1.2.3 Arithmetic Operations on Vectors

Theorem *If* **v** = (v_1, v_2) *and* **w** = (w_1, w_2) *are vectors in 2-space and k is any scalar, then*

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2)$$

$$\mathbf{v} - \mathbf{w} = (v_1 - w_1, v_2 - w_2)$$

$$k\mathbf{v} = (kv_1, kv_2)$$

Similarly, if **v** = (v_1, v_2, v_3) *and* **w** = (w_1, w_2, w_3) *are vectors in 3-space and k is any scalar, then*

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$$

$$\mathbf{v} - \mathbf{w} = (v_1 - w_1, v_2 - w_2, v_3 - w_3)$$

$$k\mathbf{v} = (kv_1, kv_2, kv_3)$$

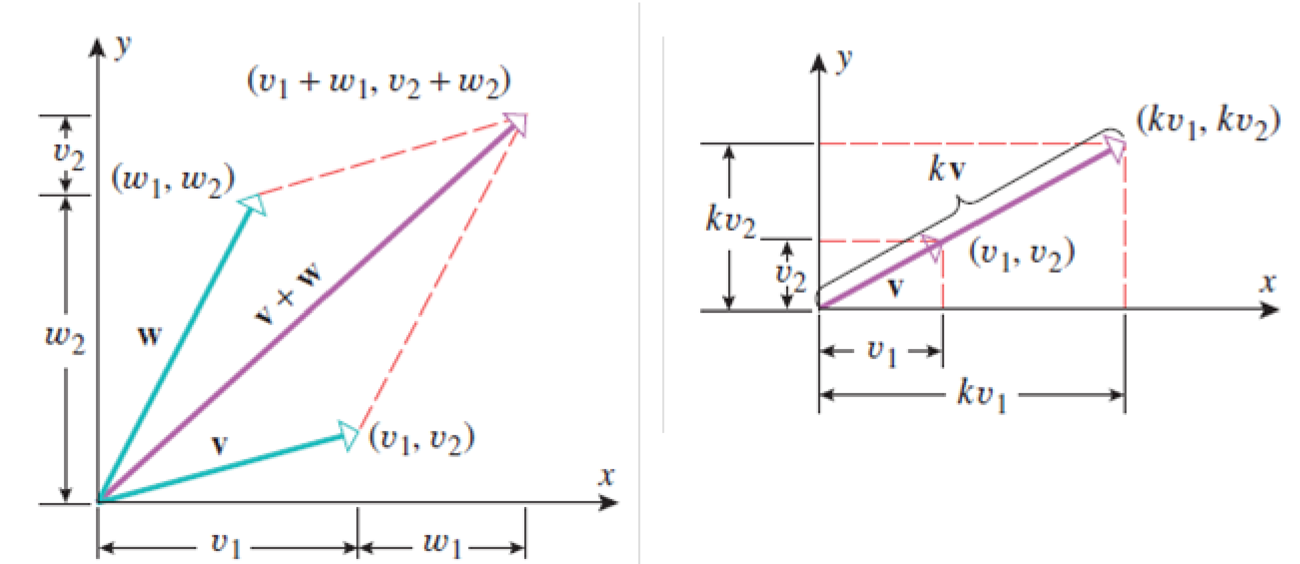

Figure 1-12

Example 1.4 If **v** = (−2, 0, 1) and **w** = (3, 5, −4), then

v + **w** = (−2, 0, 1) + (3, 5, −4) = (1, 5, -3)

3v = (-6, 0, 3)

-w = (-3, -5, 4)

w – 2v = (3, 5, −4) – (-4, 0, 2) = (7, 5, -6)

1.2.4 Vectors with Initial Point Not at the Origin

To be specific, suppose that $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in 2-space and we are interested in finding the components of the vector $\underrightarrow{P_1P_2}$. As shown in Figure 1-13, we can write this vector as

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1,\ y_2 - y_1)$$

Theorem *If $\overrightarrow{P_1P_2}$ is a vector in 2-space with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then*

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$$

Similarly, if $\overrightarrow{P_1P_2}$ is a vector in 3-space with initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$, then

