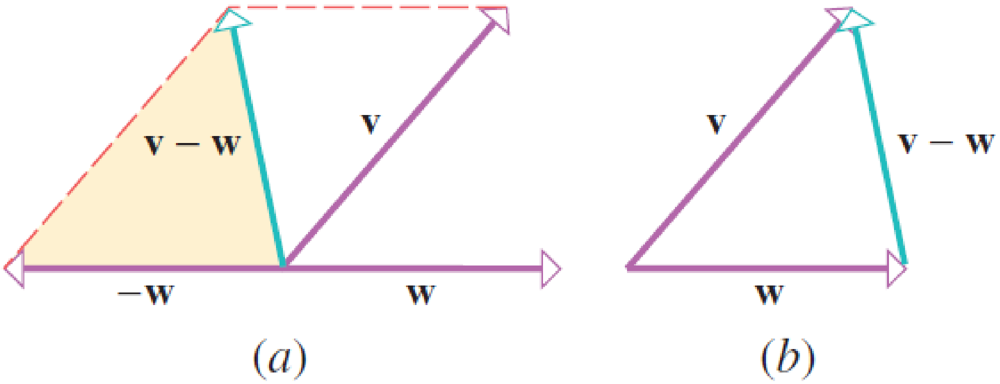


Figure 1-10

In the special case where **v** = **w** the terminal points of the vectors coincide, so their difference is **0**; that is,

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{v} - \mathbf{v} = \mathbf{0}$$

1.2.2 Vectors in coordinate systems

- As shown in figure 1-11, if a vector **v** is positioned with its initial point at the origin of a rectangular coordinate system, then its terminal point will have coordinates of the form (v_1, v_2) or (v_1, v_2, v_3), depending on whether the vector is in 2-space or 3-space.
- We call these coordinates the ***components*** of **v**, and we write **v** in *component form* using the ***bracket notation***

$$\underbrace{\mathbf{v} = \langle v_1, v_2 \rangle}_{\text{2-space}} \quad \text{or} \quad \underbrace{\mathbf{v} = \langle v_1, v_2, v_3 \rangle}_{\text{3-space}}$$

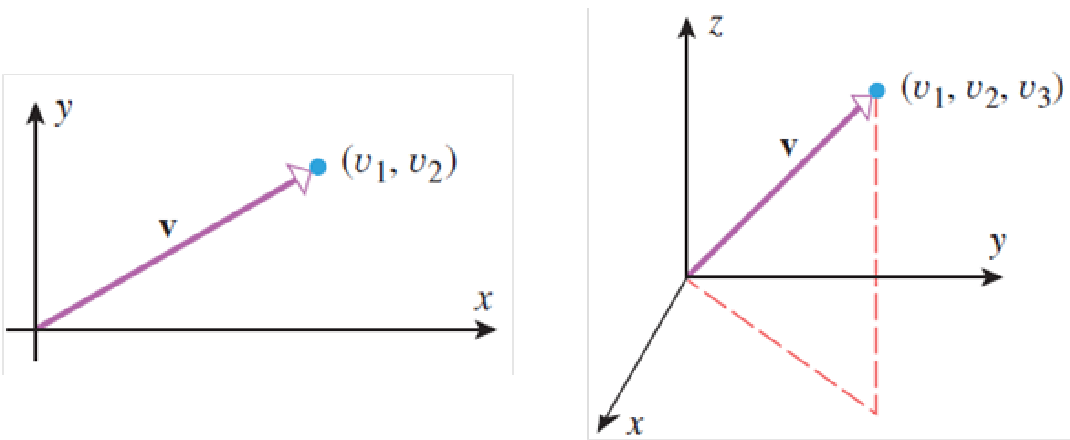


Figure 1-11

- In particular, the zero vectors in 2-space and 3-space are

$$\mathbf{0} = (0, 0) \text{ and } \mathbf{0} = (0, 0, 0)$$

- Considering the vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ in 2-space. If $\mathbf{v} = \mathbf{w}$, then the vectors have the same length and same direction, and this means that their terminal points coincide when their initial points are placed at the origin. It follows that $v_1 = w_1$ and $v_2 = w_2$, so we have shown that equivalent vectors have the same components.
- Conversely, if $v_1 = w_1$ and $v_2 = w_2$, then the terminal points of the vectors coincide when their initial points are placed at the origin. It follows that the vectors have the same length and same direction, so we have shown that vectors with the same components are equivalent.
- A similar argument holds for vectors in 3-space, so we have the following result.

Theorem Two vectors are equivalent if and only if their corresponding components are equal.