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

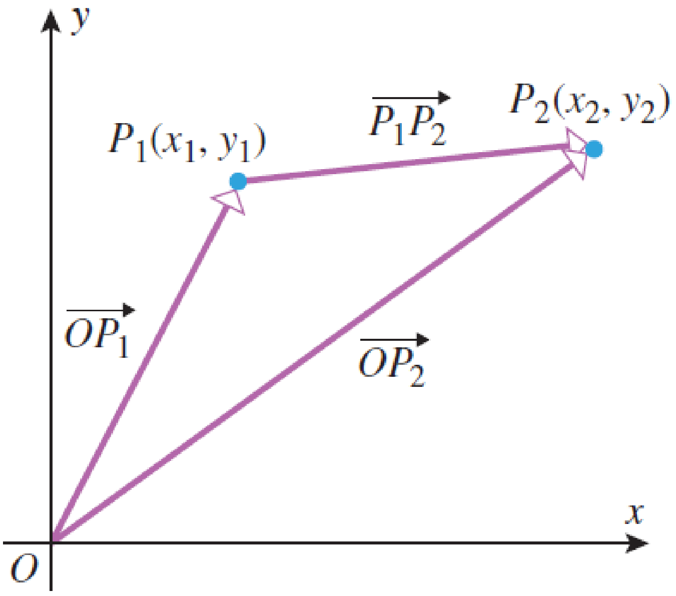

Figure 1-13

Example 1.5 In 2-space the vector from $P_1(1, 3)$ to $P_2(4,-2)$ is

$$\overrightarrow{P_1P_2} = (4 - 1, -2 - 3) = (3, -5)$$

and in 3-space the vector from A(0,−2, 5) to B(3, 4,−1) is

$$\overrightarrow{AB} = (3 - 0, 4 + 2, -1 - 5) = (3, 6, -6)$$

1.2.5 Rules of Vector Arithmetic

Theorem *For any vectors* **u**, **v**, *and* **w** *and any scalars* ***k*** *and* ***l***, *the following relationships hold:*

(a) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$	(e) $k(l\mathbf{u})=(kl)\mathbf{u}$
(b) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$	(f) $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
(c) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$	(g) $(k+l)\mathbf{u}=k\mathbf{u}+l\mathbf{u}$
(d) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$	(h) $1\mathbf{u}=\mathbf{u}$

1.2.6 Norm of a Vector

- The distance between the initial and terminal points of a vector **v** is called the ***length***, the ***norm***, or the ***magnitude*** of **v** and is denoted by ||**v**||.
- This distance does not change if the vector is translated, so for purposes of calculating the norm, we can assume that the vector is positioned with its initial point at the origin (Figure 1-14). This makes it evident that the norm of a vector **v** = (v_1, v_2) in 2-space is given by

$$\|\mathrm{v}\| = \sqrt{v_1^2 + v_2^2}$$

- and the norm of a vector v =(v_1, v_2, v_3) in 3-space is given by

$$\|\mathrm{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

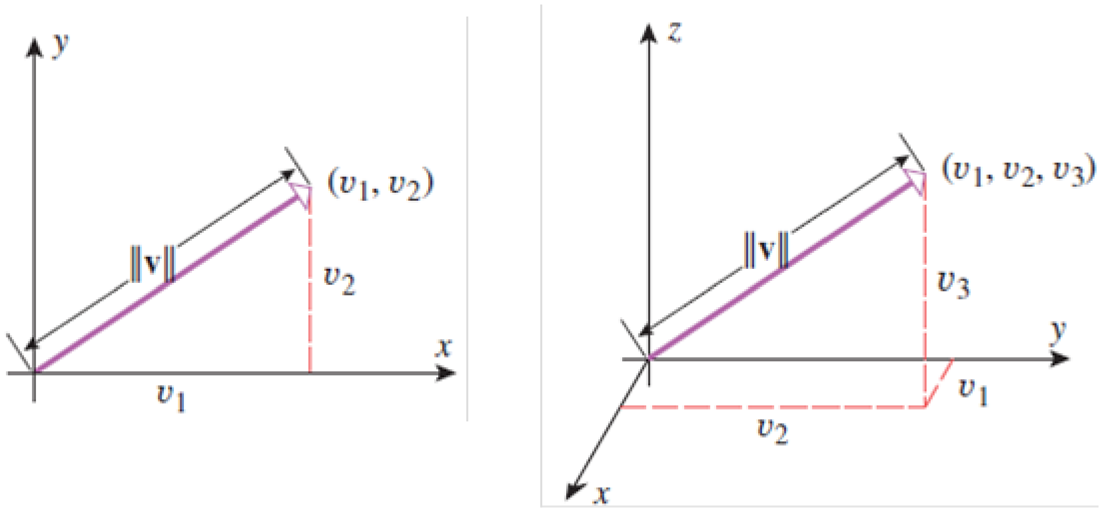

Figure 1-14

Example 1.6 Find the norms of **v** = (−2, 3), 10**v** = (−20, 30), and **w** = (2, 3, 6).

Solution

$$\|v\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$$

$$\|10v\| = \sqrt{(-20)^2 + 30^2} = \sqrt{1300} = 10\sqrt{13}$$

$$\|w\| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$$

Note: $\|kv\| = |k|\|v\|$

For example $\|3v\| = |3|\|v\| = 3\|v\|$

$$\|-2v\| = |-2|\|v\| = 2\|v\|$$

1.2.7 Unit Vectors

- A vector of length 1 is called a ***unit vector***.

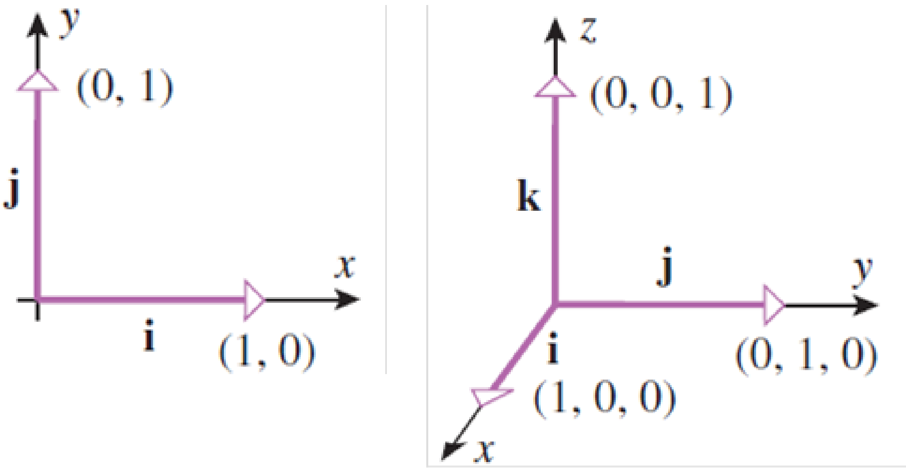

Figure 1-15

In an *xy*-coordinate system the unit vectors along the *x*- and *y*-axes are denoted by **i** and **j**, respectively; and in an *xyz*-coordinate system the unit vectors along the *x*-, *y*-, and *z*-axes are denoted by **i**, **j**, and **k**, respectively.

$\mathbf{i} = \langle 1, 0\rangle, \quad \mathbf{j} = \langle 0, 1\rangle$ In 2-space

$\mathbf{i} = \langle 1, 0, 0\rangle, \quad \mathbf{j} = \langle 0, 1, 0\rangle, \quad \mathbf{k} = \langle 0, 0, 1\rangle$ In 3-space

As shown in figure 1-15, every vector in 2-space is expressible uniquely in terms of **i** and **j**, and every vector in 3-space is expressible uniquely in terms of **i**, **j**, and **k** as follows:

$$\mathbf{v} = (v_1, v_2) = (v_1, 0) + (0, v_2) = v_1(1, 0) + v_2(0, 1) = v_1\mathbf{i} + v_2\mathbf{j}$$

$$\mathbf{v} = (v_1, v_2, v_3) = v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

Example 1.7 The following table provides some examples of vector notation in 2-space and 3-space.

2-SPACE	3-SPACE
$\langle 2, 3\rangle = 2\mathbf{i} + 3\mathbf{j}$	$\langle 2, -3, 4\rangle = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$
$\langle -4, 0\rangle = -4\mathbf{i} + 0\mathbf{j} = -4\mathbf{i}$	$\langle 0, 3, 0\rangle = 3\mathbf{j}$
$\langle 0, 0\rangle = 0\mathbf{i} + 0\mathbf{j} = \mathbf{0}$	$\langle 0, 0, 0\rangle = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$
$(3\mathbf{i} + 2\mathbf{j}) + (4\mathbf{i} + \mathbf{j}) = 7\mathbf{i} + 3\mathbf{j}$	$(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$
$5(6\mathbf{i} - 2\mathbf{j}) = 30\mathbf{i} - 10\mathbf{j}$	$2(\mathbf{i} + \mathbf{j} - \mathbf{k}) + 4(\mathbf{i} - \mathbf{j}) = 6\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$
$\lVert 2\mathbf{i} - 3\mathbf{j}\rVert = \sqrt{2^2 + (-3)^2} = \sqrt{13}$	$\lVert \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\rVert = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$
$\lVert v_1\mathbf{i} + v_2\mathbf{j}\rVert = \sqrt{v_1^2 + v_2^2}$	$\lVert\langle v_1, v_2, v_3\rangle\rVert = \sqrt{v_1^2 + v_2^2 + v_3^2}$

1.2.8 Normalizing a Vector

- A common problem in applications is to find a unit vector **u** that has the same direction as some given nonzero vector **v**. This can be done by multiplying **v** by the reciprocal of its length; that is,

$$\mathrm{u} = \frac{1}{\lVert \mathrm{v}\rVert}\,\mathrm{v} = \frac{\mathrm{v}}{\lVert \mathrm{v}\rVert}$$

is a unit vector with the same direction as **v**—the direction is the same because $k = 1/\lVert \mathrm{v}\rVert$ is a positive scalar, and the length is 1 because

$$\lVert \mathrm{u}\rVert = \lVert k\mathrm{v}\rVert = |k|\lVert \mathrm{v}\rVert = \frac{1}{\lVert \mathrm{v}\rVert}\,\lVert \mathrm{v}\rVert = 1$$

Example 1.8 find the unit vector that has the same direction as $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

$$\lVert \mathrm{v}\rVert = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

So the unit vector **u** in the same direction as **v** is

$$\mathbf{u} = \frac{1}{3}\mathbf{v} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$$

1.2.9 Vectors Determined by Length and Angle

If **v** is a nonzero vector with its initial point at the origin of an *xy*-coordinate system, and if θ is the angle from the positive *x*-axis to the radial line through **v**, then the *x*-component of **v** can be written as $\|\mathrm{v}\| \cos\theta$ and the *y*-component as $\|\mathrm{v}\| \sin\theta$ (Figure 1-16); and hence **v** can be expressed in trigonometric form as

$$\mathbf{v} = \|\mathrm{v}\| \; (\cos\theta\,, \sin\theta) \quad or \quad \mathbf{v} = \|\mathrm{v}\| \cos\theta\, \mathbf{i} + \|\mathrm{v}\| \sin\theta\, \mathbf{j}$$

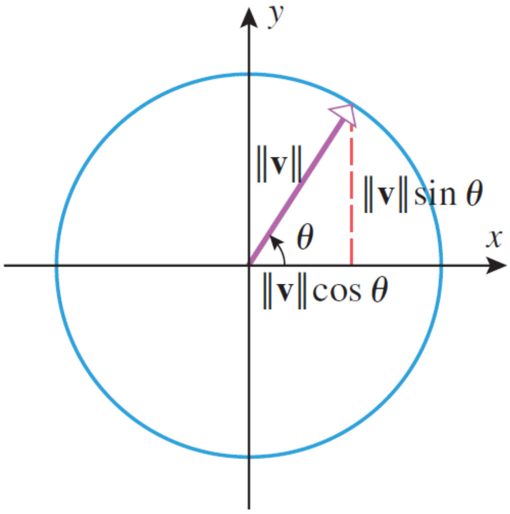

Figure 1-16

In the special case of a unit vector **u** this simplifies to

$$\mathbf{u} = (\cos\theta\,, \sin\theta) \quad or \quad \mathbf{u} = \cos\theta\, \mathbf{i} + \sin\theta\, \mathbf{j}$$

Example 1.9

(a) Find the vector of length **2** that makes an angle of ***π*/4** with the positive *x*-axis.

(b) Find the angle that the vector $\mathbf{v} = -\sqrt{3}\,\mathbf{i} + \mathbf{j}$ makes with the positive *x*-axis.

$$\mathbf{v} = 2\; \cos\frac{\pi}{4}\mathbf{i}\,, +2\,\sin\frac{\pi}{4}\mathbf{j} = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$$

We will normalize **v**, then use (previous equation) to find sin θ and cos θ, and then use these values to find θ. Normalizing **v** yields

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{-\sqrt{3}\mathbf{i} + \mathbf{j}}{\sqrt{(-\sqrt{3})^2 + 1^2}} = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

Thus, $\cos\theta = -\sqrt{3}/2$ and $\sin\theta = 1/2$, from which we conclude that $\boldsymbol{\theta = 5\pi/6}$.

1.2.10 Vectors Determined by Length and a Vector in the Same Direction

It is a common problem in many applications that a direction in 2-space or 3-space is determined by some known unit vector **u**, and it is of interest to find the components of a vector **v** that has the same direction as **u** and some specified length $\|\mathbf{v}\|$. This can be done by expressing **v** as

$$\mathbf{v} = \|\mathbf{v}\|\mathbf{u}$$

v is equal to its length times a unit vector in the same direction.

and then reading off the components of $\|\mathbf{v}\|\mathbf{u}$.

Example 1.10 Figure 1-17 shows a vector v of length $\sqrt{5}$ that extends along the line through A and B. Find the components of **v**.

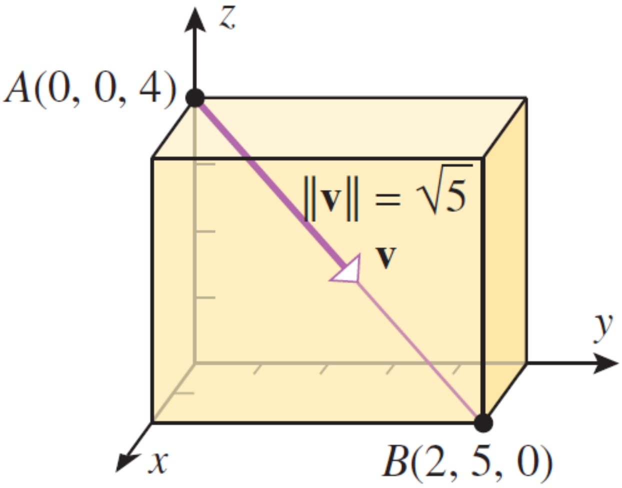

Figure 1-17

Solution:

$$\overrightarrow{AB} = \langle 2, 5, 0\rangle - \langle 0, 0, 4\rangle = \langle 2, 5, -4\rangle$$

$$\|\overrightarrow{AB}\| = \sqrt{2^2 + 5^2 + (-4)^2} = \sqrt{45} = 3\sqrt{5}$$

$$\frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \left\langle \frac{2}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}, -\frac{4}{3\sqrt{5}} \right\rangle$$

$$\mathbf{v} = \|\mathbf{v}\|\left(\frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|}\right) = \sqrt{5}\left\langle \frac{2}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}, -\frac{4}{3\sqrt{5}} \right\rangle = \left\langle \frac{2}{3}, \frac{5}{3}, -\frac{4}{3} \right\rangle$$

1.2.11 Resultant of Two Concurrent Forces

- If two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are applied at the same point on an object, then the two forces have the same effect on the object as the single force $\mathbf{F}_1 + \mathbf{F}_2$ applied at the point (Figure 1-18).
- Physicists and engineers call $\mathbf{F}_1 + \mathbf{F}_2$ the ***resultant*** of $\mathbf{F}_1$ and $\mathbf{F}_2$, and they say that the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are ***concurrent*** to indicate that they are applied at the same point.

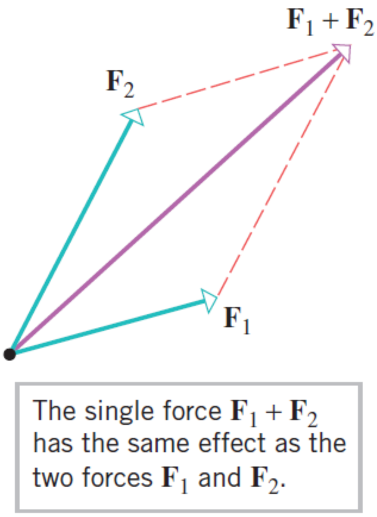

Figure 1-18

Example 1.11 Suppose that two forces are applied to an eye bracket, as shown in Figure 1-19. Find the magnitude of the resultant and the angle θ that it makes with the positive x-axis.

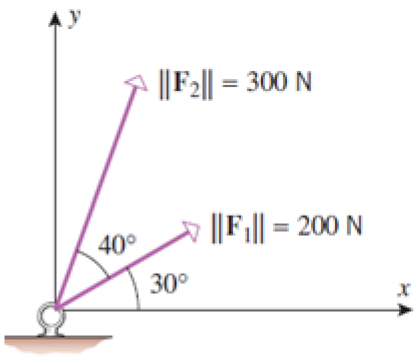

Figure 1-19

Solution. Note that $\mathbf{F}_1$ makes an angle of 30° with the positive x-axis and $\mathbf{F}_2$ makes an angle of $30^\circ + 40^\circ = 70^\circ$ with the positive x-axis. Since we are given that $\|\mathbf{F}_1\| = 200$ N and $\|\mathbf{F}_2\| = 300$ N,

$$\mathbf{F}_1 = 200\langle \cos 30^\circ, \sin 30^\circ \rangle = \langle 100\sqrt{3}, 100 \rangle$$

$$\mathbf{F}_2 = 300\langle \cos 70^\circ, \sin 70^\circ \rangle = \langle 300\cos 70^\circ, 300\sin 70^\circ \rangle$$

Therefore, the resultant $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ has component form

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \langle 100\sqrt{3} + 300\cos 70°,\ 100 + 300\sin 70°\rangle$$
$$= 100\langle\sqrt{3} + 3\cos 70°,\ 1 + 3\sin 70°\rangle \approx \langle 275.8, 381.9\rangle$$

The magnitude of the resultant is then

$$\|\mathbf{F}\| = 100\sqrt{\left(\sqrt{3} + 3\cos 70°\right)^2 + \left(1 + 3\sin 70°\right)^2} \approx 471 \text{ N}$$

Let θ denote the angle **F** makes with the positive x-axis when the initial point of **F** is at the origin.

$$\|\mathbf{F}\|\cos\theta = 100\sqrt{3} + 300\cos 70° \quad \text{or} \quad \cos\theta = \frac{100\sqrt{3} + 300\cos 70°}{\|\mathbf{F}\|}$$

Since the terminal point of F is in the first quadrant, we have

$$\theta = \cos^{-1}\left(\frac{100\sqrt{3} + 300\cos 70°}{\|\mathbf{F}\|}\right) \approx 54.2°$$

See Figure 1-20

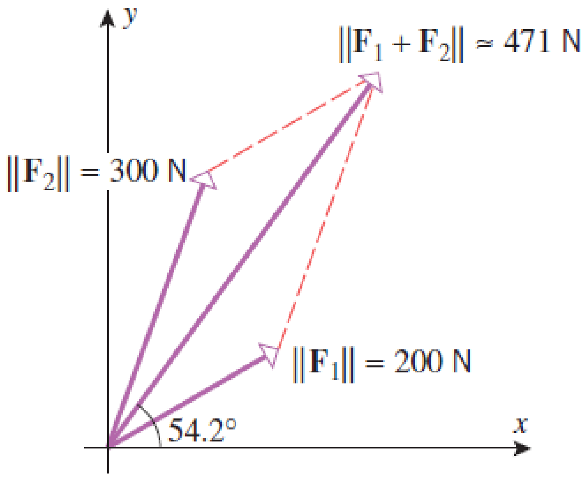

Figure 1-20